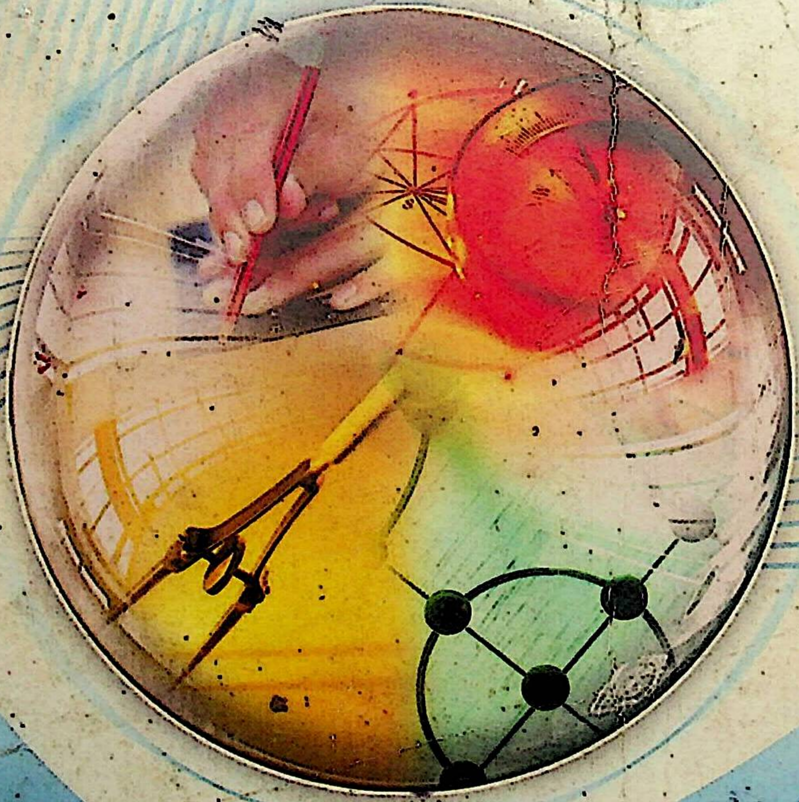
MATHEMATICS

CLASS XII VOLUME-2

R.D. SHARMA



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MATHEMATICS

CLASS XII Prashant VOLUME-2

including
Very Short Answer Questions (VSAQs)
& Multiple Choice Questions (MCQs)

Based on the latest revised syllabus prescribed by CBSE for Class XII under 10+2 Pattern of Senior School Certificate Examination

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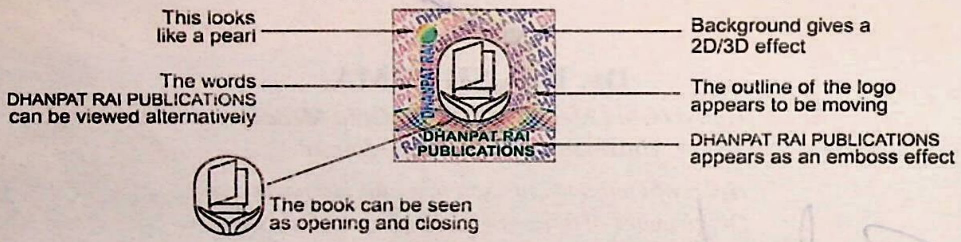

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Dear Teachers & Students

I feel deeply indebted to all of you for giving such a tremendous response to the earlier editions of this book. It will be my sincere endeavour to keep on serving you through the new editions also.

Main highlights of the present edition are:

— **The concept of addition of Determinants has been introduced.**

— **Chapters on Integration and Probability have been thoroughly revised for better understanding of the students.**

— **New problems have been added in the chapters on Definite Integrals and Areas of Bounded Regions.**

— **All NCERT textbook problems have been marked as [NCERT].**

It is my sincere advice to the students that in each chapter first they should go through the theory and concepts thoroughly, then they should attempt to solve the illustrative examples without looking at their solutions. They should consult the solutions only when they are unable to solve on their own. The exercises given at the end of each section should be attempted at the time of revision of the chapter.

Please send your feedback, suggestions or queries through e-mail to **ishkapur@vsnl.com**

With my Best Wishes

Dr. R.D. SHARMA

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MATHEMATICS - XII **Volume 2**

20

DEFINITE INTEGRALS

20.1 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

STATEMENT: *Let $\phi(x)$ be the primitive or antiderivative of a continuous function $f(x)$ defined on $[a, b]$ i.e., $\frac{d}{dx}\{\phi(x)\} = f(x)$. Then the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)\,dx$ and is equal to $[\phi(b) - \phi(a)]$.*

i.e., $$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \qquad \text{...(i)}$$

The numbers a and b are called the limits of integration, 'a' is called the lower limit and 'b' the upper limit. The interval $[a, b]$ is called the interval of integration.

If we use the notation $\left[\phi(x)\right]_a^b$ to denote $\phi(b) - \phi(a)$, then from (i), we have

$$\int_a^b f(x)\,dx = \left[\phi(x)\right]_a^b$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \left(\phi(x) \text{ at } x = b\right) - \left(\phi(x) \text{ at } x = a\right)$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = (\text{Value of antiderivative at } b, \text{ the upper limit}) - \text{Value of antiderivative at } a, \text{ the lower limit}$$

<u>REMARK 1</u> *In the above statement it does not matter which anti-derivative is used to evaluate the definite integral, because if $\int f(x)\,dx = \phi(x) + C$, then*

$$\int_a^b f(x)\,dx = \left[\phi(x) + C\right]_a^b = \{\phi(b) + C\} - \{\phi(a) + C\} = \phi(b) - \phi(a)$$

In other words, to evaluate the definite integral there is no need to keep the constant of integration.

<u>REMARK 2</u> $\int_a^b f(x)\,dx$ *is read as "the integral of $f(x)$ from a to b".*

20.2 EVALUATION OF DEFINITE INTEGRALS

To evaluate the definite integral $\int_a^b f(x)\,dx$ of a continuous function $f(x)$ defined on $[a, b]$ we use the following algorithm.

ALGORITHM

STEP I *Find the indefinite integral $\int f(x)\,dx$. Let this be $\phi(x)$. There is no need to keep the constant of integration.*

STEP II *Evaluate $\phi(b)$ and $\phi(a)$.*

STEP III *Calculate $\phi(b) - \phi(a)$.*

The number obtained in Step III is the value of the definite integral $\int_a^b f(x)\,dx$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :*

(i) $\int_1^2 x^2\,dx$ (ii) $\int_{-4}^{-1} \frac{1}{x}\,dx$

(iii) $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx$ (iv) $\int_0^1 \frac{1}{2x-3}\,dx$

SOLUTION We have,

(i) $$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

(ii) $$\int_{-4}^{-1} \frac{1}{x}\,dx = \Big[\log|x|\Big]_{-4}^{-1}$$

$$\Rightarrow \int_{-4}^{-1} \frac{1}{x}\,dx = \Big[\log|-1| - \log|-4|\Big] = \log 1 - \log 4 = 0 - \log 4 = -\log 4$$

(iii) $$\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx$$

$$= \int_0^1 \frac{\sqrt{1+x}-\sqrt{x}}{(\sqrt{1+x}+\sqrt{x})(\sqrt{1+x}-\sqrt{x})}\,dx$$

$$= \int_0^1 (\sqrt{1+x}-\sqrt{x})\,dx$$

$$= \left[\frac{2}{3}(1+x)^{3/2} - \frac{2}{3}x^{3/2}\right]_0^1$$

$$= \left[\frac{2}{3}(1+1)^{3/2} - \frac{2}{3}(1)^{3/2}\right] - \left[\frac{2}{3}(1+0)^{3/2} - \frac{2}{3}(0)^{3/2}\right]$$

$$= \frac{2}{3}[2^{3/2} - 1] - \frac{2}{3}[1 - 0]$$

$$= \frac{2}{3}[2\sqrt{2} - 2] = \frac{4}{3}[\sqrt{2} - 1]$$

(iv) $$\int_0^1 \frac{1}{2x-3}\, dx$$

$$= \frac{1}{2}\Big[\log(2x-3)\Big]_0^1$$

$$= \frac{1}{2}\Big[\log|-1| - \log|-3|\Big] = \frac{1}{2}[\log 1 - \log 3] = \frac{1}{2}[0 - \log 3] = -\frac{1}{2}\log 3$$

EXAMPLE 2 *Evaluate :*

(i) $\displaystyle\int_0^{\pi/4} \tan^2 x\, dx$ (ii) $\displaystyle\int_0^{\pi/2} \sin^2 x\, dx$

(iii) $\displaystyle\int_0^{\pi/4} \sin 3x \sin 2x\, dx$

SOLUTION We have,

(i) $$\int_0^{\pi/4} \tan^2 x\, dx$$

$$= \int_0^{\pi/4} (\sec^2 x - 1)\, dx = \Big[\tan x - x\Big]_0^{\pi/4} = \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - (\tan 0 - 0) = \left(1 - \frac{\pi}{4}\right)$$

(ii) $$\int_0^{\pi/2} \sin^2 x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right] = \frac{\pi}{4}$$

(iii) $$\int_0^{\pi/4} \sin 3x \sin 2x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/4} (2\sin 3x \sin 2x)\, dx$$

$$= \frac{1}{2}\int_0^{\pi/4} (\cos x - \cos 5x)\, dx$$

$$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(\sin\frac{\pi}{4} - \frac{\sin\frac{5\pi}{4}}{5}\right) - \left(\sin 0 - \frac{\sin 0}{5}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{(\sqrt{2})\,5}\right] = \frac{6}{2\,(5\sqrt{2})} = \frac{3\sqrt{2}}{10}$$

EXAMPLE 3 *Evaluate :*

(i) $\int_0^{\pi} \sin^3 x\, dx$ (ii) $\int_0^{\pi/2} \cos^3 x\, dx$ (iii) $\int_0^{\pi/2} \sin^4 x\, dx$

SOLUTION We have,

(i) $\int_0^{\pi} \sin^3 x\, dx$

$$= \int_0^{\pi} \frac{3\sin x - \sin 3x}{4}\, dx \qquad [\because \sin 3x = 3\sin x - 4\sin^3 x]$$

$$= \frac{1}{4}\int_0^{\pi} (3\sin x - \sin 3x)\, dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\left(-3\cos\pi + \frac{\cos 3\pi}{3}\right) - \left(-3\cos 0 + \frac{\cos 0}{3}\right)\right]$$

$$= \frac{1}{4}\left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right] = \frac{4}{3}$$

(ii) $\int_0^{\pi/2} \cos^3 x\, dx$

$$= \int_0^{\pi/2} \frac{\cos 3x + 3\cos x}{4}\, dx \qquad [\because \cos 3x = 4\cos^3 x - 3\cos x]$$

$$= \frac{1}{4}\int_0^{\pi/2} (\cos 3x + 3\cos x)\, dx$$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(\frac{\sin 3\pi/2}{3} + 3\sin\frac{\pi}{2}\right) - \left(\frac{\sin 0}{3} + 3\sin 0\right)\right]$$

$$= \frac{1}{4}\left[\left(-\frac{1}{3} + 3\right) - (0 + 0)\right] = \frac{2}{3}$$

(iii) $\int_0^{\pi/2} \sin^4 x\, dx$

$$= \frac{1}{4}\int_0^{\pi/2} (2\sin^2 x)^2\,dx = \frac{1}{4}\int_0^{\pi/2} (1-\cos 2x)^2\,dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x)\,dx = \frac{1}{4}\int_0^{\pi/2} 1 - 2\cos 2x + \frac{1+\cos 4x}{2}\,dx$$

$$= \frac{1}{8}\int_0^{\pi/2} (3 - 4\cos 2x + \cos 4x)\,dx = \frac{1}{8}\left[3x - \frac{4}{2}\sin 2x + \frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{8}\left[\left\{\frac{3\pi}{2} - 2\sin\pi + \frac{1}{4}\sin 2\pi\right\} - \{0 - 0 + 0\}\right] = \frac{1}{8}\left[\frac{3\pi}{2} - 0 + 0\right] = \frac{3\pi}{16}$$

EXAMPLE 4 *Evaluate :*

(i) $\int_0^{\pi/4} \sqrt{1+\sin 2x}\,dx$ (ii) $\int_0^{\pi/4} \sqrt{1-\sin 2x}\,dx$ **[CBSE 2004]**

SOLUTION We have,

(i) $\int_0^{\pi/4} \sqrt{1+\sin 2x}\cdot dx$

$$= \int_0^{\pi/4} \sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x}\,dx$$

$$= \int_0^{\pi/4} (\cos x + \sin x)\,dx = \Big[\sin x - \cos x\Big]_0^{\pi/4}$$

$$= \left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) - (\sin 0 - \cos 0) = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - (0-1) = 1$$

(ii) $\int_0^{\pi/4} \sqrt{1-\sin 2x}\,dx$

$$= \int_0^{\pi/4} \sqrt{\sin^2 x + \cos^2 x - 2\sin x\cos x}\,dx = \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2}$$

$$= \int_0^{\pi/4} |(\cos x - \sin x)|\,dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\,dx \qquad \left[\begin{array}{l}\because\ 0 < x < \pi/4 \\ \therefore\ \cos x > \sin x \Rightarrow \cos x - \sin x > 0\end{array}\right]$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\,dx = \Big[\sin x + \cos x\Big]_0^{\pi/4}$$

$$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1) = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

EXAMPLE 5 *Evaluate :* $\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\, dx$

SOLUTION We have,

$$\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\; dx$$

$$= \int_{\pi/4}^{\pi/2} \sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x}\; dx$$

$$= \int_{\pi/4}^{\pi/2} \sqrt{(\cos x - \sin x)^2}\; dx$$

$$= \int_{\pi/4}^{\pi/2} |\cos x - \sin x|\, dx \qquad \left[\because \sqrt{x^2} = |x|\right]$$

$$= \int_{\pi/4}^{\pi/2} -(\cos x - \sin x)\; dx \qquad \left[\begin{array}{l} \because \cos x < \sin x \text{ for } \frac{\pi}{4} < x < \frac{\pi}{2} \\ \therefore \cos x - \sin x < 0 \\ \Rightarrow |\cos x - \sin x| = -(\cos x - \sin x) \end{array}\right]$$

$$= \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\; dx$$

$$= \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2}$$

$$= \left\{-\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right\} - \left\{-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right\} = (0-1) - \left(\frac{2}{\sqrt{2}}\right) = \sqrt{2} - 1$$

EXAMPLE 6 *Evaluate :* $\int_0^{\pi/2} \sqrt{1-\cos 2x}\, dx.$

SOLUTION We have,

$$\int_0^{\pi/2} \sqrt{1-\cos 2x}\, dx$$

$$= \int_0^{\pi/2} \sqrt{2\sin^2 x}\; dx$$

$$= \sqrt{2} \int_0^{\pi/2} \sin x\, dx$$

$$= \sqrt{2}\; [-\cos x]_0^{\pi/2} = \sqrt{2}\left[\left(-\cos\left(\frac{\pi}{2}\right)\right) - (-\cos 0)\right] = \sqrt{2}\,(0+1) = \sqrt{2}$$

EXAMPLE 7 *If* $\int_0^1 (3x^2 + 2x + k)\, dx = 0$, find k.

SOLUTION We have,

$$\int_0^1 (3x^2 + 2x + k)\, dx = 0$$

$\Rightarrow \quad \left[x^3+x^2+kx\right]_0^1 = 0 \Rightarrow (1+1+k)-0 = 0 \Rightarrow k = -2$

EXAMPLE 8 *If* $\int_1^a (3x^2+2x+1)\,dx = 11$, *find a.*

SOLUTION We have,

$$\int_1^a (3x^2+2x+1)\,dx = 11$$

$\Rightarrow \quad \left[x^3+x^2+x\right]_1^a = 11$

$\Rightarrow \quad (a^3+a^2+a)-(1+1+1) = 11$

$\Rightarrow \quad a^3+a^2+a-3 = 11$

$\Rightarrow \quad a^3+a^2+a-14 = 0$

$\Rightarrow \quad (a-2)(a^2+3a+7) = 0 \Rightarrow a = 2$

EXAMPLE 9 *If* $\int_a^b x^3\,dx = 0$ *and if* $\int_a^b x^2\,dx = \frac{2}{3}$, *find a and b.*

SOLUTION We have,

$$\int_a^b x^3\,dx = 0$$

$\Rightarrow \quad \left[\frac{x^4}{4}\right]_a^b = 0 \Rightarrow \frac{1}{4}(b^4-a^4) = 0 \Rightarrow b^4-a^4 = 0$

$\Rightarrow \quad (b^2-a^2)(b^2+a^2) = 0 \Rightarrow b^2-a^2 = 0 \Rightarrow b = -a \qquad [\because b \neq a]$

Now, $\quad \int_a^b x^2\,dx = \frac{2}{3}$

$\Rightarrow \quad \left(\frac{x^3}{3}\right)_a^b = \frac{2}{3} \Rightarrow \frac{1}{3}(b^3-a^3) = \frac{2}{3}$

$\Rightarrow \quad b^3-a^3 = 2 \Rightarrow (-a)^3-a^3 = 2 \qquad [\because b = -a]$

$\Rightarrow \quad -2a^3 = 2 \Rightarrow a^3 = -1 \Rightarrow a = -1$

$\therefore \quad b = -a \Rightarrow b = 1$

EXAMPLE 10 *If* $\int_0^a \sqrt{x}\,dx = 2a\int_0^{\pi/2} \sin^3 x\,dx$, *find the value of* $\int_a^{a+1} x\,dx$.

SOLUTION We have,

$$\int_0^a \sqrt{x}\,dx = \frac{2}{3}\left[x^{3/2}\right]_0^a = \frac{2}{3}a^{3/2}$$

and, $\int_0^{\pi/2} \sin^3 x \, dx$

$$= \int_0^{\pi/2} \frac{3 \sin x - \sin 3x}{4} dx$$

$$= \frac{1}{4} \int_0^{\pi/2} (3 \sin x - \sin 3x) \, dx$$

$$= \frac{1}{4}\left[-3 \cos x + \frac{\cos 3x}{3}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(-3 \cos\left(\frac{\pi}{2}\right) + \frac{\cos \frac{3\pi}{2}}{3}\right) - \left(-3 + \frac{1}{3}\right)\right] = \frac{1}{4}\left[0 - \left(-3 + \frac{1}{3}\right)\right] = \frac{1}{4}\left[3 - \frac{1}{3}\right] = \frac{2}{3}$$

$\therefore \quad \int_0^a \sqrt{x} \, dx = 2a \int_0^{\pi/2} \sin^3 x \, dx$

$\Rightarrow \quad \frac{2}{3} a^{3/2} = 2a\left(\frac{2}{3}\right)$

$\Rightarrow \quad a^{3/2} = 2a \Rightarrow a^3 = 4a^2 \Rightarrow a^2 (a - 4) = 0 \Rightarrow a = 0, 4.$

When $a = 4$, we have

$$\int_a^{a+1} x \, dx = \int_4^5 x \, dx = \left[\frac{x^2}{2}\right]_4^5 = \frac{25}{2} - \frac{16}{2} = \frac{9}{2}.$$

When $a = 0$, we have

$$\int_a^{a+1} x \, dx = \int_a^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Hence, $\int_a^{a+1} x dx = \frac{9}{2}$ or, $\frac{1}{2}$

EXAMPLE 11 *Evaluate:*

(i) $\int_0^4 \frac{1}{\sqrt{x^2 + 2x + 3}} dx$ (ii) $\int_0^a \frac{1}{\sqrt{ax - x^2}} dx$

SOLUTION We have,

(i) $\int_0^4 \frac{1}{\sqrt{x^2 + 2x + 3}} dx$

$$= \int_0^4 \frac{1}{\sqrt{(x + 1)^2 + (\sqrt{2})^2}} dx$$

$$= \left[\log\left|x+1+\sqrt{(x+1)^2+(\sqrt{2})^2}\right|\right]_0^4$$

$$= \left[\log\left|x+1+\sqrt{x^2+2x+3}\right|\right]_0^4$$

$$= \log(5+\sqrt{16+8+3}) - \log(1+\sqrt{3})$$

$$= \log(5+3\sqrt{3}) - \log(1+\sqrt{3})$$

$$= \log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$$

(ii) $$\int_0^a \frac{1}{\sqrt{ax-x^2}}\,dx$$

$$= \int_0^a \frac{1}{\sqrt{-\left\{x^2-ax+\frac{a^2}{4}-\frac{a^2}{4}\right\}}}\,dx$$

$$= \int_0^a \frac{1}{\sqrt{\left(\frac{a}{2}\right)^2-\left(x-\frac{a}{2}\right)^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x-\frac{a}{2}}{\frac{a}{2}}\right)\right]_0^a = \left[\sin^{-1}\left(\frac{2x-a}{a}\right)\right]_0^a$$

$$= \sin^{-1}1-\sin^{-1}(-1) = 2\sin^{-1}(1) = 2\left(\frac{\pi}{2}\right) = \pi$$

EXAMPLE 12 *Evaluate:*

(i) $\displaystyle\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}}\,dx$ (ii) $\displaystyle\int_2^4 \frac{x}{x^2+1}\,dx$

(iii) $\displaystyle\int_0^1 \frac{2x}{5x^2+1}$ (iv) $\displaystyle\int_0^2 \frac{5x+1}{x^2+4}\,dx$

SOLUTION We have,

(i) $$\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}}\,dx$$

$$= \int_{1/4}^{1/2} \frac{1}{\sqrt{-\left\{x^2-x+\frac{1}{4}-\frac{1}{4}\right\}}}\,dx$$

$$= \int_{1/4}^{1/2} \frac{1}{\sqrt{-\left\{\left(x-\frac{1}{2}\right)^2-\frac{1}{4}\right\}}}$$

$$= \int_{1/4}^{1/2} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}} = \left[\sin^{-1}\left(\frac{x - 1/2}{1/2}\right)\right]_{1/4}^{1/2}$$

$$= \left[\sin^{-1}(2x-1)\right]_{1/4}^{1/2} = \left[\sin^{-1} 0 - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \left[0 + \sin^{-1}\frac{1}{2}\right] = \frac{\pi}{6}$$

(ii) $\displaystyle\int_2^4 \frac{x}{x^2+1}\, dx$

$$= \frac{1}{2}\int_2^4 \frac{2x}{x^2+1}\, dx$$

$$= \frac{1}{2}\left[\log(x^2+1)\right]_2^4$$

$$= \frac{1}{2}\left[\log(4^2+1) - \log(2^2+1)\right] = \frac{1}{2}\left[\log 17 - \log 5\right] = \frac{1}{2}\log\left(\frac{17}{5}\right)$$

(iii) $\displaystyle\int_0^1 \frac{2x}{5x^2+1}\, dx$

$$= \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}\, dx = \frac{1}{5}\left[\log(5x^2+1)\right]_0^1$$

$$= \frac{1}{5}\left[\log 6 - \log 1\right] = \frac{1}{5}\log 6$$

(iv) $\displaystyle\int_0^2 \frac{5x+1}{x^2+4}\, dx$

$$= \int_0^2 \frac{5x}{x^2+4}\, dx + \int_0^2 \frac{1}{x^2+4}\, dx$$

$$= \frac{5}{2}\int_0^2 \frac{2x}{x^2+4}\, dx + \int_0^2 \frac{1}{x^2+2^2}\, dx$$

$$= \frac{5}{2}\left[\log(x^2+4)\right]_0^2 + \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \frac{5}{2}\left[\log 8 - \log 4\right] + \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1} 0\right]$$

$$= \frac{5}{2}\log\left(\frac{8}{4}\right) + \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{5}{2}\log 2 + \frac{\pi}{8}$$

EXAMPLE 13 *Evaluate :*

(i) $\displaystyle\int_0^1 x e^x\, dx$ **[NCERT, HPSB 2001C]** (ii) $\displaystyle\int_1^2 \frac{\log x}{x^2}\, dx$

(iii) $\int_0^{\pi/2} x \sin x \, dx$ (iv) $\int_0^1 \left\{ xe^x + \sin \frac{\pi x}{4} \right\} dx$ **[NCERT, HPSB 1999C]**

SOLUTION We have,

(i) $\int_0^1 \underset{\text{I}}{x} \, \underset{\text{II}}{e^x} \, dx$

$$= \left[x e^x \right]_0^1 - \int_0^1 1 \cdot e^x \, dx = (1 \cdot e^1 - 0 \, e^0) - [e^x]_0^1 = (e - 0) - (e - e^0) = 1$$

(ii) $\int_1^2 \frac{\log x}{x^2} \, dx$

$$= \int_1^2 \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{\frac{1}{x^2}} = \left[\log x \cdot \left(-\frac{1}{x} \right) \right]_1^2 - \int_1^2 \frac{1}{x} \left(-\frac{1}{x} \right) dx \qquad \text{[Integ. by parts]}$$

$$= \left[-\frac{1}{x} \log x \right]_1^2 - \left[\frac{1}{x} \right]_1^2 = \left(-\frac{1}{2} \log 2 \right) + (1 \cdot \log 1) - \left(\frac{1}{2} - \frac{1}{1} \right)$$

$$= -\frac{1}{2} \log 2 + \frac{1}{2} = \frac{1}{2} (-\log 2 + 1) = \frac{1}{2} (-\log 2 + \log e) = \frac{1}{2} \log \left(\frac{e}{2} \right)$$

(iii) $\int_0^{\pi/2} \underset{\text{I}}{x} \, \underset{\text{II}}{\sin x} \, dx$

$$= \left[-x \cos x \right]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot (-\cos x) \, dx \qquad \text{[Integ. by parts]}$$

$$= \left[-x \cos x \right]_0^{\pi/2} + \left[\sin x \right]_0^{\pi/2} = \left(-\frac{\pi}{2} \cos \frac{\pi}{2} + 0 \cos 0 \right) + \left(\sin \frac{\pi}{2} - \sin 0 \right) = 1$$

(iv) $\int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx$

$$= \int_0^1 xe^x \, dx + \int_0^1 \sin \frac{\pi x}{4} \, dx$$

$$= \left[x e^x \right]_0^1 - \int_0^1 1 \cdot e^x \, dx - \frac{4}{\pi} \left[\cos \frac{\pi x}{4} \right]_0^1$$

$$= \left[x e^x \right]_0^1 - \left[e^x \right]_0^1 - \frac{4}{\pi} \left[\cos \frac{\pi x}{4} \right]_0^1$$

$$= (1 \cdot e^1 - 0 \, e^0) - (e^1 - e^0) - \frac{4}{\pi} \left(\cos \frac{\pi}{4} - \cos 0 \right)$$

$$= (e - 0) - (e - 1) - \frac{4}{\pi} \left(\frac{1}{\sqrt{2}} - 1 \right) = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

EXAMPLE 14 *Evaluate:*

(i) $\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$ **[NCERT, CBSE 2010]** (ii) $\int_1^3 \frac{1}{x^2(x+1)}\,dx$ **[NCERT]**

SOLUTION We have,

(i) $\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$

$$= 5\int_1^2 \frac{x^2}{x^2+4x+3}\,dx = 5\int_1^2 \left(1-\frac{4x+3}{x^2+4x+3}\right)dx$$

$$= 5\int_1^2 1\cdot dx - 5\int_1^2 \frac{4x+3}{x^2+4x+3}\,dx = 5\int_1^2 1\cdot dx - 5\int_1^2 \frac{2(2x+4)-5}{x^2+4x+3}\,dx$$

$$= 5\int_1^2 1\cdot dx - 5\left[\int_1^2 \left\{\frac{2(2x+4)}{x^2+4x+3}-\frac{5}{x^2+4x+3}\right\}dx\right]$$

$$= 5\int_1^2 1\cdot dx - 10\int_1^2 \frac{2x+4}{x^2+4x+3}\,dx + 25\int_1^2 \frac{1}{x^2+4x+3}\,dx$$

$$= 5\int_1^2 1\cdot dx - 10\int_1^2 \frac{2x+4}{x^2+4x+3}\,dx + 25\int_1^2 \frac{1}{(x+2)^2-1^2}\,dx$$

$$= 5\,[x]_1^2 - 10\left[\log(x^2+4x+3)\right]_1^2 + 25\cdot\frac{1}{2(1)}\left[\log\left|\frac{x+2-1}{x+2+1}\right|\right]_1^2$$

$$= 5(2-1) - 10\,[\log 15 - \log 8] + \frac{25}{2}\left[\log\left(\frac{3}{5}\right)-\log\left(\frac{2}{4}\right)\right]$$

$$= 5 - 10\log\left(\frac{15}{8}\right) + \frac{25}{2}\log\left(\frac{3}{5}\times\frac{4}{2}\right) = 5 - 10\log\frac{15}{8} + \frac{25}{2}\log\left(\frac{6}{5}\right)$$

(ii) Let $\frac{1}{x^2(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2}$ ...(i)

Then, $1 = Ax^2 + (Bx+C)(x+1)$ ...(ii)

Putting $x=0, x=-1$ respectively in (ii), we get

$C = 1$ and $A = 1$

Equating coefficients of x^2 on both sides of (ii), we get

$0 = A + B \Rightarrow B = -A = -1$

Substituting the values of A, B and C in (i), we obtain

$$\frac{1}{x^2(x+1)} = \frac{1}{x+1} + \frac{-x+1}{x^2}$$

$$\Rightarrow \quad \frac{1}{x^2(x+1)} = \frac{1}{x+1} - \frac{x}{x^2} + \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{1}{x^2(x+1)} = \frac{1}{x+1} - \frac{1}{x} + \frac{1}{x^2}$$

$$\Rightarrow \quad \int_1^3 \frac{1}{x^2(x+1)}\,dx = \int_1^3 \left(\frac{1}{x+1} - \frac{1}{x} + \frac{1}{x^2}\right) dx$$

$$\Rightarrow \quad \int_1^3 \frac{1}{x^2(x+1)}\,dx = \left[\log|x+1| - \log|x| - \frac{1}{x}\right]_1^3$$

$$\Rightarrow \quad \int_1^3 \frac{1}{x^2(x+1)}\,dx = \left(\log 4 - \log 3 - \frac{1}{3}\right) - (\log 2 - \log 1 - 1)$$

$$\Rightarrow \quad \int_1^3 \frac{1}{x^2(x+1)}\,dx = \log\left(\frac{4}{2\times 3}\right) - \frac{1}{3} + 1 = \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

EXAMPLE 15 *Evaluate:*

(i) $\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx$ (ii) $\int_0^{\pi/6} (2+3x^2)\cos 3x\,dx$

SOLUTION We have,

(i) $$\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx$$

$$= \int_0^{\sqrt{2}} \sqrt{(\sqrt{2})^2 - x^2}\,dx$$

$$= \left[\frac{1}{2}x\sqrt{2-x^2} + \frac{1}{2}(\sqrt{2})^2 \sin^{-1}\frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}}$$

$$= \{0 + \sin^{-1}(1)\} - \{0 + \sin^{-1} 0\} = \frac{\pi}{2}$$

(ii) $$\int_0^{\pi/6} \underset{I}{(2+3x^2)}\ \underset{II}{\cos 3x}\,dx$$

$$= \left[\frac{(2+3x^2)}{3}\sin 3x\right]_0^{\pi/6} - \int_0^{\pi/6} 6x\frac{\sin 3x}{3}\,dx$$

$$= \left[\frac{(2+3x^2)}{3}\sin 3x\right]_0^{\pi/6} - 2\int_0^{\pi/6} \underset{I}{x}\ \underset{II}{\sin 3x}\,dx$$

$$= \left[\frac{1}{3}(2+3x^2)\sin 3x\right]_0^{\pi/6} - 2\left[\left[\frac{-x\cos 3x}{3}\right]_0^{\pi/6} - \int_0^{\pi/6} -\frac{\cos 3x}{3}\,dx\right]$$

$$= \left[\frac{1}{3}(2+3x^2)\sin 3x\right]_0^{\pi/6} - 2\left[\left[\frac{-x\cos 3x}{3}\right]_0^{\pi/6} + \frac{1}{9}[\sin 3x]_0^{\pi/6}\right]$$

$$= \left[\frac{1}{3}\left(2+\frac{\pi^2}{12}\right)\sin\frac{\pi}{2} - \frac{1}{3}(2+0)\,0\right] - 2\left[\left\{\left(-\frac{\pi}{18}\cos\frac{\pi}{2}\right) + \frac{0\cos 0}{3}\right\} + \frac{1}{9}\left\{\sin\frac{\pi}{2} - \sin 0\right\}\right]$$

$$= \left[\frac{1}{3}\left(2+\frac{\pi^2}{12}\right)\sin\frac{\pi}{2} - \frac{1}{3}(2+0)\,0\right] - 2\left[\left\{\left(-\frac{\pi}{18}\cos\frac{\pi}{2}\right) + \frac{0\cos 0}{3}\right\} + \frac{1}{9}\left\{\sin\frac{\pi}{2} - \sin 0\right\}\right]$$

$$= \left[\frac{1}{3}\left(2+\frac{\pi^2}{12}\right) - \frac{2}{3}\times 0\right] - 2\left[(0-0) + \frac{1}{9}(1-0)\right]$$

$$= \frac{2}{3} + \frac{\pi^2}{36} - \frac{2}{9} = \frac{\pi^2}{36} + \frac{4}{9} = \frac{1}{36}(\pi^2 + 16)$$

EXAMPLE 16 *Evaluate :* $\int_0^\infty \frac{1}{(x^2+a^2)\,(x^2+b^2)}\,dx$.

SOLUTION Let $x^2 = y$. Then,

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(y+a^2)(y+b^2)}$$

Let $$\frac{1}{(y+a^2)(y+b^2)} = \frac{A}{y+a^2} + \frac{B}{y+b^2} \quad \text{...(i)}$$

$$\Rightarrow \quad 1 = A(y+b^2) + B(y+a^2) \quad \text{...(ii)}$$

Putting $y = -a^2$ and $y = -b^2$ successively in (ii), we get

$$A = \frac{1}{b^2 - a^2} \text{ and } B = \frac{1}{a^2 - b^2}$$

Substituting the values of A and B in (i), we obtain

$$\frac{1}{(y+a^2)\,(y+b^2)} = \frac{1}{a^2-b^2}\left[\frac{1}{y+b^2} - \frac{1}{y+a^2}\right]$$

$$\Rightarrow \quad \frac{1}{(x^2+a^2)\,(x^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right] \quad [\because\ y = x^2]$$

$$\therefore \quad \int_0^\infty \frac{1}{(x^2+a^2)\,(x^2+b^2)}\,dx$$

$$= \frac{1}{a^2-b^2}\int_0^\infty \left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right)dx$$

$$= \frac{1}{a^2-b^2}\left[\left(\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right)_0^\infty\right]$$

$$= \frac{1}{a^2-b^2}\left[\left(\frac{1}{b}\tan^{-1}\infty - \frac{1}{a}\tan^{-1}\infty\right) - \left(\frac{1}{b}\tan^{-1}0 - \frac{1}{a}\tan^{-1}0\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[\left(\frac{\pi}{2b} - \frac{\pi}{2a}\right) - (0-0)\right] = \frac{\pi}{2ab\,(a+b)}$$

EXAMPLE 17 *If $f(x)$ is of the form $f(x) = a + bx + cx^2$, show that*

$$\int_0^1 f(x)\,dx = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

SOLUTION We have,

$$f(x) = a + bx + cx^2$$

$$\Rightarrow \quad f(0) = a,\ f\left(\frac{1}{2}\right) = a + \frac{b}{2} + \frac{c}{4} \text{ and } f(1) = a + b + c$$

$$\Rightarrow \quad \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] = \frac{1}{6}\left[a + 4\left(a + \frac{b}{2} + \frac{c}{4}\right) + (a + b + c)\right]$$

$$\Rightarrow \quad \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] = \frac{1}{6}\,[6a + 3b + 2c] \qquad \text{...(i)}$$

Now,

$$\int_0^1 f(x)\,dx = \int_0^1 (a + bx + cx^2)\,dx$$

$$\Rightarrow \quad \int_0^1 f(x)\,dx = \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]_0^1$$

$$\Rightarrow \quad \int_0^1 f(x)\,dx = \left(a + \frac{b}{2} + \frac{c}{3}\right) - 0 = \frac{1}{6}\,[6a + 3b + 2c] \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\int_0^1 f(x)\,dx = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

EXAMPLE 18 *Evaluate :*

(i) $\displaystyle\int_1^2 \frac{1}{(x+1)(x+2)}\,dx$ (ii) $\displaystyle\int_1^2 \frac{1}{x(1+x^2)}\,dx$

SOLUTION (i) Let $\dfrac{1}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Then,

$$1 = A(x+2) + B(x+1) \qquad \text{...(i)}$$

Putting $x + 2 = 0$ or $x = -2$ in (i), we get

$$B = -1$$

Putting $x + 1 = 0$ or $x = -1$ in (i), we get

$$A = 1$$

$$\therefore \quad \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\Rightarrow \quad \int_1^2 \frac{1}{(x+1)(x+2)}\,dx = \int_1^2 \frac{1}{x+1}\,dx - \int_1^2 \frac{1}{x+2}\,dx$$

$$\Rightarrow \quad \int_1^2 \frac{1}{(x+1)(x+2)}\,dx = [\log(x+1)]_1^2 - [\log(x+2)]_1^2$$

$$\Rightarrow \quad \int_1^2 \frac{1}{(x+1)(x+2)}\,dx = (\log 3 - \log 2) - (\log 4 - \log 3)$$

$$\Rightarrow \quad \int_1^2 \frac{1}{(x+1)(x+2)}\,dx = 2\log 3 - \log 2 - \log 4 = \log 9 - \log 8 = \log\left(\frac{9}{8}\right)$$

(ii) Let $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$. Then,

$$1 = A(1+x^2) + (Bx+C)x \quad \text{...(i)}$$

Putting $x = 0$ in (i), we get $A = 1$

Comparing the coefficients of x^2 and x, we get

$$A + B = 0 \text{ and } C = 0 \Rightarrow B = -1 \text{ and } C = 0 \quad [\because A = 1]$$

$$\therefore \quad \frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$$

$$\Rightarrow \int_1^2 \frac{1}{x(1+x^2)}\,dx = \int_1^2 \frac{1}{x}\,dx - \frac{1}{2}\int_1^2 \frac{2x}{1+x^2}\,dx$$

$$\Rightarrow \int_1^2 \frac{1}{x(1+x^2)}\,dx = [\log x]_1^2 - \frac{1}{2}[\log(1+x^2)]_1^2$$

$$\Rightarrow \int_1^2 \frac{1}{x(1+x^2)}\,dx = (\log 2 - \log 1) - \frac{1}{2}[\log 5 - \log 2]$$

$$\Rightarrow \int_1^2 \frac{1}{x(1+x^2)}\,dx = \log 2 - \frac{1}{2}\log 5 + \frac{1}{2}\log 2 = \frac{3}{2}\log 2 - \frac{1}{2}\log 5$$

EXAMPLE 19 *Evaluate:* $\int_{\pi/4}^{\pi/2} \cos 2x \log \sin x\,dx$ **[CBSE 2003]**

SOLUTION We have,

$$I = \int_{\pi/4}^{\pi/2} \underset{II}{\cos 2x}\, \underset{I}{\log \sin x}\,dx$$

$$\Rightarrow I = \left[\frac{1}{2}(\log \sin x)\sin 2x\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \frac{1}{2}\cot x \sin 2x\,dx$$

$$\Rightarrow I = \left[0 - \frac{1}{2}\log\left(\frac{1}{\sqrt{2}}\right)\right] - \int_{\pi/4}^{\pi/2} \cos^2 x\,dx$$

$$\Rightarrow I = \frac{1}{4}\log 2 - \frac{1}{2}\int_{\pi/4}^{\pi/2} (1 + \cos 2x)\,dx$$

$$\Rightarrow I = \frac{1}{4}\log 2 - \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$$

$$\Rightarrow I = \frac{1}{4}\log 2 - \frac{1}{2}\left[\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right]$$

$$\Rightarrow I = \frac{1}{4}\log 2 - \frac{\pi}{8} + \frac{1}{4}$$

EXAMPLE 20 *Evaluate* : $\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4}+\frac{x}{2}\right) dx.$

SOLUTION Let $I = \int_0^{2\pi} \underset{II}{e^x} \underset{I}{\sin}\left(\frac{\pi}{4}+\frac{x}{2}\right) dx$

On integrating by parts, we get

$$I = \left[\sin\left(\frac{\pi}{4}+\frac{x}{2}\right) . e^x\right]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} \underset{II}{e^x} \underset{I}{\cos}\left(\frac{\pi}{4}+\frac{x}{2}\right) dx$$

$$\Rightarrow \quad I = \left[\sin\frac{5\pi}{4} e^{2\pi} - \sin\frac{\pi}{4}\right] - \frac{1}{2}\left[\left\{e^x \cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4}+\frac{x}{2}\right) dx\right]$$

$$\Rightarrow \quad I = \left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left[\left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \frac{1}{2} I\right]$$

$$\Rightarrow \quad I = -\left(-\frac{e^{2\pi}+1}{\sqrt{2}}\right) + \left(\frac{e^{2\pi}+1}{2\sqrt{2}}\right) - \frac{1}{4} I$$

$$\Rightarrow \quad I + \frac{1}{4} I = \frac{e^{2\pi}+1}{2\sqrt{2}} (1-2)$$

$$\Rightarrow \quad \frac{5I}{4} = -\frac{e^{2\pi}+1}{2\sqrt{2}}$$

$$\Rightarrow \quad I = -\frac{\sqrt{2}}{5}\left(e^{2\pi}+1\right)$$

EXERCISE 20.1

Evaluate the following definite integrals:

1. $\int_4^9 \frac{1}{\sqrt{x}} dx$
2. $\int_{-2}^3 \frac{1}{x+7} dx$
3. $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx$
4. $\int_0^1 \frac{1}{1+x^2} dx$
5. $\int_2^3 \frac{x}{x^2+1} dx$
6. $\int_0^\infty \frac{1}{a^2+b^2 x^2} dx$
7. $\int_{-1}^1 \frac{1}{1+x^2} dx$
8. $\int_0^\infty e^{-x} dx$
9. $\int_0^1 \frac{x}{x+1} dx$
10. $\int_0^{\pi/2} (\sin x + \cos x) \, dx$
11. $\int_{\pi/4}^{\pi/2} \cot x \, dx$
12. $\int_0^{\pi/4} \sec x \, dx$

13. $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x \, dx$ [NCERT]

14. $\int_0^1 \frac{1-x}{1+x} \, dx$

15. $\int_0^{\pi} \frac{1}{1+\sin x} \, dx$

16. $\int_{-\pi/4}^{\pi/4} \frac{1}{1+\sin x} \, dx$

17. $\int_0^{\pi/2} \cos^2 x \, dx$ [NCERT, CBSE 2002]

18. $\int_0^{\pi/2} \cos^3 x \, dx$

19. $\int_0^{\pi/6} \cos x \cos 2x \, dx$

20. $\int_0^{\pi/2} \sin x \sin 2x \, dx$

21. $\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 \, dx$

22. $\int_0^{\pi/2} \cos^4 x \, dx$

23. $\int_0^{\pi/2} (a^2 \cos^2 x + b^2 \sin^2 x) \, dx$

24. $\int_0^{\pi/2} \sqrt{1+\sin x} \, dx$

25. $\int_0^{\pi/2} \sqrt{1+\cos x} \, dx$

26. $\int_0^{\pi/2} x \sin x \, dx$

27. $\int_0^{\pi/2} x \cos x \, dx$

28. $\int_0^{\pi/2} x^2 \cos x \, dx$

29. $\int_0^{\pi/4} x^2 \sin x \, dx$

30. $\int_0^{\pi/2} x^2 \cos 2x \, dx$

31. $\int_0^{\pi/2} x^2 \cos^2 x \, dx$

32. $\int_1^2 \log x \, dx$

33. $\int_1^3 \frac{\log x}{(x+1)^2} \, dx$

34. $\int_1^e \frac{e^x}{x} (1 + x \log x) \, dx$

35. $\int_1^e \frac{\log x}{x} \, dx$

36. $\int_e^{e^2} \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \, dx \right\}$

37. $\int_1^2 \frac{x+3}{x(x+2)} \, dx$

38. $\int_0^1 \frac{2x+3}{5x^2+1} \, dx$ [NCERT]

39. $\int_0^2 \frac{1}{4+x-x^2} \, dx$ [NCERT]

40. $\int_0^1 \frac{1}{2x^2+x+1} \, dx$

41. $\int_0^1 \sqrt{x(1-x)} \, dx$

42. $\int_0^2 \frac{1}{\sqrt{3+2x-x^2}} \, dx$

43. $\int_0^4 \frac{1}{\sqrt{4x-x^2}}\,dx$

44. If $\int_0^k \frac{1}{2+8x^2}\,dx = \frac{\pi}{16}$, find the value of k.

45. $\int_1^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx$

46. $\int_0^1 x(1-x)^5\,dx$

47. $\int_1^2 \left(\frac{x-1}{x^2}\right)e^x\,dx$ **[CBSE 2000, 02]**

48. $\int_0^1 \left(xe^{2x}+\sin\frac{\pi x}{2}\right)dx$

49. $\int_0^1 \left(xe^{x}+\cos\frac{\pi x}{4}\right)dx$

50. $\int_{\pi/2}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$ **[NCERT]**

51. $\int_0^{2\pi} e^{x/2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx$

52. $\int_0^{2\pi} e^x\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$

53. If $\int_0^a 3x^2\,dx = 8$, Find the value of a.

54. $\int_1^2 \frac{x}{(x+1)(x+2)}\,dx$ **[NCERT]**

55. $\int_0^{\pi/2} \sin^3 x\,dx$ **[NCERT]**

56. $\int_0^{\pi} \left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)dx$ **[NCERT]**

57. $\int_1^2 e^{2x}\left(\frac{1}{x}-\frac{1}{2x^2}\right)dx$ **[NCERT]**

58. $\int_{-1}^1 \frac{1}{x^2+2x+5}\,dx$ **[NCERT]**

59. $\int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}}\,dx$ **[NCERT]**

ANSWERS

1. 2 **2.** $\log 2$ **3.** $\frac{\pi}{6}$

4. $\frac{\pi}{4}$ **5.** $\frac{1}{2}\log 2$ **6.** $\frac{\pi}{2ab}$

7. $\frac{\pi}{2}$ **8.** 1 **9.** $\log\left(\frac{e}{2}\right)$

10. 2 **11.** $\frac{1}{2}\log 2$ **12.** $\log(\sqrt{2}+1)$

13. $\log(\sqrt{2}-1)-\log(2-\sqrt{3})$ **14.** $2\log 2-1$

15. 2 **16.** 2 **17.** $\frac{\pi}{4}$

18. $\frac{2}{3}$ **19.** $\frac{5}{12}$ **20.** $\frac{2}{3}$

21. $-\frac{2}{\sqrt{3}}$ **22.** $\frac{3\pi}{16}$ **23.** $\frac{\pi}{4}(a^2+b^2)$

24. 2 **25.** 2 **26.** 1

27. $\frac{\pi}{2}-1$ **28.** $\left(\frac{\pi^2}{4}-2\right)$ **29.** $\sqrt{2}+\frac{\pi}{2\sqrt{2}}-\frac{\pi^2}{16\sqrt{2}}-2$

30. $-\dfrac{\pi}{4}$

31. $\left(\dfrac{\pi^3}{48}-\dfrac{\pi}{8}\right)$

32. $2\log 2-1$

33. $\dfrac{3}{4}\log 3-\log 2$

34. e^e

35. $\dfrac{1}{2}$

36. $\dfrac{e^2}{2}-e$

37. $\dfrac{1}{2}\log 6$

38. $\dfrac{1}{5}\log 6+\dfrac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$

39. $\dfrac{1}{\sqrt{17}}\log\left(\dfrac{21+5\sqrt{17}}{4}\right)$

40. $\dfrac{2}{\sqrt{7}}\left\{\tan^{-1}\dfrac{5}{\sqrt{7}}-\tan^{-1}\dfrac{1}{\sqrt{7}}\right\}$

41. $\dfrac{\pi}{8}$

42. $\dfrac{\pi}{3}$

43. π

44. $\dfrac{1}{2}$

45. $\dfrac{57-\sqrt{3}}{5}$

46. $\dfrac{1}{42}$

47. $\dfrac{e^2}{2}-e$

48. $\dfrac{e^2}{4}+\dfrac{1}{4}+\dfrac{2}{\pi}$

49. $\dfrac{1+2\sqrt{2}}{\pi}$

50. $e^{\pi/2}$

51. 0

52. $-\dfrac{3\sqrt{2}}{5}\left(e^{2\pi}+1\right)$

53. 2

54. $\log\left(\dfrac{32}{27}\right)$

55. $\dfrac{2}{3}$

56. 0

57. $\dfrac{e^4-2e^2}{4}$

58. $\dfrac{\pi}{8}$

59. $\dfrac{2^{5/2}}{3}$

HINTS TO SELECTED PROBLEMS

9. $I=\displaystyle\int_0^1 \frac{x+1-1}{x+1}\,dx=\int_0^1\left(1-\frac{1}{x+1}\right)dx=\Big[x-\log(x+1)\Big]_0^1$

14. $I=\displaystyle\int_0^1 \frac{2-(1+x)}{1+x}\,dx=\int_0^1\left(\frac{2}{1+x}-1\right)dx=\Big[2\log(x+1)-x\Big]_0^1$

15. $I=\displaystyle\int_0^\pi \frac{1-\sin x}{(1+\sin x)\,(1-\sin x)}\,dx=\int_0^\pi\frac{1-\sin x}{\cos^2 x}\,dx=\int_0^\pi(\sec^2 x-\sec x\tan x)\,dx$

23. Use $\cos^2 x=\dfrac{1+\cos 2x}{2}$, $\sin^2 x=\dfrac{1-\cos 2x}{2}$

31. Use $\cos^2 x=\dfrac{1+\cos 2x}{2}$

36. $I=\displaystyle\int_e^{e^2}\underset{\text{I}}{\frac{1}{\log x}}\cdot\underset{\text{II}}{1}\,dx-\int_e^{e^2}\frac{1}{(\log x)^2}\,dx=\left[\frac{x}{\log x}\right]_e^{e^2}-\int_e^{e^2}-\frac{1}{x(\log x)^2}\,x\,dx-\int_e^{e^2}\frac{1}{(\log x)^2}\,dx$

20.2 EVALUATION OF DEFINITE INTEGRALS BY SUBSTITUTION

When the variable in a definite integral is changed, the substitution in terms of new variable should be effected at three places.

(i) in the integrand,

(ii) in the differential, say, dx

(iii) in the limits

The limits of the new variable, say, t are simply the values of t corresponding to the values of the original variable, say, x, and so they can be easily obtained by putting the values of x in the substitutional relation between x and t. The method is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\displaystyle\int_0^4 \frac{1}{x+\sqrt{x}}\,dx$ (ii) $\displaystyle\int_0^1 \frac{2x}{5x^2+1}\,dx$

SOLUTION (i) Let $x = t^2$. Then, $dx = d(t^2) \Rightarrow dx = 2t\,dt$.

When $x = 0, x = t^2 \Rightarrow t = 0$

When $x = 4, t^2 = x \Rightarrow t^2 = 4 \Rightarrow t = 2$

$$\therefore \int_0^4 \frac{1}{x+\sqrt{x}}\,dx$$

$$= \int_0^2 \frac{2t\,dt}{t^2+t} = 2\int_0^2 \frac{1}{t+1}\,dt$$

$$= 2\Big[\log(t+1)\Big]_0^2 = 2[\log 3 - \log 1] = 2\log 3$$

(ii) Let $5x^2 + 1 = t$. Then, $d(5x^2+1) = dt \Rightarrow 10\,x dx = dt$

When $x = 0, t = 5x^2 + 1 \Rightarrow t = 1$

When $x = 1, t = 5x^2 + 1 \Rightarrow t = 6$

$$\therefore \int_0^1 \frac{2x}{5x^2+1}\,dx$$

$$= \int_1^6 \frac{2x}{t}\cdot\frac{dt}{10x} = \frac{1}{5}\int_1^6 \frac{1}{t}\,dt = \frac{1}{5}\Big[\log t\Big]_1^6 = \frac{1}{5}(\log 6 - \log 1) = \frac{1}{5}\log 6$$

EXAMPLE 2 *Evaluate :*

(i) $\displaystyle\int_0^1 \sin^{-1} x\,dx$ [NCERT] (ii) $\displaystyle\int_0^{\pi/2} \sqrt{\cos\theta}\,\sin^3\theta\,d\theta$

(iii) $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}\,d\theta$ [CBSE 2004]

SOLUTION (i) Let $t = \sin^{-1} x$ or, $x = \sin t$. Then, $dx = d(\sin t) = \cos t\,dt$

When $x = 0, t = \sin^{-1} x \Rightarrow t = \sin^{-1} 0 \Rightarrow t = 0$

and, when $x = 1, t = \sin^{-1} x \Rightarrow t = \sin^{-1} 1 = \dfrac{\pi}{2}$

$$\therefore \int_0^1 \sin^{-1} x\,dx = \int_0^{\pi/2} \underset{\mathrm{I}}{t}\ \underset{\mathrm{II}}{\cos t}\,dt = \Big[t\sin t\Big]_0^{\pi/2} - \int_0^{\pi/2} \sin t\,dt$$

$$\Rightarrow \int_0^1 \sin^{-1} x\,dx = \Big[t\sin t\Big]_0^{\pi/2} + \Big[\cos t\Big]_0^{\pi/2} = \frac{\pi}{2}\sin\frac{\pi}{2} - 0 + \cos\frac{\pi}{2} - \cos 0 = \frac{\pi}{2} - 1$$

(ii) Let $\cos\theta = t$. Then, $d(\cos\theta) = dt \Rightarrow -\sin\theta\,d\theta = dt$

When $\theta = 0, t = \cos 0 = 1$

When $\theta = \dfrac{\pi}{2}, t = \cos\dfrac{\pi}{2} = 0$

$$\therefore \quad \int_0^{\pi/2} \sqrt{\cos\theta}\,\sin^3\theta\,d\theta = \int_1^0 \sqrt{t}\,\sin^3\theta\left(\frac{-dt}{\sin\theta}\right) = -\int_1^0 \sqrt{t}\,\sin^2\theta\,dt$$

$$= -\int_1^0 \sqrt{t}\,(1-t^2)\,dt = -\int_1^0 \left(\sqrt{t} - t^{5/2}\right)dt$$

$$= -\left[\frac{2}{3}t^{3/2} - \frac{2}{7}t^{7/2}\right]_1^0 = -\left[0 - \left(\frac{2}{3} - \frac{2}{7}\right)\right] = \frac{8}{21}$$

(iii) Let $\sin\theta = t$. Then, $d(\sin\theta) = dt \Rightarrow \cos\theta\,d\theta = dt$

When $\theta = 0, t = \sin 0 = 0$. When $\theta = \frac{\pi}{2},\ t = \sin\frac{\pi}{2} = 1$

$$\therefore \quad \int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}\,d\theta$$

$$= \int_0^1 \frac{1}{(1+t)(2+t)}\,dt$$

$$= \int_0^1 \left\{\frac{1}{1+t} - \frac{1}{2+t}\right\}dt \qquad \text{[By using partial fractions]}$$

$$= \Big[\log(1+t)\Big]_0^1 - \Big[\log(2+t)\Big]_0^1$$

$$= (\log 2 - \log 1) - (\log 3 - \log 2)$$

$$= \log 2 - \log 3 + \log 2 = 2\log 2 - \log 3 = \log\left(\frac{4}{3}\right)$$

EXAMPLE 3 *Evaluate:*

(i) $\displaystyle\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}\,dx$ (ii) $\displaystyle\int_0^{1/\sqrt{2}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$ [CBSE2007]

(iii) $\displaystyle\int_0^1 \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$ (iv) $\displaystyle\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$ [NCERT, CBSE 2002]

SOLUTION (i) Let $x = a\sin\theta$. Then, $dx = d(a\sin\theta) = a\cos\theta\,d\theta$

Also, $x = 0 \Rightarrow a\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

And, $x = a \Rightarrow a\sin\theta = a \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^a \frac{x^4}{\sqrt{a^2-x^2}}\,dx$$

$$I = \int_0^{\pi/2} \frac{(a\sin\theta)^4}{\sqrt{a^2 - a^2\sin^2\theta}}\,a\cos\theta\,d\theta$$

$$= a^4 \int_0^{\pi/2} \sin^4 \theta \, d\theta$$

$$= a^4 \cdot \frac{3\pi}{16} = \frac{3\pi a^4}{16}$$ [See Example 3 (iii) on page 20.4]

(ii) Let $\sin^{-1} x = \theta$ or, $x = \sin \theta$. Then, $dx = d(\sin \theta) = \cos \theta \, d\theta$

Now, $x = 0 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0$ and $x = \frac{1}{\sqrt{2}} \Rightarrow \sin \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} \, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\theta}{\cos^3 \theta} \cos \theta \, d\theta = \int_0^{\pi/4} \underset{I}{\theta} \underset{II}{\sec^2 \theta} \, d\theta$$

$$\Rightarrow \quad I = \Big[\theta \tan \theta\Big]_0^{\pi/4} - \int_0^{\pi/4} 1 \cdot \tan \theta \, d\theta$$

$$\Rightarrow \quad I = \Big[\theta \tan \theta\Big]_0^{\pi/4} + \Big[\log \cos \theta\Big]_0^{\pi/4}$$

$$\Rightarrow \quad I = \left(\frac{\pi}{4} - 0\right) + \left(\log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right) = \frac{\pi}{4} - \frac{1}{2} \log 2$$

(iii) Let $\tan^{-1} x = \theta$ or, $x = \tan \theta$. Then, $dx = \sec^2 \theta \, d\theta$

Now, $x = 0 \Rightarrow \tan \theta = 0 \Rightarrow \theta = 0$, and $x = 1 \Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^1 \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} \, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\theta \tan \theta}{\sec^3 \theta} \sec^2 \theta \, d\theta$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \underset{I}{\theta} \underset{II}{\sin \theta} \, d\theta$$

$$\Rightarrow \quad I = \Big[-\theta \cos \theta\Big]_0^{\pi/4} - \int_0^{\pi/4} (-\cos \theta) \, d\theta = \Big[-\theta \cos \theta\Big]_0^{\pi/4} + \Big[\sin \theta\Big]_0^{\pi/4}$$

$$\Rightarrow \quad I = \left(-\frac{\pi}{4\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} - 0\right) = \frac{4-\pi}{4\sqrt{2}}$$

(iv) We have,

$$I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

$$\Rightarrow \quad I = \int_0^1 2 \tan^{-1} x \, dx \qquad \left[\because \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2 \tan^{-1} x\right]$$

$$\Rightarrow \quad I = 2\int_0^1 \underset{I}{\tan^{-1} x} \cdot \underset{II}{1}\, dx = 2\left[\left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot x\, dx\right]$$

$$\Rightarrow \quad I = 2\left[[x \tan^{-1} x]_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\, dx\right]$$

$$\Rightarrow \quad I = 2\left[[x \tan^{-1} x]_0^1 - \frac{1}{2}[\log(1+x^2)]_0^1\right]$$

$$\Rightarrow \quad I = 2\left[(1 \cdot \tan^{-1} 1 - 0 \tan^{-1} 0) - \frac{1}{2}(\log 2 - \log 1)\right]$$

$$\Rightarrow \quad I = 2\left[\left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\log 2\right] = \frac{\pi}{2} - \log 2$$

ALITER Let $x = \tan\theta$. Then, $dx = d(\tan\theta) = \sec^2\theta\, d\theta$

Now, $x = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \sin^{-1}(\sin 2\theta)\ \sec^2\theta\, d\theta$$

$$\Rightarrow \quad I = \int_0^{\pi/4} 2\theta \sec^2\theta\ d\theta = 2\int_0^{\pi/4} \underset{I}{\theta} \cdot \underset{II}{\sec^2\theta}\ d\theta = 2\left[[\theta\tan\theta]_0^{\pi/4} - \int_0^{\pi/4} 1 \cdot \tan\theta\, d\theta\right]$$

$$\Rightarrow \quad I = 2\left[[\theta\tan\theta]_0^{\pi/4} + [\log\cos\theta]_0^{\pi/4}\right] = 2\left[\left(\frac{\pi}{4}\tan\frac{\pi}{4} - 0\right) + \left(\log\cos\frac{\pi}{4} - \log 1\right)\right]$$

$$\Rightarrow \quad I = 2\left[\frac{\pi}{4} + \log\frac{1}{\sqrt{2}}\right] = \frac{\pi}{2} + 2\log\frac{1}{\sqrt{2}} = \frac{\pi}{2} + \log\frac{1}{2} = \frac{\pi}{2} - \log 2$$

EXAMPLE 4 *Evaluate :*

(i) $\int_0^{\pi/4} \tan^3 x\, dx$ **[NCERT, CBSE 2004]** (ii) $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$ **[CBSE 2002, 2003]**

SOLUTION (i) We have,

$$I = \int_0^{\pi/4} \tan^3 x\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \tan^2 x\ \tan x\, dx = \int_0^{\pi/4} (\sec^2 x - 1)\ \tan x\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \sec^2 x \tan x\ dx - \int_0^{\pi/4} \tan x\, dx$$

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\, dx = dt$

Now $\quad x = 0, \Rightarrow t = 0$, and $x = \frac{\pi}{4} \Rightarrow t = 1$

$\therefore \quad \int_0^{\pi/4} \tan^3 x \, dx$

$$= \int_0^1 t \, dt - \int_0^{\pi/4} \tan x \, dx = \left[\frac{t^2}{2}\right]_0^1 - \left[\log \sec x\right]_0^{\pi/4}$$

$$= \left(\frac{1}{2} - 0\right) - \log \sec \frac{\pi}{4} + \log \sec 0$$

$$= \frac{1}{2} - \log \sqrt{2} + \log 1 = \frac{1}{2} - \frac{1}{2} \log 2 = \frac{1}{2}(1 - \log 2)$$

(ii) $\quad \int_0^{\pi/2} \left\{\sqrt{\tan x} + \sqrt{\cot x}\right\} dx$

$$= \int_0^{\pi/2} \left\{\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right\} dx$$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} dx$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx$$

Let $\quad \sin x - \cos x = t$. Then, $d(\sin x - \cos x) = dt \Rightarrow (\cos x + \sin x)\, dx = dt$

Now $\quad x = 0, \Rightarrow t = -1$, and $x = \frac{\pi}{2} \Rightarrow t = 1$

$\therefore \quad \int_0^{\pi/2} \left\{\sqrt{\tan x} + \sqrt{\cot x}\right\} dx$

$$= \sqrt{2} \int_{-1}^{1} \frac{dt}{\sqrt{1 - t^2}} = \sqrt{2}\left[\sin^{-1} t\right]_{-1}^{1}$$

$$= \sqrt{2}\,[\sin^{-1} 1 - \sin^{-1}(-1)] = \sqrt{2}\,[2 \sin^{-1}(1)] = 2\sqrt{2}\left(\frac{\pi}{2}\right) = \sqrt{2}\,\pi$$

EXAMPLE 5 *Evaluate :*

(i) $\int_0^{\pi} \frac{1}{5 + 4 \cos x} dx$ **[CBSE 2005]**

(ii) $\int_0^{\pi/2} \frac{1}{3 + 2 \cos x} dx$

SOLUTION We have,

(i) $$I = \int_0^{\pi} \frac{1}{5+4\cos x}\, dx$$

$$= \int_0^{\pi} \frac{1}{5+4\left(\dfrac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)}\, dx$$

$$= \int_0^{\pi} \frac{1+\tan^2 \frac{x}{2}}{5\left(1+\tan^2 \frac{x}{2}\right)+4\left(1-\tan^2 \frac{x}{2}\right)}\, dx$$

$$= \int_0^{\pi} \frac{1+\tan^2 \frac{x}{2}}{9+\tan^2 \frac{x}{2}}\, dx = \int_0^{\pi} \frac{\sec^2 \frac{x}{2}}{9+\tan^2 \frac{x}{2}}\, dx$$

Let $\tan \frac{x}{2} = t$. Then, $d\left(\tan \frac{x}{2}\right) = dt \Rightarrow \frac{1}{2}\sec^2 \frac{x}{2}\, dx = dt \Rightarrow dx = \dfrac{2dt}{\sec^2 \frac{x}{2}}$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \pi \Rightarrow t = \tan \frac{\pi}{2} = \infty$

$\therefore$ $$I = \int_0^{\infty} \frac{\sec^2 \frac{x}{2}}{9+t^2} \cdot \frac{2dt}{\sec^2 \frac{x}{2}}$$

$\Rightarrow$ $$I = 2\int_0^{\infty} \frac{dt}{3^2+t^2}$$

$\Rightarrow$ $$I = \frac{2}{3}\left[\tan^{-1} \frac{t}{3}\right]_0^{\infty} = \frac{2}{3}\left[\tan^{-1} \infty - \tan^{-1} 0\right] = \frac{2}{3}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{3}$$

(ii) $$I = \int_0^{\pi/2} \frac{1}{3+2\cos x}\, dx$$

$\Rightarrow$ $$I = \int_0^{\pi/2} \frac{1}{3+2\left(\dfrac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)}\, dx$$

$\Rightarrow$ $$I = \int_0^{\pi/2} \frac{1+\tan^2 \frac{x}{2}}{3\left(1+\tan^2 \frac{x}{2}\right)+2\left(1-\tan^2 \frac{x}{2}\right)}\, dx = \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{5+\tan^2 \frac{x}{2}}\, dx$$

Let $\tan\frac{x}{2} = t$. Then, $d\left(\tan\frac{x}{2}\right) = dt \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt\;\; dx = \dfrac{2\,dt}{\sec^2\frac{x}{2}}$

Now, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$

$$\therefore \qquad I = \int_0^1 \frac{\sec^2\frac{x}{2}}{5+t^2}\cdot\frac{2dt}{\sec^2\frac{x}{2}}$$

$$\Rightarrow \qquad I = 2\int_0^1 \frac{dt}{(\sqrt{5})^2 + t^2}$$

$$\Rightarrow \qquad I = 2\times\frac{1}{\sqrt{5}}\left[\tan^{-1}\left[\frac{t}{\sqrt{5}}\right]\right]_0^1 = \frac{2}{\sqrt{5}}\left[\tan^{-1}\frac{1}{\sqrt{5}} - \tan^{-1}0\right] = \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

EXAMPLE 6 *Evaluate :*

(i) $\displaystyle\int_0^{\pi/2} \frac{1}{2\cos x + 4\sin x}\,dx$ **[HSB 2001, PSB 2001C, 2002]**

(ii) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$ **[NCERT]**

(iii) $\displaystyle\int_0^{\pi/2} \frac{1}{4\sin^2 x + 5\cos^2 x}\,dx$

SOLUTION We have,

$$\text{(i)} \qquad I = \int_0^{\pi/2} \frac{1}{2\cos x + 4\sin x}\,dx$$

$$\Rightarrow \qquad I = \int_0^{\pi/2} \frac{1}{\dfrac{2\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}} + \dfrac{4\left(2\tan\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}}\,dx$$

$$\Rightarrow \qquad I = \int_0^{\pi/2} \frac{1+\tan^2\frac{x}{2}}{2-2\tan^2\frac{x}{2}+8\tan\frac{x}{2}}\,dx = \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{2-2\tan^2\frac{x}{2}+8\tan\frac{x}{2}}\,dx$$

Let $\tan\frac{x}{2} = t$. Then, $d\left(\tan\frac{x}{2}\right) = dt \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$ or, $dx = 2\dfrac{dt}{\sec^2\frac{x}{2}}$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$

$$\therefore \qquad I = \int_0^1 \frac{\sec^2 \frac{x}{2}}{2 - 2t^2 + 8t} \cdot \frac{2dt}{\sec^2 \frac{x}{2}}$$

$$\Rightarrow \qquad I = \int_0^1 \frac{1}{1 - t^2 + 4t}\, dt$$

$$\Rightarrow \qquad I = \int_0^1 \frac{1}{-[t^2 - 4t - 1]}\, dt$$

$$\Rightarrow \qquad I = \int_0^1 \frac{dt}{-[t^2 - 4t + 4 - 4 - 1]}$$

$$\Rightarrow \qquad I = \int_0^1 \frac{dt}{-[(t-2)^2 - 5]}$$

$$\Rightarrow \qquad I = \int_0^1 \frac{1}{(\sqrt{5})^2 - (t-2)^2}\, dt$$

$$\Rightarrow \qquad I = \frac{1}{2\sqrt{5}} \left[\log \left|\frac{\sqrt{5} + t - 2}{\sqrt{5} - t + 2}\right|\right]_0^1 = \frac{1}{2\sqrt{5}} \left[\log\left(\frac{\sqrt{5} - 1}{\sqrt{5} + 1}\right) - \log\left(\frac{\sqrt{5} - 2}{\sqrt{5} + 2}\right)\right]$$

$$\Rightarrow \qquad I = \frac{1}{2\sqrt{5}} \left[\log\left\{\frac{(\sqrt{5} - 1)(\sqrt{5} + 2)}{(\sqrt{5} + 1)(\sqrt{5} - 2)}\right\}\right] = \frac{1}{2\sqrt{5}} \log\left(\frac{3 + \sqrt{5}}{3 - \sqrt{5}}\right)$$

$$\Rightarrow \qquad I = \frac{1}{2\sqrt{5}} \log\left(\frac{3 + \sqrt{5}}{3 - \sqrt{5}} \times \frac{3 + \sqrt{5}}{3 + \sqrt{5}}\right) = \frac{1}{2\sqrt{5}} \log\left(\frac{3 + \sqrt{5}}{2}\right)^2 = \frac{2}{2\sqrt{5}} \log\left(\frac{3 + \sqrt{5}}{2}\right)$$

$$\Rightarrow \qquad I = \frac{1}{\sqrt{5}} \log\left(\frac{3 + \sqrt{5}}{2}\right)$$

(ii) Let $I = \displaystyle\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx$

Let $\quad \cos x = t$ and $-\sin x\, dx = dt$

Now, $\quad x = 0 \Rightarrow t = \cos 0 = 1$ and $x = \frac{\pi}{2} \Rightarrow t = \cos \frac{\pi}{2} = 0$

$$\therefore \qquad I = \int_1^0 \frac{\sin x}{1 + t^2} \left(\frac{-dt}{\sin x}\right)$$

$$\Rightarrow \qquad I = -\int_1^0 \frac{dt}{1 + t^2}$$

$$\Rightarrow \qquad I = -\left[\tan^{-1} t\right]_1^0 = -[\tan^{-1} 0 - \tan^{-1} 1] = -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$$

(ii) We have,

$$I = \int_0^{\pi/2} \frac{1}{4\sin^2 x + 5\cos^2 x}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sec^2 x}{4\tan^2 x + 5}\,dx \qquad \text{[Dividing num. and denom. by } \cos^2 x]$$

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow sec^2 x\,dx = dt$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{2} = \infty$

$$\therefore \quad I = \int_0^{\infty} \frac{dt}{4t^2 + 5}$$

$$\Rightarrow \quad I = \frac{1}{4}\int_0^{\infty} \frac{1}{t^2 + \left(\frac{\sqrt{5}}{2}\right)^2}\,dt = \frac{1}{4}\cdot\frac{1}{\left(\frac{\sqrt{5}}{2}\right)}\left[\tan^{-1}\left(\frac{t}{\frac{\sqrt{5}}{2}}\right)\right]_0^{\infty}$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{5}}\left[\tan^{-1}\left(\frac{2t}{\sqrt{5}}\right)\right]_0^{\infty} = \frac{1}{2\sqrt{5}}[\tan^{-1}\infty - \tan^{-1}0] = \frac{1}{2\sqrt{5}} \times \frac{\pi}{2} = \frac{\pi}{4\sqrt{5}}$$

EXAMPLE 7 *Evaluate*

(i) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(3\cos x + \sin x)}\,dx$ (ii) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x}\,dx$

(iii) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3}\,dx$ (iv) $\displaystyle\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$

SOLUTION (i) Let $\cos x = K(3\cos x + \sin x) + L\,\frac{d}{dx}(3\cos x + \sin x)$. Then,

$$\cos x = K(3\cos x + \sin x) + L\,(-3\sin x + \cos x)$$

Comparing coefficients of $\cos x$ and $\sin x$, we get

$$3K + L = 1 \text{ and } K - 3L = 0$$

Solving these two equations, we have

$$K = \frac{3}{10} \text{ and } L = \frac{1}{10}$$

$$\therefore \quad \cos x = \frac{3}{10}(3\cos x + \sin x) + \frac{1}{10}(-3\sin x + \cos x)$$

$$\therefore \quad \int_0^{\pi/2} \frac{\cos x}{(3\cos x + \sin x)}\,dx$$

$$= \int_0^{\pi/2} \frac{\frac{3}{10}(3\cos x + \sin x) + \frac{1}{10}(-3\sin x + \cos x)}{3\cos x + \sin x}\,dx$$

$$= \frac{3}{10}\int_0^{\pi/2} \frac{3\cos x + \sin x}{3\cos x + \sin x}\,dx + \frac{1}{10}\int_0^{\pi/2} \frac{-3\sin x + \cos x}{3\cos x + \sin x}\,dx$$

$$= \frac{3}{10}\int_0^{\pi/2} 1\cdot dx + \frac{1}{10}\int_0^{\pi/2} \frac{-3\sin x+\cos x}{3\cos x+\sin x}\,dx$$

$$= \frac{3}{10}\Big[x\Big]_0^{\pi/2} + \frac{1}{10}\Big[\log|3\cos x+\sin x|\Big]_0^{\pi/2}$$

$$= \frac{3}{10}\left(\frac{\pi}{2}-0\right)+\frac{1}{10}(\log 1-\log 3) = \frac{3\pi}{20}-\frac{1}{10}\log 3$$

(ii) We have,

$$\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{(1+\cos x)+\sin x}\,dx$$

$$= \int_0^{\pi/2} \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$$

$$= \int_0^{\pi/2} \frac{1-\tan^2\frac{x}{2}}{2+2\tan\frac{x}{2}}\,dx \qquad \left[\text{Dividing num. and denom. by } \cos^2\frac{x}{2}\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\left(1-\tan\frac{x}{2}\right)\left(1+\tan\frac{x}{2}\right)}{1+\tan\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(1-\tan\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left[x+2\log\cos\frac{x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}+2\log\cos\frac{\pi}{4}\right)-(0+2\log 1)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+2\log\frac{1}{\sqrt{2}}\right] = \frac{1}{2}\left[\frac{\pi}{2}+\log\frac{1}{2}\right] = \frac{1}{2}\left[\frac{\pi}{2}-\log 2\right]$$

(iii) We have,

$$I = \int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3}\,dx$$

$$= \int_0^{\pi/2} \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3}\,dx = \int_0^{\pi/2} \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\,dx$$

Let $\cos\frac{x}{2}+\sin\frac{x}{2} = t$. Then,

$\Rightarrow$ $d\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right) = dt \Rightarrow \frac{1}{2}\left(-\sin\frac{x}{2}+\cos\frac{x}{2}\right)dx = dt \Rightarrow \left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)dx = -2dt$

Also, $x=0 \Rightarrow t=\cos 0+\sin 0=1$ and $x=\frac{\pi}{2} \Rightarrow t=\cos\frac{\pi}{4}+\sin\frac{\pi}{4}=\frac{2}{\sqrt{2}}=\sqrt{2}$

$$\therefore \quad I = \int_1^{\sqrt{2}} \frac{2dt}{t^2} = 2\int_1^{\sqrt{2}} \frac{1}{t^2}\,dt = 2\left[-\frac{1}{t}\right]_1^{\sqrt{2}} = 2\left[-\frac{1}{\sqrt{2}}+1\right] = (2-\sqrt{2})$$

(iv) Let $I = \int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}\,dx$

Here, we express denominator in terms $\sin x - \cos x$ which is integration of numerator.

We have, $(\sin x-\cos x)^2 = \sin^2 x+\cos^2 x-2\sin x\cos x = 1-\sin 2x$

$\Rightarrow$ $\sin 2x = 1-(\sin x-\cos x)^2$

$$\therefore \quad I = \int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\{1-(\sin x-\cos x)^2\}}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x+\cos x}{25-16(\sin x-\cos x)^2}\,dx$$

Let $\sin x-\cos x = t$. Then, $d(\sin x-\cos x) = dt \Rightarrow (\cos x+\sin x)\,dx = dt$.

Also, $x=0 \Rightarrow t=\sin 0-\cos 0 = -1$ and $x=\frac{\pi}{4} \Rightarrow t=\sin\frac{\pi}{4}-\cos\frac{\pi}{4}=0$

$$\therefore \quad I = \int_{-1}^{0} \frac{dt}{25-16t^2}$$

$$\Rightarrow \quad I = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\frac{25}{16}-t^2} = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{4}\right)^2-t^2}$$

$$\Rightarrow \quad I = \frac{1}{16}\cdot\frac{1}{2(5/4)}\left[\log\left|\frac{5/4+t}{5/4-t}\right|\right]_{-1}^{0}$$

$$\Rightarrow \quad I = \frac{1}{40}\left[\log 1-\log\left(\frac{1/4}{9/4}\right)\right]$$

$$\Rightarrow \quad I = \frac{1}{40}\left[\log 1-\log\left(\frac{1}{9}\right)\right] = \frac{1}{40}[\log 1+\log 9] = \frac{1}{40}\log 9$$

EXAMPLE 8 *Evaluate :* $\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$ **[CBSE 2003C]**

SOLUTION Let $I = \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx = \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x+(1-\sin^2 x)^2}\,dx$

Let $\sin^2 x = t$. Then, $d(\sin^2 x) = dt \Rightarrow 2\sin x\cos x\,dx = dt \Rightarrow \sin 2x\,dx = dt$

Also, $x = 0 \Rightarrow t = \sin^2 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \sin^2 \frac{\pi}{2} = 1$

$$\therefore \quad I = \int_0^1 \frac{dt}{t^2 + (1-t)^2}$$

$$\Rightarrow \quad I = \int_0^1 \frac{dt}{2t^2 - 2t + 1}$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^1 \frac{dt}{t^2 - t + \frac{1}{2}}$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^1 \frac{dt}{t^2 - t + \frac{1}{4} - \frac{1}{4} + \frac{1}{2}} = \frac{1}{2}\int_0^1 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$\Rightarrow \quad I = \frac{1}{2} \cdot \frac{1}{\left(\frac{1}{2}\right)}\left[\tan^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$\Rightarrow \quad I = \left[\tan^{-1}(2t-1)\right]_0^1 = \tan^{-1} 1 - \tan^{-1}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

EXAMPLE 9 *Evaluate :* $\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\, dx$

SOLUTION Let $I = \int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\, dx$. Let $x^2 = t$. Then, $d(x^2) = dt \Rightarrow 2x\,dx = dt$

Also, $x = 0 \Rightarrow t = 0$ and $x = 1 \Rightarrow t = 1$

$$\therefore \quad I = \int_0^1 x\sqrt{\frac{1-t}{1+t}} \cdot \frac{dt}{2x}$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^1 \sqrt{\frac{1-t}{1+t}}\, dt$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^1 \sqrt{\frac{1-t}{1+t} \times \frac{1-t}{1-t}}\, dt$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^1 \frac{1-t}{\sqrt{1-t^2}}\, dt$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-t^2}}\, dt + \frac{1}{4}\int_0^1 \frac{-2t}{\sqrt{1-t^2}}\, dt$$

$$\Rightarrow \quad I = \frac{1}{2}\left[\sin^{-1} t\right]_0^1 + \frac{1}{4}\left[2\sqrt{1-t^2}\right]_0^1 \qquad \left[\because \int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)}\right]$$

$$\Rightarrow \quad I = \frac{1}{2}[\sin^{-1} 1 - \sin^{-1} 0] + \frac{1}{4}[2\times 0 - 2\times 1] = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] + \frac{1}{4}[0-2] = \frac{\pi}{4} - \frac{1}{2}$$

EXAMPLE 10 *Evaluate:* $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$ **[CBSE 2010]**

SOLUTION We have,

$$I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - (\cos x - \sin x)^2}}\, dx$$

Let $t = (\cos x - \sin x)$. Then, $dt = d(\cos x - \sin x) \Rightarrow -(\sin x + \cos x)\, dx = dt$

Also, $x = \frac{\pi}{6} \Rightarrow t = \cos\frac{\pi}{6} - \sin\frac{\pi}{6} = \frac{\sqrt{3}-1}{2}$

and, $x = \frac{\pi}{3} \Rightarrow t = \cos\frac{\pi}{3} - \sin\frac{\pi}{3} = \frac{1-\sqrt{3}}{2}$

$$\therefore \quad I = -\int_{\frac{\sqrt{3}-1}{2}}^{\frac{1-\sqrt{3}}{2}} \frac{1}{\sqrt{1-t^2}}\, dt$$

$$\Rightarrow \quad I = -\left[\sin^{-1} t\right]_{\frac{\sqrt{3}-1}{2}}^{\frac{1-\sqrt{3}}{2}}$$

$$\Rightarrow \quad I = -\left[\sin^{-1}\frac{1-\sqrt{3}}{2} - \sin^{-1}\frac{\sqrt{3}-1}{2}\right]$$

$$\Rightarrow \quad I = -\sin^{-1}\frac{1-\sqrt{3}}{2} + \sin^{-1}\frac{\sqrt{3}-1}{2}$$

$$\Rightarrow \quad I = 2\sin^{-1}\frac{\sqrt{3}-1}{2} \qquad \left[\because \sin^{-1}(-x) = -\sin^{-1} x\right]$$

EXAMPLE 11 *Evaluate:* $\displaystyle\int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$

SOLUTION Let $I = \displaystyle\int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$. Then,

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4(1-\cos^2 x)}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^2 x}{4 - 3\cos^2 x}\, dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\int_0^{\pi/2} \frac{-3\cos^2 x}{4 - 3\cos^2 x}\, dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\int_0^{\pi/2} \frac{(4-3\cos^2 x)-4}{4-3\cos^2 x}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\int_0^{\pi/2} \left\{1-\frac{4}{4-3\cos^2 x}\right\} dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\int_0^{\pi/2} 1\cdot dx + \frac{4}{3}\int_0^{\pi/2} \frac{1}{4-3\cos^2 x}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\int_0^{\pi/2} 1\cdot dx + \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2 x}{4(1+\tan^2 x)-3}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\Big[x\Big]_0^{\pi/2} + \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2 x}{1+4\tan^2 x}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\left(\frac{\pi}{2}-0\right)+\frac{4}{3}\int_0^{\infty} \frac{1}{1+4t^2}\,dt, \text{ where } t = \tan x$$

$$\Rightarrow \quad I = -\frac{\pi}{6}+\frac{4}{3}\times\frac{1}{2}\Big[\tan^{-1} 2t\Big]_0^{\infty} = -\frac{\pi}{6}+\frac{2}{3}\times\left(\frac{\pi}{2}-0\right) = \frac{\pi}{6}$$

EXAMPLE 12 *Evaluate:* $\displaystyle\int_0^{\pi/4} \frac{\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx$

SOLUTION Let $\displaystyle I = \int_0^{\pi/4} \frac{\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx$

Dividing numerator and denominator by $\cos^4 x$, we get

$$I = \int_0^{\pi/4} \frac{\tan x\sec^2 x}{1+\tan^4 x}\,dx$$

Let $\tan^2 x = t$. Then, $d(\tan^2 x) = dt \Rightarrow 2\tan x\sec^2 x\,dx = dt$

Also, $x = 0 \Rightarrow t = \tan^2 0 = 0$ and $x = \frac{\pi}{4} \Rightarrow t = \tan^2\frac{\pi}{4} = 1$

Substituting $t = \tan^2 x$ and $\tan x\sec^2 x\,dx = \frac{1}{2}dt$, we get

$$I = \frac{1}{2}\int_0^1 \frac{1}{1+t^2}\,dt$$

$$\Rightarrow \quad I = \frac{1}{2}\Big[\tan^{-1} t\Big]_0^1 = \frac{1}{2}\left(\frac{\pi}{4}-0\right) = \frac{\pi}{8}$$

EXAMPLE 13 *If* $\displaystyle I_n = \int_0^{\pi/4} \tan^n x\,dx$, *prove that* $I_n + I_{n+2} = \dfrac{1}{n+1}$

SOLUTION We have, $\displaystyle I_n = \int_0^{\pi/4} \tan^n x$

$$\therefore \quad I_{n+2} = \int_0^{\pi/4} \tan^{n+2} x$$

Now, $$I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x \, dx + \int_0^{\pi/4} \tan^{n+2} x \, dx$$

$$\Rightarrow \quad I_n + I_{n+2} = \int_0^{\pi/4} \left(\tan^n x + \tan^{n+2} x\right) dx$$

$$\Rightarrow \quad I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x \, (1 + \tan^2 x) \, dx$$

$$\Rightarrow \quad I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x \sec^2 x \, dx$$

Let $t = \tan x$. Then, $dt = \sec^2 x \, dx$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and, $x = \frac{\pi}{4} \Rightarrow t = \tan \frac{\pi}{4} = 1$

$$\therefore \quad I_n + I_{n+2} = \int_0^1 t^n \, dt$$

$$\Rightarrow \quad I_n + I_{n+2} = \left[\frac{t^{n+1}}{n+1}\right]_0^1 \Rightarrow I_n + I_{n+2} = \frac{1}{n+1}$$

EXAMPLE 14 *If* $I_n = \int_0^{\pi/4} \tan^n x \, dx$, *show that* $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}, \frac{1}{I_5 + I_7}, \ldots$ *from an A.P. Find the common difference of this progression.*

SOLUTION We have,

$$I_n = \int_0^{\pi/4} \tan^n x \, dx \Rightarrow I_{n+2} = \int_0^{\pi/4} \tan^{n+2} x \, dx$$

$$\therefore \quad I_n + I_{n+2} = \int_0^{\pi} \tan^n x \, dx + \int_0^{\pi/4} \tan^{n+2} x \, dx$$

$$\Rightarrow \quad I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x \, (1 + \tan^2 x) \, dx$$

$$\Rightarrow \quad I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x \sec^2 x \, dx$$

$$\Rightarrow \quad I_n + I_{n+2} = \int_0^1 t^n \, dt, \text{ where } t = \tan x.$$

$$\Rightarrow \quad I_n + I_{n+2} = \left[\frac{t^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}, \quad n = 2, 3, 4, 5, \ldots$$

$$\Rightarrow \quad \frac{1}{I_n + I_{n+2}} = n+1, n = 2, 3, 4, 5, ...$$

$$\Rightarrow \quad \frac{1}{I_2 + I_4} = 3, \frac{1}{I_3 + I_5} = 4, \frac{1}{I_4 + I_6} = 5, \frac{1}{I_5 + I_7} = 6, ...$$

Clearly, 3, 4, 5, 6, ... is an AP with common difference 1.

Hence, $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}$, ... is an AP with common difference 1.

EXERCISE 20.2

Evaluate the following integrals:

1. $\int_2^4 \frac{x}{x^2+1} dx$ [NCERT]

2. $\int_1^2 \frac{1}{x(1+\log x)^2} dx$

3. $\int_1^2 \frac{3x}{9x^2-1} dx$

4. $\int_0^{\pi/2} \frac{1}{5\cos x + 3\sin x} dx$ [PSB 2001C,

5. $\int_0^a \frac{x}{\sqrt{a^2+x^2}} dx$

6. $\int_0^1 \frac{e^x}{1+e^{2x}} dx$

7. $\int_0^1 xe^{x^2} dx$

8. $\int_1^3 \frac{\cos(\log x)}{x} dx$

9. $\int_0^1 \frac{2x}{1+x^4} dx$

10. $\int_0^a \sqrt{a^2-x^2}\, dx$

11. $\int_0^{\pi/2} \sqrt{\sin\phi}\, \cos^5\phi\, d\phi$ [NCERT]

12. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx$

13. $\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1+\cos\theta}} d\theta$

14. $\int_0^{\pi/3} \frac{\cos x}{3+4\sin x} dx$

15. $\int_0^1 \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx$

16. $\int_0^2 x\sqrt{x+2}\, dx$ [NCERT]

17. $\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

18. $\int_0^{\pi/2} \frac{\sin x \cos x}{1+\sin^4 x} dx$

19. $\int_0^{\pi/2} \frac{dx}{a\cos x + b\sin x}$ $a, b > 0$

20. $\int_0^{\pi/2} \frac{1}{5+4\sin x} dx$

21. $\int_0^{\pi} \frac{\sin x}{\sin x + \cos x} dx$

22. $\int_0^{\pi} \frac{1}{3+2\sin x + \cos x} dx$

23. $\int_0^1 \tan^{-1} x\, dx$

24. $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\, dx$

25. $\int_0^1 (\cos^{-1} x)^2\, dx$

26. $\int_0^{\pi/4} \frac{\tan^3 x}{1+\cos 2x}\, dx$

27. $\int_0^{\pi} \frac{1}{5+3\cos x}\, dx$

28. $\int_0^{\pi/2} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}\, dx$ [HSB 2001]

29. $\int_0^a \sin^{-1} \sqrt{\frac{x}{a+x}}\, dx$

30. $\int_0^{\pi} \sin^3 x\,(1+2\cos x)\,(1+\cos x)^2\, dx$

31. $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{3/2}}\, dx$

32. $\int_0^1 x \tan^{-1} x\, dx$

33. $\int_0^1 \frac{1-x^2}{x^4+x^2+1}\, dx$

34. $\int_0^1 \frac{24x^3}{(1+x^2)^4}\, dx$

35. $\int_4^{12} x\,(x-4)^{1/3}\, dx$

36. $\int_0^{\pi/2} x^2 \sin x\, dx$

37. $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$ [CBSE 2004]

38. $\int_0^1 \frac{1-x^2}{(1+x^2)^2}\, dx$

39. $\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}\, dx$

40. $\int_0^a x \sqrt{\frac{a^2-x^2}{a^2+x^2}}\, dx$

41. $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$

42. $\int_0^{\pi} 5\,(5-4\cos\theta)^{1/4} \sin\theta\, d\theta$

43. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta$

44. $\int_0^{(\pi)^{2/3}} \sqrt{x} \cos^2 x^{3/2}\, dx$

45. $\int_1^2 \frac{1}{x\,(1+\log x)^2}\, dx$ [CBSE 2003]

46. $\int_0^{\pi/2} \cos^5 x\, dx$

47. $\int_4^9 \frac{\sqrt{x}}{(30-x^{3/2})^2}\, dx$ [NCERT]

48. $\int_0^{\pi/4} \sin^3 2t \cos 2t\, dt$ [NCERT]

49. $\int_{-1}^1 5x^4 \sqrt{x^5+1}\, dx$ [NCERT]

50. $\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$ [NCERT]

ANSWERS

1. $\frac{1}{2}\log\left(\frac{17}{5}\right)$
2. $\frac{\log 2}{\log 2e}$
3. $\frac{1}{6}(\log 35 - \log 8)$
4. $\frac{1}{\sqrt{34}}\log\left|\frac{8+\sqrt{34}}{8-\sqrt{34}}\right|$
5. $a(\sqrt{2}-1)$
6. $\tan^{-1} e - \frac{\pi}{4}$
7. $\frac{e-1}{2}$
8. $\sin(\log 3)$
9. $\frac{\pi}{4}$

10. $\frac{\pi a^2}{4}$ 11. $\frac{64}{231}$ 12. $\frac{\pi}{4}$

13. $2(\sqrt{2}-1)$ 14. $\frac{1}{4}\log\left(\frac{3+2\sqrt{3}}{3}\right)$ 15. $\frac{1}{12}\pi^{3/2}$

16. $\frac{16}{15}(2+\sqrt{2})$ 17. $\frac{\pi}{2}-\log 2$ 18. $\frac{\pi}{8}$

19. $\frac{1}{\sqrt{a^2+b^2}}\log\left(\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}}\right)$ 20. $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$

21. $\frac{\pi}{2}$ 22. $\frac{\pi}{4}$ 23. $\frac{\pi}{4}-\frac{1}{2}\log 2$

24. $\frac{\pi}{2}-1$ 25. $\pi-2$ 26. $\frac{1}{8}$

27. $\frac{\pi}{4}$ 28. $\frac{\pi}{2ab}$ 29. $a\left(\frac{\pi}{2}-1\right)$

30. $\frac{8}{3}$ 31. 1 32. $\frac{\pi}{4}-\frac{1}{2}$ 33. $\frac{1}{2}\log 3$

34. 1 35. $\frac{720}{7}$ 36. $\pi-2$ 37. $\frac{\pi}{2}-1$

38. $\frac{1}{2}$ 39. $\log\left(\frac{9}{8}\right)$ 40. $a^2\left(\frac{\pi}{4}-\frac{1}{2}\right)$ 41. πa

42. $9\sqrt{3}-1$ 43. $\frac{3}{4}$ 44. $\frac{\pi}{3}$ 45. $\frac{\log 2}{1+\log 2}$

46. $\frac{8}{15}$ 47. $\frac{19}{99}$ 48. $\frac{1}{8}$ 49. $\frac{4\sqrt{2}}{3}$ 50. $\frac{\pi^2}{32}$

HINTS TO SELECTED PROBLEMS

1. Put $x^2+1=t$
3. Put $9x^2-1=t$
5. Put $a^2+x^2=t^2$
6. Put $e^x=t$
7. Put $x^2=t$
8. Put $\log x=t$
9. Put $x^2=t$
10. Put $x=a\sin\theta$
11. Put $\sin\phi=t$
12. Put $\sin x=t$
13. Put $\cos\theta=t$
14. Put $\sin x = t$
15. Put $\tan^{-1}x=t$
16. Put $x+2=t^2$
17. Use : $\tan^{-1}\left(\frac{2x}{1-x^2}\right)=2\tan^{-1}x$
18. Put $\sin^2 x=t$
19. Use : $\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, $\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$
20. Put : $\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\tan\frac{x}{2} = t$
21. Put $\sin x = K(\sin x+\cos x)+L\frac{d}{dx}(\sin x+\cos x)$

22. Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

24. Put $\sin x = t$

25. Put $\cos^{-1} x = \theta$ or $x = \cos\theta$

26. $I = \displaystyle\int_0^{\pi/4} \frac{\tan^3 x}{2\cos^2 x}\,dx = \frac{1}{2}\int_0^{\pi/4} \tan^3 x \sec^2 x\; dx$. Now, put $\tan x = t$

27. Put $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

28. Divide numerator and denominator by $\cos^2 x$ and then put $\tan x = t$

34. Put $1 + x^2 = t$

35. Put $x - 4 = t^3$

37. $I = \displaystyle\int_0^1 \frac{1-x}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx + \frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1-x^2}}\,dx$

38. $I = \displaystyle\int_0^1 \frac{-\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2}\,dx$. Now, put $x + \dfrac{1}{x} = t$

39. Put $\cos x = t$

20.5 PROPERTIES OF DEFINITE INTEGRALS

In this section, we will study some fundamental properties of definite integrals which are very useful in evaluating integrals.

PROPERTY I $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ *i.e., integration is independent of the change of variable.*

PROOF Let $\phi(x)$ be a primitive of $f(x)$. Then,

$$\frac{d}{dx}\{\phi(x)\} = f(x) \Rightarrow \frac{d}{dt}\{\phi(t)\} = f(t).$$

Hence, $\displaystyle\int_a^b f(x)\,dx = \Big[\phi(x)\Big]_a^b = \phi(b) - \phi(a)$ ...(i)

and, $\displaystyle\int_a^b f(t)\,dt = \Big[\phi(t)\Big]_a^b = \phi(b) - \phi(a)$ ...(ii)

From (i) and (ii), we have

$$\int_a^b f(x)\,dx = \int_b^a f(t)\,dt$$

PROPERTY II $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

i.e., if the limits of a definite integral are interchanged then its value changes by minus sign only.

<u>PROOF</u> Let $\phi(x)$ be a primitive of $f(x)$. Then,

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

and, $$-\int_b^a f(x)\,dx = -[(\phi(a) - \phi(b)] = \phi(b) - \phi(a)$$

$\therefore$ $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

PROPERTY III $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$.

<u>PROOF</u> Let $\phi(x)$ be a primitive of $f(x)$. Then,

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \qquad \text{...(i)}$$

and, $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = [\phi(c) - \phi(a)] + [\phi(b), -\phi(c)] = \phi(b) - \phi(a) \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

<u>GENERALIZATION</u> *The above property can be generalized into the following form*

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \dots + \int_{c_n}^b f(x)\,dx, \text{ where } a < c_1 < c_2 < c_3 \dots < c_{n-1} < c_n < b$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :*

(i) $\int_{-1}^{1} f(x)\,dx$, where $f(x) = \begin{cases} 1 - 2x, & x \le 0 \\ 1 + 2x, & x \ge 0 \end{cases}$

(ii) $\int_{1}^{4} f(x)\,dx$, where $f(x) = \begin{cases} 2x + 8, & 1 \le x \le 2 \\ 6x\ , & 2 \le x \le 4 \end{cases}$

SOLUTION (i) We have,

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$$

$$\Rightarrow \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (1 - 2x)\,dx + \int_{0}^{1} (1 + 2x)\,dx \qquad \text{[By def. of } f(x)]$$

$$\Rightarrow \int_{-1}^{1} f(x)\,dx = [x - x^2]_{-1}^{0} + [x + x^2]_0^1 = [0 - (-1 - 1)] + [(1 + 1) - (0)] = 4$$

(ii) $$\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx$$

$$\Rightarrow \quad \int_1^4 f(x)\,dx = \int_1^2 (2x+8)\,dx + \int_2^4 6x\,dx \qquad \text{[Using def. of } f(x)\text{]}$$

$$\Rightarrow \quad \int_1^4 f(x)\,dx = [x^2+8x]_1^2 + [3x^2]_2^4 = [(4+16)-(1+8)] + [48-12] = 47$$

EXAMPLE 2 *Evaluate :*

(i) $\displaystyle\int_0^1 |\,5x-3\,|\,dx$ (ii) $\displaystyle\int_0^\pi |\cos x|\,dx$

(iii) $\displaystyle\int_{-5}^5 |\,x-2\,|\,dx$ (iv) $\displaystyle\int_{-1}^1 e^{|x|}\,dx$

(v) $\displaystyle\int_0^2 |\,x^2+2x-3\,|\,dx$ (vi) $\displaystyle\int_1^4 \left(|\,x-1\,|+|\,x-2\,|+|\,x-3\,|\right)dx$ [NCERT]

(vii) $\displaystyle\int_{-1}^2 |x^3-x|\,dx$ [NCERT]

SOLUTION (i) Clearly,

$$|\,5x-3\,| = \begin{cases} -(5x-3) & \text{when } 5x-3<0 \text{ i.e., } x<\dfrac{3}{5} \\ 5x-3 & \text{when } 5x-3\ge 0 \text{ i.e., } x\ge\dfrac{3}{5} \end{cases}$$

$$\therefore \quad I = \int_0^1 |\,5x-3\,|\,dx$$

$$\Rightarrow \quad I = \int_0^{3/5} |\,5x-3\,|\,dx + \int_{3/5}^1 |5x-3|\,dx$$

$$\Rightarrow \quad I = \int_0^{3/5} -(5x-3)\,dx + \int_{3/5}^1 (5x-3)\,dx$$

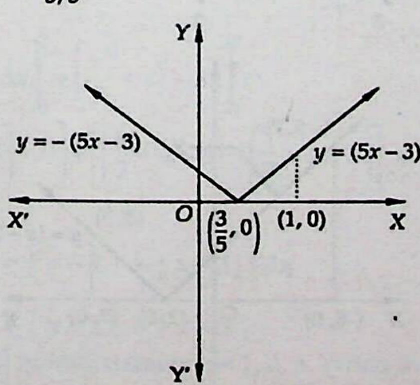

Fig. 20.1

$$\Rightarrow \quad I = \left[3x - \frac{5x^2}{2}\right]_0^{3/5} + \left[\frac{5x^2}{2} - 3x\right]_{3/5}^{1} = \left(\frac{9}{5} - \frac{9}{10}\right) + \left(-\frac{1}{2} + \frac{9}{10}\right) = \frac{13}{10}$$

(ii) Clearly, $|\cos x| = \begin{cases} \cos x \text{ when } 0 \leq x \leq \frac{\pi}{2} \\ -\cos x \text{ when } \frac{\pi}{2} \leq x \leq \pi \end{cases}$

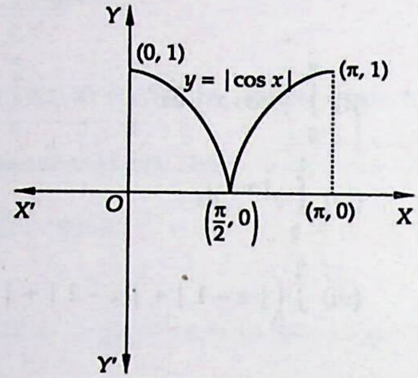

Fig. 20.2

$$\therefore \quad \int_0^{\pi} |\cos x|\, dx = \int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx$$

$$\Rightarrow \quad \int_0^{\pi} |\cos x|\, dx = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)\, dx$$

$$\Rightarrow \quad \int_0^{\pi} |\cos x|\, dx = \left[\sin x\right]_0^{\pi/2} - \left[\sin x\right]_{\pi/2}^{\pi} = 1 + 1 = 2$$

(iii) Clearly, $|x - 2| = \begin{cases} x - 2 \text{ when } x - 2 \geq 0 \text{ i.e., } x \geq 2 \\ -(x-2) \text{ when } x - 2 < 0 \text{ i.e., } x < 2 \end{cases}$

$$\therefore \quad \int_{-5}^{5} |x - 2|\, dx = \int_{-5}^{2} |x - 2|\, dx + \int_{2}^{5} |x - 2|\, dx$$

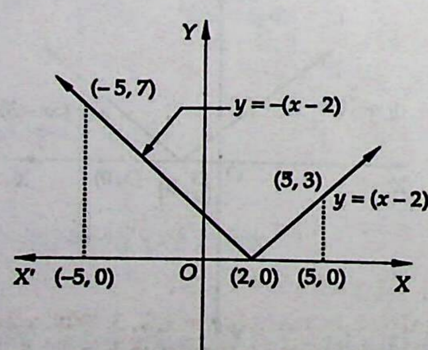

Fig. 20.3

$$\Rightarrow \quad \int_{-5}^{5} |x-2|\, dx = \int_{-5}^{2} -(x-2)\, dx + \int_{2}^{5} (x-2)\, dx$$

$$\Rightarrow \quad \int_{-5}^{5} |x-2|\, dx = \left[2x - \frac{x^2}{2}\right]_{-5}^{2} + \left[\frac{x^2}{2} - 2x\right]_{2}^{5} = \left(2 + \frac{45}{2}\right) + \left(\frac{5}{2} + 2\right) = 29$$

(iv) Clearly, $|x| = \begin{cases} x \text{ when } x \geq 0 \\ -x \text{ when } x < 0 \end{cases}$

$$\therefore \quad \int_{-1}^{1} e^{|x|}\, dx = \int_{-1}^{0} e^{|x|}\, dx + \int_{0}^{1} e^{|x|}\, dx$$

$$\Rightarrow \quad \int_{-1}^{1} e^{|x|}\, dx = \int_{-1}^{0} e^{-x}\, dx + \int_{0}^{1} e^{x}\, dx$$

$$\Rightarrow \quad \int_{-1}^{1} e^{|x|}\, dx = \left[-e^{-x}\right]_{-1}^{0} + \left[e^{x}\right]_{0}^{1}$$

$$\Rightarrow \quad \int_{-1}^{1} e^{|x|}\, dx = (-1 + e^1) + (e^1 - 1) = 2e - 2$$

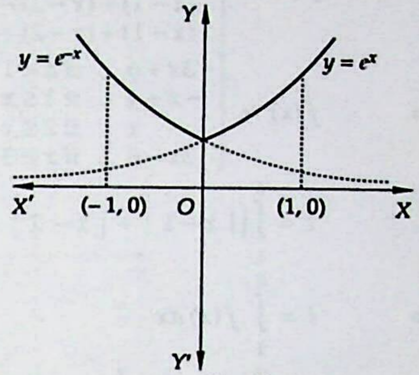

Fig. 20.4

(v) Let $I = \int_{0}^{2} |x^2 + 2x - 3|\, dx$

We have, $x^2 + 2x - 3 = (x+3)(x-1)$

The signs of $x^2 + 2x - 3$ for different values of x are shown in Fig. 20.5.

$$\therefore \quad |x^2 + 2x - 3| = \begin{cases} -(x^2 + 2x - 3), & \text{if } 0 < x < 1 \\ (x^2 + 2x - 3), & \text{if } 1 \leq x \leq 2 \end{cases}$$

$$\Rightarrow \quad I = \int_{0}^{2} |x^2 + 2x - 3|\, dx$$

$$\Rightarrow \quad I = \int_{0}^{1} |x^2 + 2x - 3|\, dx + \int_{1}^{2} |x^2 + 2x - 3|\, dx$$

$$\Rightarrow \quad I = \int_{0}^{1} -(x^2 + 2x - 3)\, dx + \int_{1}^{2} (x^2 + 2x - 3)\, dx$$

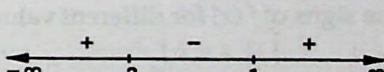

Fig. 20.5

$$\Rightarrow \quad I = -\left[\frac{x^3}{3} + x^2 - 3x\right]_{0}^{1} + \left[\frac{x^3}{3} + x^2 - 3x\right]_{1}^{2}$$

$$\Rightarrow \quad I = -\left[\left(\frac{1}{3} + 1 - 3\right) - 0\right] + \left[\left(\frac{8}{3} + 4 - 6\right) - \left(\frac{1}{3} + 1 - 3\right)\right] = \frac{5}{3} + \frac{2}{3} + \frac{5}{3} = 4$$

(vi) We have,

$$I = \int_{1}^{4} (|x-1| + |x-2| + |x-3|)\, dx$$

Let $\quad f(x) = |x-1| + |x-2| + |x-3|$

Here, we have three critical points, namely $x = 1, 2, 3$. When these points are marked on real line, it is divided into four parts as shown in Fig. 20.6. Therefore, to remove the modulus sign, we consider the following four cases.

(i) When $x < 1$
(ii) When $1 \le x < 2$
(iii) When $2 \le x < 3$
(iv) When $x \ge 3$.

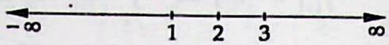

Fig. 20.6

$$\therefore \quad f(x) = \begin{cases} -(x-1)-(x-2)-(x-3) &, \text{ if } x < 1 \\ (x-1)-(x-2)-(x-3) &, \text{ if } 1 \le x < 2 \\ (x-1)+(x-2)-(x-3) &, \text{ if } 2 \le x < 3 \\ (x-1)+(x-2)+(x-3) &, \text{ if } x \ge 3 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -3x+6 &, \text{ if } x < 1 \\ -x+4 &, \text{ if } 1 \le x < 2 \\ x &, \text{ if } 2 \le x < 3 \\ 3x-6 &, \text{ if } x \ge 3 \end{cases}$$

$$\therefore \quad I = \int_1^4 \{|x-1| + |x-2| + |x-3|\}\, dx$$

$$\Rightarrow \quad I = \int_1^4 f(x)\, dx$$

$$\Rightarrow \quad I = \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + \int_3^4 f(x)\, dx$$

$$\Rightarrow \quad I = \int_1^2 (-x+4)\, dx + \int_2^3 x\, dx + \int_3^4 (3x-6)\, dx$$

$$\Rightarrow \quad I = \left[-\frac{x^2}{2} + 4x\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 + \left[\frac{3x^2}{2} - 6x\right]_3^4$$

$$\Rightarrow \quad I = (-2+8) - \left(-\frac{1}{2} + 4\right) + \left(\frac{9}{2} - \frac{4}{2}\right) + (24-24) - \left(\frac{27}{2} - 18\right) = 6 - \frac{7}{2} + \frac{5}{2} + \frac{9}{2} = \frac{19}{2}$$

(vii) Let $I = \int_{-1}^{2} |x^3 - x|\, dx$ and $f(x) = x^3 - x$

Cleraly, $f(x) = x^3 - x = x(x-1)(x+1)$

The signs of $f(x)$ for different values of x are as shown below.

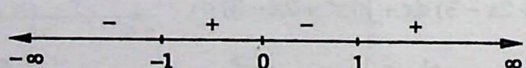

Fig. 20.7

$\therefore$ $\quad f(x) = x^3 - x > 0$ for all $x \in (-1, 0) \cup (1, 2)$

and, $\quad f(x) = x^3 - x < 0$ for all $x \in (0, 1)$

$$\text{Hence,} \quad |x^3 - x| = \begin{cases} x^3 - x, & x \in (-1, 0) \cup (1, 2) \\ -(x^3 - x), & x \in (0, 1) \end{cases}$$

$$\therefore \quad I = \int_{-1}^{2} |x^3 - x|\, dx$$

$$\Rightarrow \quad I = \int_{-1}^{0} |x^3 - x|\, dx + \int_{0}^{1} |x^3 - x|\, dx + \int_{1}^{2} |x^3 - x|\, dx$$

$$\Rightarrow \quad I = \int_{-1}^{0} (x^3 - x)\, dx - \int_{0}^{1} (x^3 - x)\, dx + \int_{1}^{2} (x^3 - x)\, dx$$

$$\Rightarrow \quad I = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$\Rightarrow \quad I = -\left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{16}{4} - \frac{4}{4}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$\Rightarrow \quad I = \frac{3}{4} + (4 - 2) = \frac{11}{4}$$

EXAMPLE 3 *Evaluate:* $\displaystyle\int_{1/e}^{e} |\log_e x|\, dx$

SOLUTION We have,

$$|\log_e x| = \begin{cases} -\log_e x, & \text{if } 1/e < x < 1 \\ \log_e x, & \text{if } 1 < x < e \end{cases}$$

$$\therefore \quad I = \int_{1/e}^{e} |\log_e x|\, dx$$

$$\Rightarrow \quad I = \int_{1/e}^{1} -\log_e x\, dx + \int_{1}^{e} \log_e x\, dx$$

$$\Rightarrow \quad I = -\int_{1/e}^{e} \log_e x\, dx + \int_{1}^{e} \log_e x\, dx$$

$$\Rightarrow \quad I = -\Big[x\left(\log_e x - 1\right)\Big]_{1/e}^{1} + \Big[x\,(\log_e x - 1)\Big]_{1}^{e} \qquad \left[\because \int \log_e x\, dx = x\,(\log_e x - 1)\right]$$

$$\Rightarrow \quad I = -\left[1\,(0 - 1) - \frac{1}{e}\,(-1 - 1)\right] + \Big[e\,(1 - 1) - 1\,(0 - 1)\Big]$$

$$\Rightarrow \quad I = -\left[-1 + \frac{2}{e}\right] + \Big[0 + 1\Big] = 2 - \frac{2}{e}$$

EXAMPLE 4 *Evaluate:*

(i) $\displaystyle\int_0^3 [x]\, dx$ (ii) $\displaystyle\int_0^2 [x^2]\, dx$ (iii) $\displaystyle\int_0^{1.5} [x^2]\, dx$

SOLUTION (i) Clearly,

$$\int_0^3 [x]\, dx = \int_0^1 [x]\, dx + \int_1^2 [x]\, dx + \int_2^3 [x]\, dx$$

$$\Rightarrow \quad \int_0^3 [x]\, dx = \int_0^1 0\, dx + \int_1^2 1\,.\, dx + \int_2^3 2\,.\, dx$$

$$\Rightarrow \quad \int_0^3 [x]\, dx = 0 + \Big[\,x\,\Big]_1^2 + \Big[\,2x\,\Big]_2^3 = (2 - 1) + (6 - 4) = 3$$

(ii) Clearly,

$$\int_0^2 [x^2]\,dx = \int_0^1 [x^2]\,dx + \int_1^{\sqrt{2}} [x^2]\,dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]\,dx + \int_{\sqrt{3}}^2 [x^2]\,dx$$

$$\Rightarrow \int_0^2 [x^2]\,dx = \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,.\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^2 3\,dx$$

$$\Rightarrow \int_0^2 [x^2]\,dx = 0 + \Big[x\Big]_1^{\sqrt{2}} + 2\Big[x\Big]_{\sqrt{2}}^{\sqrt{3}} + 3\Big[x\Big]_{\sqrt{3}}^2$$

$$\Rightarrow \int_0^2 [x^2]\,dx = 0 + (\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3})$$

$$\Rightarrow \int_0^2 [x^2]\,dx = \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3} = 5 - \sqrt{2} - \sqrt{3}$$

(iii) Clearly,

$$\int_0^{1.5} [x^2]\,dx = \int_0^1 [x^2]\,dx + \int_1^{\sqrt{2}} [x^2]\,dx + \int_{\sqrt{2}}^{1.5} [x^2]\,dx$$

$$\int_0^{1.5} [x^2]\,dx = \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,.\,dx + \int_{\sqrt{2}}^{1.5} 2\,dx$$

$$\int_0^{1.5} [x^2]\,dx = 0 + \Big[x\Big]_1^{\sqrt{2}} + 2\Big[x\Big]_{\sqrt{2}}^{1.5} = 0 + (\sqrt{2}-1) + 2(1.5-\sqrt{2}) = 2 - \sqrt{2}$$

EXAMPLE 5 *If* $[\cdot]$ *denotes the greatest integer function, then find the value of* $\int_1^2 [3x]\,dx$.

SOLUTION We observe that when $x \in [1, 2]$, $3x \in [3, 6]$.

$$\therefore \int_1^2 [3x]\,dx = \int_1^{4/3} [3x]\,dx + \int_{4/3}^{5/3} [3x]\,dx + \int_{5/3}^{6/3} [3x]\,dx$$

$$\Rightarrow \int_1^2 [3x]\,dx = \int_1^{4/3} 3\,dx + \int_{4/3}^{5/3} 4\,dx + \int_{5/3}^{6/3} 5\,dx$$

$$\Rightarrow \int_1^2 [3x]\,dx = 3\left(\frac{4}{3}-1\right) + 4\left(\frac{5}{3}-\frac{4}{3}\right) + 5\left(\frac{6}{3}-\frac{5}{3}\right) = 1 + \frac{4}{3} + \frac{5}{3} = 4.$$

EXAMPLE 6 *Let* $f(x) = x - [x]$, *for every real number* x, *where* $[x]$ *is the integral part of* x. *Then, evaluate* $\int_{-1}^1 f(x)\,dx$.

SOLUTION We have,

$$\int_{-1}^1 \big(x - [x]\big)\,dx = \int_{-1}^0 \big(x - [x]\big)\,dx + \int_0^1 \big(x - [x]\big)\,dx = \int_{-1}^0 (x+1)\,dx + \int_0^1 (x-0)\,dx$$

$$\Rightarrow \quad \int_{-1}^{1} \left(x - [x]\right) dx = \left[\frac{(x+1)^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

ALITER It is evident from the graph of $y = f(x)$ that

$$\int_{-1}^{1} f(x)\, dx = \text{Area of } \Delta\, OLA + \text{Area of } \Delta\, OMB$$

$$\Rightarrow \quad \int_{-1}^{1} f(x)\, dx = 2\, (\text{Area of } \Delta\, OMB)$$

$$\Rightarrow \quad \int_{-1}^{1} f(x)\, dx = 2 \times \frac{1}{2}\, (OM \times MB) = 1 \times 1 = 1.$$

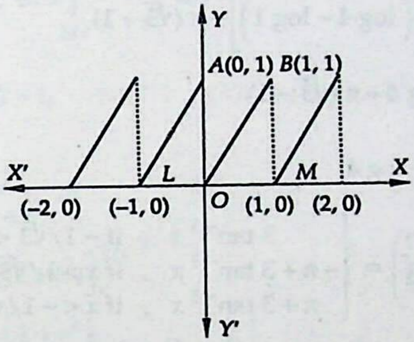

Fig. 20.8

EXAMPLE 7 *Evaluate:*

(i) $\displaystyle\int_{0}^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$ (ii) $\displaystyle\int_{0}^{1} \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) dx$

SOLUTION (i) We have,

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} 2\tan^{-1} x \ , & \text{if} -1 < x < 1 \\ -\pi + 2\tan^{-1} x \ , & \text{if}\ x > 1 \\ \pi + 2\tan^{-1} x \ , & \text{if}\ x < -1 \end{cases}$$

$$\therefore \quad I = \int_{0}^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

$$\Rightarrow \quad I = \int_{0}^{1} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx + \int_{1}^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

$$\Rightarrow \quad I = \int_{0}^{1} 2\tan^{-1} x\, dx + \int_{1}^{\sqrt{3}} (-\pi + 2\tan^{-1} x)\, dx$$

$$\Rightarrow \quad I = \int_0^1 2\tan^{-1}x\,dx + \int_1^{\sqrt{3}} -\pi\,dx + \int_1^{\sqrt{3}} 2\tan^{-1}x\,dx$$

$$\Rightarrow \quad I = \left\{\int_0^1 2\tan^{-1}x\,dx + \int_1^{\sqrt{3}} 2\tan^{-1}x\,dx\right\} - \pi\int_1^{\sqrt{3}} 1\,.\,dx$$

$$\Rightarrow \quad I = 2\int_0^{\sqrt{3}} \tan^{-1}x\,dx - \pi\int_1^{\sqrt{3}} 1\,.\,dx$$

$$\Rightarrow \quad I = 2\left[\left\{x\tan^{-1}x\right\}_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx\right] - \pi\left[x\right]_1^{\sqrt{3}}$$

$$\Rightarrow \quad I = 2\left[\left\{\sqrt{3}\tan^{-1}\sqrt{3} - 0\right\} - \frac{1}{2}\left\{\log(1+x^2)\right\}_0^{\sqrt{3}}\right] - \pi(\sqrt{3}-1)$$

$$\Rightarrow \quad I = 2\left[\frac{\pi}{3}\sqrt{3} - \frac{1}{2}\left(\log 4 - \log 1\right)\right] - \pi(\sqrt{3}-1)$$

$$\Rightarrow \quad I = \frac{2\pi}{3}\sqrt{3} - \log 4 - \pi(\sqrt{3}-1)$$

$$\Rightarrow \quad I = \pi\left(1 - \frac{1}{\sqrt{3}}\right) - \log 4$$

(ii) Clearly, $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) = \begin{cases} 3\tan^{-1}x & , \text{ if } -1/\sqrt{3} < x < 1/\sqrt{3} \\ -\pi + 3\tan^{-1}x & , \text{ if } x > 1/\sqrt{3} \\ \pi + 3\tan^{-1}x & , \text{ if } x < -1/\sqrt{3} \end{cases}$

$$\therefore \quad I = \int_0^1 \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$$

$$\Rightarrow \quad I = \int_0^{1/\sqrt{3}} \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx + \int_{1/\sqrt{3}}^1 \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \quad I = \int_0^{1/\sqrt{3}} 3\tan^{-1}x\,dx + \int_{1/\sqrt{3}}^1 (-\pi + 3\tan^{-1}x)\,dx$$

$$\Rightarrow \quad I = \left\{\int_0^{1/\sqrt{3}} 3\tan^{-1}x\,dx + \int_{1/\sqrt{3}}^1 3\tan^{-1}x\,dx\right\} + \int_{1/\sqrt{3}}^1 -\pi\,dx$$

$$\Rightarrow \quad I = \int_0^1 3\tan^{-1}x - \pi\int_{1/\sqrt{3}}^1 dx$$

$$\Rightarrow \quad I = 3\left[x\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right]_0^1 - \pi\left[x\right]_{1/\sqrt{3}}^1$$

$$\left[\because \int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right]$$

$$\Rightarrow \quad I = 3\left[\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) - 0\right] - \pi\left[1 - \frac{1}{\sqrt{3}}\right] = \pi\left(\frac{1}{\sqrt{3}} - \frac{1}{4}\right) - \frac{3}{2}\log 2$$

EXAMPLE 8 *Prove that* $\int\limits_{1/e}^{\tan x} \frac{t}{1+t^2}\,dt + \int\limits_{1/e}^{\cot x} \frac{1}{t(1+t^2)}\,dt = 1$ *for all x for which* $\tan x$ *and* $\cot x$ *are defined.*

SOLUTION Let $I_1 = \int\limits_{1/e}^{\tan x} \frac{t}{1+t^2}\,dt$ and $I_2 = \int\limits_{1/e}^{\cot x} \frac{1}{t(1+t^2)}\,dt$

Putting $t = \frac{1}{u}$ and $dt = -\frac{1}{u^2}\,du$ in I_2, we get

$$I_2 = \int\limits_{e}^{\tan x} \frac{u^3}{1+u^2} \times -\frac{1}{u^2}\,du = -\int\limits_{e}^{\tan x} \frac{u}{1+u^2}\,du$$

$$\Rightarrow \quad I_2 = -\int\limits_{e}^{\tan x} \frac{t}{1+t^2}\,dt \qquad [\because \text{ integration is independent of change of variable}]$$

$$\Rightarrow \quad I_2 = -\left\{\int\limits_{e}^{1/e} \frac{t}{1+t^2}\,dt + \int\limits_{1/e}^{\tan x} \frac{t}{1+t^2}\,dt\right\}$$

$$\Rightarrow \quad I_2 = -\int\limits_{e}^{1/e} \frac{t}{1+t^2}\,dt - I_1$$

$$\Rightarrow \quad I_1 + I_2 = -\frac{1}{2}\int\limits_{e}^{1/e} \frac{2t}{1+t^2}\,dt$$

$$\Rightarrow \quad I_1 + I_2 = -\frac{1}{2}\left[\log(1+t^2)\right]_e^{1/e}$$

$$\Rightarrow \quad I_1 + I_2 = -\frac{1}{2}\left[\log\left(\frac{1+e^2}{e^2}\right) - \log(1+e^2)\right]$$

$$\Rightarrow \quad I_1 + I_2 = -\frac{1}{2}\left\{\log\frac{1}{e^2}\right\} = -\frac{1}{2} \times -2\log_e = 1$$

PROPERTY IV *If f(x) is a continuous function defined on [a, b], then*

$$\int\limits_a^b f(x)\,dx = \int\limits_a^b f(a+b-x)\,dx$$

<u>**PROOF**</u> Let $x = a+b-t$. Then, $dx = -dt$

Also, $\quad x = a \Rightarrow t = b$ and $x = b \Rightarrow t = a$

$$\therefore \quad \int\limits_a^b f(x)\,dx = -\int\limits_b^a f(a+b-t)\,dt$$

$$\Rightarrow \quad \int\limits_a^b f(x)\,dx = \int\limits_a^b f(a+b-t)\,dt \qquad \text{[By prop. II]}$$

$$\Rightarrow \quad \int\limits_a^b f(x)\,dx = \int\limits_a^b f(a+b-x)\,dx \qquad \text{[By prop. I]}$$

Hence, $\int\limits_a^b f(x)\,dx = \int\limits_a^b f(a+b-x)\,dx$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate*

(i) $\displaystyle\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx$ (ii) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx$ **[NCERT]**

SOLUTION (i) Let $I = \displaystyle\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx$ ...(i)

Then,

$$I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{3-(3-x)}+\sqrt{3-x}}\,dx \quad \text{[Using Prop. IV]}$$

$$\Rightarrow \quad I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\,dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_1^2 \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\,dx = \int_1^2 1\cdot dx = \Big[\,x\,\Big]_1^2 = 2-1 = 1$$

$$\Rightarrow \quad I = 1/2.$$

(ii) Let $\displaystyle I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ ...(i)

Then,

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} 1\cdot dx = \Big[\,x\,\Big]_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$$

$$\Rightarrow \quad I = \pi/12.$$

EXAMPLE 2 *Prove that* $\displaystyle\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

SOLUTION Let $I = \displaystyle\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$ ...(i)

Then,

$$I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(a+b-(a+b-x))}\,dx$$

$$\Rightarrow \qquad I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}\,dx$$

$$\Rightarrow \qquad 2I = \int_a^b 1\,dx = (b-a)$$

$$\Rightarrow \qquad I = \left(\frac{b-a}{2}\right)$$

PROPERTY V *If $f(x)$ is a continuous function defined on $[0, a]$, then*

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

PROOF Let $x = a - t$. Then, $dx = d(a-t) \Rightarrow dx = -dt$

Also, $x = 0 \Rightarrow t = a$ and $x = a \Rightarrow t = 0$

$$\therefore \qquad \int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt$$

$$\Rightarrow \qquad \int_0^a f(x)\,dx = \int_0^a f(a-t)\,dt \qquad \text{[By prop. II]}$$

$$\Rightarrow \qquad \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \qquad \text{[By prop. I]}$$

Hence, $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Prove that:* $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$ **[CBSE 2002C, 2007]**

SOLUTION Let $I = \displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$. Then, ...(i)

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left[\text{Using: } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \qquad I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx$$

$$\Rightarrow \qquad 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx = \int_0^{\pi/2} 1 \cdot dx = \Big[x \Big]_0^{\pi/2} = \frac{\pi}{2} - 0$$

$$\Rightarrow \qquad I = \frac{\pi}{4}$$

EXAMPLE 2 *Evaluate:* $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ **[NCERT]**

SOLUTION Let $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ ...(i)

Then, $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\sin (\pi/2 - x)}}{\sqrt{\sin (\pi/2 - x)} + \sqrt{\cos (\pi/2 - x)}}\, dx$ $\displaystyle\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$

$$\Rightarrow \qquad I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$\Rightarrow \qquad 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \int_0^{\pi/2} 1 \cdot dx = \Big[x \Big]_0^{\pi/2} = \frac{\pi}{2} - 0$$

$$\Rightarrow \qquad I = \frac{\pi}{4} \Rightarrow \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \frac{\pi}{4}$$

EXAMPLE 3 *Evaluate :*

(i) $\displaystyle\int_0^{\pi/2} \log \tan x\, dx$ **[CBSE 2007]**

(ii) $\displaystyle\int_0^{\pi/4} \log (1 + \tan x)\, dx$ **[NCERT, CBSE 2002C, 2003, 2004, PSB 2001; HPSB 2001C]**

SOLUTION (i) Let $\displaystyle I = \int_0^{\pi/2} \log \tan x\, dx$ ...(i)

Then,

$$I = \int_0^{\pi/2} \log \tan\left(\frac{\pi}{2} - x\right) dx \qquad \left[\text{Using : } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \cot x \, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} (\log \tan x + \log \cot x)\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log (\tan x \cdot \cot x)\, dx = \int_0^{\pi/2} \log 1 \cdot dx = \int_0^{\pi/2} 0 \cdot dx = 0$$

$$\Rightarrow \quad I = 0$$

(ii) Let $I = \int_0^{\pi/4} \log (1 + \tan x)\, dx$ ...(i)

Then, $I = \int_0^{\pi/4} \log\left\{1 + \tan\left(\frac{\pi}{4} - x\right)\right\} dx \qquad \left[\text{Using : } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$

$$\Rightarrow \quad I = \int_0^{\pi/4} \log\left\{1 + \frac{\tan \pi/4 - \tan x}{1 + \tan \pi/4 \tan x}\right\} dx = \int_0^{\pi/4} \log\left\{1 + \frac{1 - \tan x}{1 + \tan x}\right\} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \log\left\{\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right\} dx = \int_0^{\pi/4} \log\left\{\frac{2}{1 + \tan x}\right\} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \{\log 2 - \log (1 + \tan x)\}\, dx = \int_0^{\pi/4} \log 2 \, dx - \int_0^{\pi/4} \log (1 + \tan x)\, dx$$

$$\Rightarrow \quad I = (\log 2)\, [x]_0^{\pi/4} - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{4} \log 2 \Rightarrow I = \frac{\pi}{8} \log 2$$

EXAMPLE 4 *Evaluate*

(i) $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx$ **[NCERT]**

(ii) $\int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\, dx$ **[NCERT, CBSE 2009]**

SOLUTION (i) Let $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx$ ...(i)

Then,

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\cos x \sin x}\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we obtain

$$2I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x \cos x}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \left(\frac{\sin x - \cos x}{1+\sin x \cos x} + \frac{\cos x - \sin x}{1+\sin x \cos x}\right)dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sin x - \cos x + \cos x - \sin x}{1+\sin x \cos x}\,dx = 0$$

$$\Rightarrow \quad I = 0$$

(ii) We have,

$$I = \int_0^{\pi/2} (2\log \sin x - \log \sin 2x)\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \{2\log \sin x - \log(2\sin x \cos x)\}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \{2\log \sin x - \log 2 - \log \sin x - \log \cos x\}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \sin x\,dx - \int_0^{\pi/2} \log 2\,dx - \int_0^{\pi/2} \log \cos x\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \sin x\,dx - (\log 2)\int_0^{\pi/2} 1 \cdot dx - \int_0^{\pi/2} \log \cos\left(\frac{\pi}{2}-x\right)dx \qquad \text{[Using prop. V]}$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \sin x\,dx - (\log 2)\left[x\right]_0^{\pi/2} - \int_0^{\pi/2} \log \sin x\,dx$$

$$\Rightarrow \quad I = -(\log 2)\left(\frac{\pi}{2}-0\right) = -\frac{\pi}{2}\log 2$$

EXAMPLE 5 *Evaluate* : $\displaystyle\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx$ **[CBSE 2009]**

SOLUTION Let $I = \displaystyle\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx$ ...(i)

Then,

$$I = \int_0^{\pi} \frac{e^{\cos (\pi - x)}}{e^{\cos (\pi - x)} + e^{-\cos (\pi - x)}}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}} dx$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} 1\, dx = \pi \Rightarrow I = \frac{\pi}{2}$$

EXAMPLE 6 *Prove that:* $\displaystyle\int_0^{2a} f(x)\, dx = \int_0^{2a} f(2a - x)\, dx.$ **[CBSE 2002C]**

SOLUTION Let $I = \displaystyle\int_0^{2a} f(x)\, dx$

Let $\quad 2a - x = t$. Then, $d\,(2a - x) = dt \Rightarrow -dx = dt \Rightarrow dx = -dt$

Also, $\quad x = 0 \Rightarrow t = 2a - 0 = 2a$ and $x = a \Rightarrow t = 2a - a = a$

$$\therefore \quad I = \int_{2a}^{0} f(2a - t)\,(-dt) = -\int_{2a}^{0} f(2a - t)\, dt$$

$$\Rightarrow \quad I = \int_0^{2a} f(2a - t)\, dt \qquad \left[\because \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{2a} f(2a - x)\, dx \qquad \left[\because \int_a^b f(x)\, dx = \int_a^b f(t)\, dt\right]$$

Hence, $\displaystyle\int_0^{2a} f(x)\, dx = \int_0^{2a} f(2a - x)\, dx$

EXAMPLE 7 *Evaluate* : $\displaystyle\int_0^{1} x(1 - x)^n\, dx$ **[NCERT, HPSB 2000C]**

SOLUTION We have,

$$I = \int_0^1 x\,(1 - x)^n\, dx$$

$$\Rightarrow \quad I = \int_0^1 (1-x)\{1-(1-x)\}^n\, dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^1 (1-x)\,x^n\, dx = \int_0^1 (x^n - x^{n+1})\,dx = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$$

$$\Rightarrow \quad I = \left\{\frac{1}{n+1} - \frac{1}{n+2}\right\} - (0-0) = \frac{1}{(n+1)(n+2)}$$

EXAMPLE 8 *Prove that :* $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

SOLUTION Let $I = \int_0^{2a} f(x)\,dx$. Then,

$$I = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \qquad \left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^a f(x)\,dx + I_1, \text{ where } I_1 = \int_a^{2a} f(x)\,dx \qquad \text{...(i)}$$

Let $2a - t = x$. Then, $dx = -dt$. Also, $x = a \Rightarrow t = a$ and $x = 2a \Rightarrow t = 0$

$$\therefore \quad I_1 = \int_a^{2a} f(x)\,dx = \int_a^0 f(2a-t)(-dt) = -\int_a^0 f(2a-t)\,dt$$

$$\Rightarrow \quad I_1 = \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

$$\therefore \quad I = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx \qquad \text{[Using (i)]}$$

EXAMPLE 9 *Evaluate:* $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$

SOLUTION Let $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$. Then,

$$I = \int_0^{\pi} \left\{\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{\sin(2\pi - x)}}\right\} dx \qquad \left[\because \int_0^{2a} f(x) = \int_0^a \{f(x) + f(2a-x)\}\,dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \left\{\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}}\right\} dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \left\{\frac{1}{1+e^{\sin x}} + \frac{e^{\sin x}}{1+e^{\sin x}}\right\} dx$$

$$\Rightarrow \quad I = \int_0^{\pi} 1\,dx = \left[x\right]_0^{\pi} = \pi$$

EXAMPLE 10 *Prove that :*

(i) $\int_0^{\pi/2} \sin 2x \log \tan x \, dx = 0$ **[HPSB 2002]**

(ii) $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx = 0$

SOLUTION (i) We have,

$$I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx \quad \text{...(i)}$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \sin 2x \log \cot x \, dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \sin 2x \{\log \tan x + \log \cot x\} \, dx = \int_0^{\pi/2} \sin 2x \log (\tan x \cdot \cot x) \, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \sin 2x \cdot \log 1 \cdot dx = 0 \quad [\because \log 1 = 0]$$

$$\Rightarrow \quad I = 0$$

(ii)
$$I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \quad \text{...(i)}$$

$$\Rightarrow \quad I = \int_0^1 \log\left\{\frac{1-(1-x)}{1-x}\right\} dx \quad \left[\text{Using : } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^1 \log\left(\frac{x}{1-x}\right) dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^1 \log\left(\frac{1-x}{x}\right) + \log\left(\frac{x}{1-x}\right) dx = \int_0^1 \log 1 \cdot dx = 0$$

EXAMPLE 11 *Evaluate*

(i) $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ **[CBSE 2002, 2003]**

(ii) $\int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x} dx.$

SOLUTION (i) Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\, dx$. Then,

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\, dx \qquad \left[\text{Using}: \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}\, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} + \frac{\cos^2 x}{\sin x + \cos x}\, dx = \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}\, dx = \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}\, dx$$

Let $\tan\frac{x}{2} = t$. Then, $d\left(\tan\frac{x}{2}\right) = dt \Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}\, dx = dt \Rightarrow \sec^2\frac{x}{2}\, dx = 2\, dt$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$

$$\therefore \quad 2I = \int_0^1 \frac{2dt}{2t + 1 - t^2} = 2\int_0^1 \frac{1}{(\sqrt{2})^2 - (t-1)^2}\, dt$$

$$\Rightarrow \quad 2I = 2 \times \frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1$$

$$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}}{\sqrt{2}}\right) - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\} = \frac{1}{\sqrt{2}}\left\{0 - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$$

$$\Rightarrow \quad 2I = -\frac{1}{\sqrt{2}}\log\left\{\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}\right\} = -\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)^2 = -\frac{2}{\sqrt{2}}\log(\sqrt{2}-1)$$

$$\Rightarrow \quad I = -\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)$$

(ii) Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x}\, dx$. Then, ...(i)

$$I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}\, dx \qquad \left[\text{Using}: \int_0^a f(x)\, dx = \int_0^a f(a-x)\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^2 x}{1 + \cos x \sin x}\, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{1 + \sin x \cos x}\, dx = \int_0^{\pi/2} \frac{1}{1 + \sin x \cos x}\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + \tan x}\, dx \qquad \text{[Dividing num. and denom. by } \cos^2 x]$$

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\, dx = dt$

Now, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan \frac{\pi}{2} = \infty$.

$$\therefore \quad 2I = \int_0^{\infty} \frac{dt}{1 + t^2 + t}$$

$$\Rightarrow \quad 2I = \int_0^{\infty} \frac{dt}{(t + 1/2)^2 + (\sqrt{3}/2)^2}\, dt$$

$$\Rightarrow \quad 2I = \frac{1}{\sqrt{3}/2}\left[\tan^{-1}\left(\frac{t + 1/2}{\sqrt{3}/2}\right)\right]_0^{\infty}$$

$$\Rightarrow \quad 2I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^{\infty} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\infty - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] = \frac{2}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{6}\right] = \frac{2\pi}{3\sqrt{3}}$$

$$\Rightarrow \quad I = \frac{\pi}{3\sqrt{3}}.$$

EXAMPLE 12 *Evalue:* $\int_0^1 \cot^{-1}(1 - x + x^2)\, dx$ **[CBSE 2008]**

SOLUTION Let

$$I = \int_0^1 \cot^{-1}(1 - x + x^2)\, dx$$

Then,

$$I = \int_0^1 \tan^{-1}\left(\frac{1}{1 - x + x^2}\right) dx \qquad \left[\because \cot^{-1} x = \tan^{-1}\frac{1}{x}\right]$$

$$\Rightarrow \quad I = \int_0^1 \tan^{-1}\left\{\frac{1}{1 - x(1-x)}\right\} dx$$

$$\Rightarrow \quad I = \int_0^1 \tan^{-1}\left\{\frac{x + (1-x)}{1 - x(1-x)}\right\} dx$$

$$\Rightarrow \quad I = \int_0^1 \left\{\tan^{-1} x + \tan^{-1}(1-x)\right\} dx$$

$$\Rightarrow \quad I = \int_0^1 \tan^{-1} x \, dx + \int_0^1 \tan^{-1} (1-x) \, dx$$

$$\Rightarrow \quad I = \int_0^1 \tan^{-1} x \, dx + \int_0^1 \tan^{-1} \{1-(1-x)\} \, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^1 \tan^{-1} x \, dx + \int_0^1 \tan^{-1} x \, dx$$

$$\Rightarrow \quad I = 2 \int_0^1 \tan^{-1} x \, dx$$

$$\Rightarrow \quad I = 2 \int_0^1 \underset{\mathrm{I}}{\tan^{-1} x} \cdot \underset{\mathrm{II}}{1} \, dx$$

$$\Rightarrow \quad I = 2 \left[x \tan^{-1} x\right]_0^1 - 2 \int_0^1 \frac{x}{1+x^2} \, dx$$

$$\Rightarrow \quad I = 2 \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{2x}{1+x^2} \, dx$$

$$\Rightarrow \quad I = 2 \left[x \tan^{-1} x\right]_0^1 - \left[\log (1+x^2)\right]_0^1$$

$$\Rightarrow \quad I = 2 \left(\frac{\pi}{4} - 0\right) - (\log_2 - \log_1)$$

$$\Rightarrow \quad I = \frac{\pi}{2} - \log 2$$

PROBLEMS ON REMOVAL OF x: Let $I = \int_0^a x f(x)\, dx$ where $f(x)$ is a function of x whose integral is known and $f(a-x) = f(x)$. Then, we have

$$I = \int_0^a (a-x) f(a-x)\, dx \qquad \text{[By Prop. IV]}$$

$$\Rightarrow \quad I = \int_0^a (a-x) f(x)\, dx \qquad [\because f(a-x) = f(x)]$$

$$\Rightarrow \quad I = a \int_0^a f(x)\, dx - \int_0^a x f(x)\, dx$$

$$\Rightarrow \quad I = a \int_0^a f(x)\, dx - I$$

$$\Rightarrow \quad 2I = a \int_0^a f(x)\, dx \Rightarrow I = \frac{a}{2} \int_0^a f(x)\, dx.$$

Now, $\int_0^a f(x)$ can be evaluated as $f(x)$ is known integrable function.

EXAMPLE 13 *Evaluate :*

(i) $\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$ **[NCERT, CBSE 2008]**

(ii) $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx.$ **[NCERT, CBSE 2003, 2005, 2008 PSB 2001]**

SOLUTION We have,

(i) $$I = \int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \quad \text{...(i)}$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)}{a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)} dx \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{x + \pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx$$

$$\Rightarrow \quad 2I = 2\pi \int_0^{\pi/2} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx \quad \text{[Using Prop. VI]}$$

$$\Rightarrow \quad 2I = 2\pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx \quad \text{[Dividing num. and denom. by } \cos^2 x\text{]}$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx.$$

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\, dx = dt$.

Also $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan \frac{\pi}{2} = \infty$.

$$\therefore \quad I = \pi \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} = \frac{\pi}{b^2} \int_0^{\infty} \frac{dt}{(a/b)^2 + t^2} = \frac{\pi}{b^2} \times \frac{1}{(a/b)} \left[\tan^{-1}\left(\frac{t}{a/b}\right)\right]_0^{\infty}$$

$$\Rightarrow \quad I = \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty} = \frac{\pi}{ab} (\tan^{-1} \infty - \tan^{-1} 0) = \frac{\pi}{ab}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{2\,ab}.$$

(ii) $$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \quad \text{...(i)}$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1 + \cos^2 (\pi - x)} dx \quad \left[\text{Using} : \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{(x+\pi-x)\sin x}{1+\cos^2 x}\,dx = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx.$$

Let $\cos x = t$. Then, $d(\cos x) = dt \Rightarrow -\sin x\,dx = dt$

Also $\quad x = 0 \Rightarrow t = \cos 0 = 1$ and $x = \pi \Rightarrow t = \cos \pi = -1$

$$\therefore \quad I = \frac{\pi}{2}\int_1^{-1} \frac{-dt}{1+t^2} = -\frac{\pi}{2}\int_1^{-1} \frac{1}{1+t^2}\,dt = -\frac{\pi}{2}\Big[\tan^{-1} t\Big]_1^{-1}$$

$$\Rightarrow \quad I = -\frac{\pi}{2}\Big[\tan^{-1}(-1) - \tan^{-1}(1)\Big] = -\frac{\pi}{2}\Big[-\frac{\pi}{4}-\frac{\pi}{4}\Big] = \frac{\pi^2}{4}.$$

EXAMPLE 14 *Evaluate :*

(i) $\displaystyle\int_0^{\pi} \frac{x}{1+\sin x}\,dx$ **[NCERT, CBSE 2001C, 2004, 2001, 2010]**

(ii) $\displaystyle\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\,dx$ ← **[NCERT, CBSE 2008, 2010]**

(iii) $\displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$

SOLUTION (i) Let $\displaystyle I = \int_0^{\pi} \frac{x}{1+\sin x}\,dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{x+\pi-x}{1+\sin x}\,dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx = \pi\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}\,dx \quad \text{...(iii)}$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} (\sec^2 x - \tan x \sec x)\,dx = \pi\Big[\tan x - \sec x\Big]_0^{\pi}$$

$$\Rightarrow \quad 2I = \pi[(\tan\pi - \sec\pi) - (\tan 0 - \sec 0)] = \pi[(0-(-1)-(0-1)] = 2\pi$$

$$\Rightarrow \quad I = \pi.$$

(ii) Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx = \int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx$ ...(i)

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \sin(\pi - x)}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \sin x}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi \sin x - x \sin x}{1 + \sin x}\, dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1 + \sin x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sin x(1 - \sin x)}{1 - \sin^2 x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} (\tan x \sec x - \tan^2 x)\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \{\tan x \sec x - (\sec^2 x -)\}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} (\sec x \tan x - \sec^2 x + 1)\, dx$$

$$\Rightarrow \quad 2I = \pi \left[\sec x - \tan x + x\right]_0^{\pi}$$

$$\Rightarrow \quad 2I = \pi\, [(\sec \pi - \tan \pi + \pi) - (\sec 0 + \tan 0 + 0)]$$

$$\Rightarrow \quad 2I = \pi\, [(-1 - 0 + \pi) - (1 - 0 + 0)]$$

$$\Rightarrow \quad 2I = \pi(\pi - 2)$$

$$\Rightarrow \quad I = \frac{\pi}{2}(\pi - 2)$$

(iii) Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x}\, dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\cos x + \sin x}\, dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{x + \frac{\pi}{2} - x}{\sin x + \cos x}\, dx = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\, dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}\, dx = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{-\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1}\, dx$$

Let $\tan\frac{x}{2} = t$. Then, $d\left(\tan\frac{x}{2}\right) = dt \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\, dx = dt \Rightarrow dx = \frac{2\, dt}{\sec^2\frac{x}{2}}$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$.

$$\therefore \quad 2I = \frac{\pi}{2} \int_0^1 \frac{2\, dt}{-t^2 + 2t + 1} = \pi \int_0^1 \frac{dt}{-[t^2 - 2t - 1]} = \pi \int_0^1 \frac{dt}{-[(t-1)^2 - 2]}$$

$$\Rightarrow \quad 2I = \pi \int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2} = \pi \cdot \frac{1}{2\sqrt{2}} \left[\log\left|\frac{\sqrt{2} + t - 1}{\sqrt{2} - t + 1}\right|\right]_0^1$$

$$\Rightarrow \quad 2I = \frac{\pi}{2\sqrt{2}} \left[\log 1 - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right] = -\frac{\pi}{2\sqrt{2}} \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = \frac{\pi}{2\sqrt{2}} \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\Rightarrow \quad 2I = \frac{\pi}{2\sqrt{2}} \log\left\{\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)}\right\} = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1)^2 = \frac{\pi}{\sqrt{2}} \log(\sqrt{2}+1)$$

$$\Rightarrow \quad I = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1).$$

EXAMPLE 15 *If f and g are continuous on $[0, a]$ and satisfy $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$, show that*

$$\int_0^a f(x)\, g(x)\, dx = \int_0^a f(x)\, dx$$

[NCERT]

SOLUTION We have,

$$\int_0^a f(x)\, g(x)\, dx = \int_0^a f(a-x)\, g(a-x)\, dx$$

$$\Rightarrow \quad \int_0^a f(x)\, g(x)\, dx = \int_0^a f(x) \{2 - g(x)\}\, dx \qquad [\because g(a-x) = 2 - g(x)]$$

$$\Rightarrow \quad \int_0^a f(x)\, g(x)\, dx = 2\int_0^a f(x)\, dx - \int_0^a f(x)\, g(x)\, dx$$

$$\Rightarrow \quad 2\int_0^a f(x)\, g(x)\, dx = 2\int_0^a f(x)\, dx$$

$$\Rightarrow \quad \int_0^a f(x)\, g(x)\, dx = \int_0^a f(x)\, dx$$

EXAMPLE 16 *Evaluate* : $\displaystyle\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx.$ **[CBSE 2010]**

SOLUTION Let $I = \displaystyle\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$. Then,

$$I = \int_0^{\pi/2} \frac{(\pi/2 - x) \sin(\pi/2 - x) \cos(\pi/2 - x)}{\sin^4(\pi/2 - x) + \cos^4(\pi/2 - x)}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{(\pi/2 - x) \sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx - \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan x \sec^2 x}{1 + \tan^4 x}\, dx \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } \cos^4 x]$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_0^{\pi/2} \frac{2 \tan x \sec^2 x\, dx}{1 + (\tan^2 x)^2}$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_0^{\pi/2} \frac{1}{1 + (\tan^2 x)^2}\, d(\tan^2 x)$$

$$\therefore \quad 2I = \frac{\pi}{4}\int_0^{\infty} \frac{1}{1 + t^2}\, dt, \text{ where } t = \tan^2 x.$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\left[\tan^{-1} t\right]_0^{\infty} = \frac{\pi}{4}\left[\tan^{-1} \infty - \tan^{-1} 0\right]$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\left[\frac{\pi}{2} - 0\right] \Rightarrow I = \frac{\pi^2}{16}$$

EXAMPLE 17 *Prove that* : $\displaystyle\int_0^{\pi} \frac{x}{1 - \cos\alpha \sin x}\, dx = \frac{\pi(\pi - \alpha)}{\sin\alpha}$

SOLUTION Let $I = \displaystyle\int_0^{\pi} \frac{x}{1 - \cos\alpha \sin x}\, dx$. Then,

$$I = \int_0^{\pi} \frac{(\pi - x)}{1 - \cos\alpha \sin(\pi - x)}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi}{1 - \cos\alpha \sin x}\, dx - \int_0^{\pi} \frac{x}{1 - \cos\alpha \sin x}\, dx$$

$$\Rightarrow \quad I = \pi \int_0^{\pi} \frac{1}{1 - \cos\alpha \sin x}\, dx - I$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{1}{1 - \cos\alpha \sin x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{1 + \tan^2 x/2}{(1 + \tan^2 x/2) - 2\cos\alpha \tan x/2}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sec^2 x/2}{\tan^2 x/2 - 2\cos\alpha \tan x/2 + 1}\, dx$$

$$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi} \frac{\sec^2 x/2}{\tan^2 x/2 - 2\cos\alpha \tan x/2 + 1}\, dx$$

Let $\tan\frac{x}{2} = t$. Then, $d\left(\tan\frac{x}{2}\right) = dt \Rightarrow \sec^2\frac{x}{2}\, dx = 2\, dt.$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \pi \Rightarrow t = \tan\frac{\pi}{2} = \infty.$

$$\therefore \quad I = \frac{\pi}{2} \int_0^{\infty} \frac{2\, dt}{t^2 - 2t\cos\alpha + 1}$$

$$\Rightarrow \quad I = \pi \int_0^{\infty} \frac{1}{(t - \cos\alpha)^2 + (1 - \cos^2\alpha)}\, dt$$

$$\Rightarrow \quad I = \pi \int_0^{\infty} \frac{1}{\sin^2\alpha + (t - \cos\alpha)^2}\, dt$$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha} \left[\tan^{-1}\left(\frac{t - \cos\alpha}{\sin\alpha}\right)\right]_0^{\infty}$$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha} \left[\tan^{-1}\infty - \tan^{-1}(-\cot\alpha)\right]$$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha} \left[\frac{\pi}{2} + \tan^{-1}(\cot\alpha)\right]$$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha} \left[\frac{\pi}{2} + \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \alpha\right)\right\}\right]$$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha} \left[\frac{\pi}{2} + \frac{\pi}{2} - \alpha\right] = \frac{\pi(\pi - \alpha)}{\sin\alpha}.$$

EXAMPLE 18 *Evaluate :* $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$

SOLUTION Let $\displaystyle I = \int_1^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$ ...(i)

Then,

$$I = \int_0^{\pi/2} \frac{\cos(\pi/2-x)}{1+\cos(\pi/2-x)+\sin(\pi/2-x)}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin x}{1+\sin x+\cos x}\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\cos x+\sin x}{1+\sin x+\cos x}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{1+\sin x+\cos x-1}{1+\sin x+\cos x}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \left\{1-\frac{1}{1+\sin x+\cos x}\right\} dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} 1\,.\,dx - \int_0^{\pi/2} \frac{1}{1+\sin x+\cos x}\,dx$$

$$\Rightarrow \quad 2I = \Big[x\Big]_0^{\pi/2} - \int_0^{\pi/2} \frac{1+\tan^2 x/2}{1+\tan^2 x/2+2\tan x/2+1-\tan^2 x/2}\,dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} - \int_0^{\pi/2} \frac{\sec^2 x/2}{2+2\tan x/2}\,dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} - \int_0^{1} \frac{2\,dt}{2+2t}, \text{ where } t = \tan\frac{x}{2}$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} - \Big[\log(1+t)\Big]_0^1$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} - \log 2 \Rightarrow I = \frac{\pi}{4} - \frac{1}{2}\log 2$$

PROPERTY VI *If $f(x)$ is a continuous function defined on $[-a, a]$, then*

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

<u>PROOF</u> Clearly,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad \text{[Using prop. III]}$$

Let $x = -t$. Then, $dx = -dt$.

Also, $x = -a \Rightarrow t = a$ and $x = 0 \Rightarrow t = 0$.

$$\therefore \quad \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-t)\,(-dt) = -\int_{a}^{0} f(-t)\,dt$$

$$\Rightarrow \quad \int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-t)\,dt \qquad \text{[By prop. II]}$$

$$\Rightarrow \quad \int_{-a}^{0} f(x) = \int_{0}^{a} f(-x)\,dx \qquad \text{[By prop. I]}$$

$$\therefore \quad \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\Rightarrow \quad \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(-x) + f(x)\}\,dx$$

$$\Rightarrow \quad \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & , \text{ if } f(-x) = f(x) \\ 0 & , \text{ if } f(-x) = -f(x) \end{cases}$$

$$\Rightarrow \quad \int_{-a}^{a} f(x)\,dx = \begin{cases} 2 \cdot \int_{0}^{a} f(x)\,dx & , \text{ if } f(x) \text{ is an even function} \\ 0 & , \text{ if } f(x) \text{ is an odd function.} \end{cases}$$

REMARK *The graph of an even function is symmetric about y-axis that is the curve on left side of y-axis is exactly identical to curve on its right side.*

So, $\int_{0}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx$ (see Fig. 20.8).

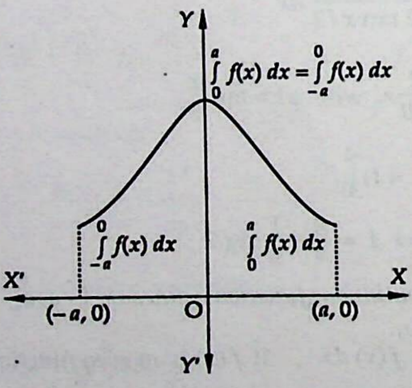

Fig. 20.9

In case of an odd function the curve is symmetric in opposite quadrants, so

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx \qquad \text{(see Fig. 20.9)}$$

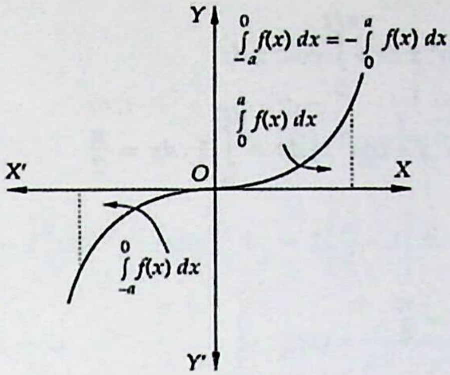

Fig. 20.10

NOTE: *In the above property, we have proved that* $\int_{-a}^{a} f(x)\ dx = \int_{0}^{a} \{f(x) + f(-x)\}\ dx$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int_{-\pi/2}^{\pi/2} \sin^7 x\, dx$ **[NCERT]** (ii) $\int_{-\pi/2}^{\pi/2} \sin^2 x\, dx$ **[NCERT]**

SOLUTION (i) Let $f(x) = \sin^7 x$. Then,

$$f(-x) = \sin^7(-x) = \{\sin(-x)\}^7 = \{-\sin x\}^7 = -\sin^7 x = -f(x)$$

$\therefore$ $f(x)$ is odd function.

$$\Rightarrow \int_{-\pi/2}^{\pi/2} f(x)\, dx = 0 \Rightarrow \int_{-\pi/2}^{\pi/2} \sin^7 x\, dx = 0$$

(ii) Let $f(x) = \sin^2 x$. Then,

$$f(-x) = \sin^2(-x) = \{\sin(-x)\}^2 = \{-\sin x\}^2 = \sin^2 x = f(x)$$

$\therefore$ $f(x)$ is an even function

$$\Rightarrow \int_{-\pi/2}^{\pi/2} f(x)\, dx = 2\int_{0}^{\pi/2} f(x)\, dx$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \sin^2 x\, dx = 2\int_{0}^{\pi/2} \sin^2 x\, dx \qquad \text{...(i)}$$

Let $I = \int_{0}^{\pi/2} \sin^2 x\, dx$. Then,

$$I = \int_{0}^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right) dx \Rightarrow I = \int_{0}^{\pi/2} \cos^2 x\, dx$$

$$\therefore \quad I + I = \int_0^{\pi/2} \sin^2 x\, dx + \int_0^{\pi/2} \cos^2 x\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} (\sin^2 x + \cos^2 x)\, dx = \int_0^{\pi/2} 1 \,.\, dx = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

$$\Rightarrow \quad \int_0^{\pi/2} \sin^2 x\, dx = \frac{\pi}{4}$$

Substituting this value in (i), we get $\int_{-\pi/2}^{\pi/2} \sin^2 x\, dx = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$.

EXAMPLE 2 *Evaluate:*

(i) $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\, dx$ (ii) $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$ **[CBSE 2002, 2008]**

SOLUTION (i) Let $f(x) = x^3 \sin^4 x$. Then,

$$f(-x) = (-x)^3 \sin^4 (-x) = -x^3 \left\{\sin(-x)\right\}^4 = -x^3 (-\sin x)^4 = -x^3 \sin^4 x = -f(x).$$

So, $f(x)$ is an odd function.

Hence, $\int_{-\pi/4}^{\pi/4} f(x)\, dx = 0 \Rightarrow \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\, dx = 0$.

(ii) Let $I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$. Then,

$$I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}}\, dx = \int_{-a}^{a} \frac{a-x}{\sqrt{a^2 - x^2}}\, dx$$

$$\Rightarrow \quad I = \int_{-a}^{a} \frac{a}{\sqrt{a^2 - x^2}}\, dx - \int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}}\, dx$$

$$\Rightarrow \quad I = a \int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}}\, dx - \int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}}\, dx$$

$$\Rightarrow \quad I = a I_1 - I_2, \text{ where } I_1 = \int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}}\, dx \text{ and } I_2 = \int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}}\, dx$$

Let $f(x) = \frac{1}{\sqrt{a^2 - x^2}}$ and $g(x) = \frac{x}{\sqrt{a^2 - x^2}}$. Then,

$$f(-x) = \frac{1}{\sqrt{a^2 - (-x)^2}} = \frac{1}{\sqrt{a^2 - x^2}} = f(x) \text{ and } g(-x) = \frac{-x}{\sqrt{a^2 - (-x)^2}} = \frac{-x}{\sqrt{a^2 - x^2}} = -g(x)$$

$\Rightarrow$ $f(x)$ is an even function and $g(x)$ is an odd function.

$\therefore \qquad I_1 = \int_{-a}^{a} \frac{1}{\sqrt{a^2 - x^2}}\, dx$

$\Rightarrow \qquad I_1 = 2\int_{0}^{a} \frac{1}{\sqrt{a^2 - x^2}}\, dx \qquad \left[\because f(x) = \frac{1}{\sqrt{a^2 - x^2}} \text{ is an even function}\right]$

$\Rightarrow \qquad I_1 = 2\left[\sin^{-1}\frac{x}{a}\right]_0^a = 2\left[\sin^{-1} 1 - \sin^{-1} 0\right] = 2\left[\frac{\pi}{2} - 0\right] = \pi$

and, $\qquad I_2 = \int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}}\, dx = 0. \qquad \left[\because f(x) = \frac{x}{\sqrt{a^2 - x^2}} \text{ is an odd function}\right]$

Hence, $\quad I = a\pi - 0 = a\pi.$

EXAMPLE 3 *Evaluate :* $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{x + \pi/4}{2 - \cos 2x}\, dx$

SOLUTION Let $I = \displaystyle\int_{-\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x}\, dx$. Then,

$$I = \int_{-\pi/4}^{\pi/4} \frac{x}{2 - \cos 2x}\, dx + \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{1}{2 - \cos 2x}\, dx$$

$$\Rightarrow \qquad I = 0 + \frac{\pi}{2}\int_{0}^{\pi/4} \frac{1}{2 - \cos 2x}\, dx \qquad \left[\begin{array}{l}\frac{x}{2 - \cos 2x} \text{ is an odd function and} \\ \frac{1}{2 - \cos 2x} \text{ is an even function}\end{array}\right]$$

$$\Rightarrow \qquad I = \frac{\pi}{2}\int_{0}^{\pi/4} \frac{1 + \tan^2 x}{1 + 3\tan^2 x}\, dx$$

$$\Rightarrow \qquad I = \frac{\pi}{2}\int_{0}^{\pi/4} \frac{\sec^2 x}{1^2 + (\sqrt{3}\tan x)^2}\, dx$$

$$\Rightarrow \qquad I = \frac{\pi}{2\sqrt{3}}\int_{0}^{\pi/4} \frac{1}{1^2 + (\sqrt{3}\tan x)^2}\, d(\sqrt{3}\tan x)$$

$$\Rightarrow \qquad I = \frac{\pi}{2\sqrt{3}}\left[\tan^{-1}(\sqrt{3}\tan x)\right]_0^{\pi/4}$$

$$\Rightarrow \qquad I = \frac{\pi}{2\sqrt{3}}(\tan^{-1}\sqrt{3} - \tan^{-1} 0) = \frac{\pi}{2\sqrt{3}} \times \frac{\pi}{3} = \frac{\pi^2}{6\sqrt{3}}.$$

EXAMPLE 4 *Evaluate:* $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{1 + e^{\sin x}}\, dx$

SOLUTION Let $I = \displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{1 + e^{\sin x}}\, dx$. Then,

$$I = \int_0^{\pi/2} \left\{\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}}\right\} dx \qquad \left[\because \int_{-a}^{a} f(x)\,dx = \int_0^a \{f(x)+f(-x)\}\,dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \left\{\frac{1}{1+e^{\sin x}} + \frac{e^{\sin x}}{1+e^{\sin x}}\right\} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} = \frac{1+e^{\sin x}}{1+e^{\sin x}}\,dx = \int_0^{\pi/2} 1\,dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2}$$

EXAMPLE 5 *Evaluate:* $\displaystyle\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$

SOLUTION We have,

$$I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx.$$

$$\Rightarrow \quad I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx + \int_{-\pi}^{\pi} \frac{2x \sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow \quad I = I_1 + I_2, \text{ where } I_1 = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx \text{ and } I_2 = \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$$

Since $f(x) = \dfrac{2x}{1+\cos^2 x}$ is an odd function and $\dfrac{2x\sin x}{1+\cos^2 x}$ is an even function.

$$I_1 = 0 \text{ and } I_2 = 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx.$$

Now,

$$I_2 = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx \qquad \text{...(i)}$$

$$\Rightarrow \quad I_2 = 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

$$\Rightarrow \quad I_2 = 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I_2 = 4\pi\int_0^{\pi} \frac{1}{1+\cos^2 x}\,\sin x\,dx$$

$$2I_2 = -4\pi\int_1^{-1} \frac{1}{1+t^2}\,dt, \text{ where } t = \cos x$$

$$\Rightarrow \quad 2I_2 = -4\pi\left[\tan^{-1} t\right]_1^{-1}$$

$\Rightarrow \quad 2I_2 = -4\pi\{-\pi/4 - \pi/4\} = 2\pi^2$

$\Rightarrow \quad I_2 = \pi^2$

Hence, $\quad I = 0 + \pi^2 = \pi^2.$

EXAMPLE 6 *Evaluate* $\displaystyle\int_{-\pi/2}^{\pi/2} |\sin x|\, dx$

SOLUTION Let $f(x) = |\sin x|$. Then,

$$f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x).$$

So, $f(x)$ is an even function.

$$\therefore \quad \int_{-\pi/2}^{\pi/2} |\sin x|\, dx = 2\int_0^{\pi/2} |\sin x|\, dx = 2\int_0^{\pi/2} \sin x\, dx \qquad \left[\because \sin x \geq 0 \text{ for } 0 \leq x \leq \frac{\pi}{2}\right]$$

$$\Rightarrow \quad \int_{-\pi/2}^{\pi/2} |\sin x|\, dx = 2\left[-\cos x\right]_0^{\pi/2} = 2\left\{-\cos\frac{\pi}{2} + \cos 0\right\} = 2$$

EXAMPLE 7 *Evaluate* $\displaystyle\int_{-1}^{3/2} |x \sin \pi x|\, dx.$ **[NCERT]**

SOLUTION We have,

$$-1 < x < \frac{3}{2} \Rightarrow -\pi < \pi x < \frac{3\pi}{2}$$

Now, $\quad -1 < x < 0$

$\Rightarrow \quad -\pi < \pi x < 0$

$\Rightarrow \quad \sin \pi x < 0 \Rightarrow x \sin \pi x > 0 \qquad [\because -1 < x < 0]$

$\Rightarrow \quad |x \sin \pi x| = x \sin \pi x$

and, $\quad 0 < x < 1$

$\Rightarrow \quad 0 < \pi x < \pi \Rightarrow \sin \pi x > 0 \Rightarrow x \sin \pi x > 0 \Rightarrow |x \sin \pi x| = x \sin \pi x$

and, $\quad 1 < x < \frac{3}{2}$

$\Rightarrow \quad \pi < \pi x < \frac{3\pi}{2} \Rightarrow \sin \pi x < 0 \Rightarrow x \sin \pi x < 0 \Rightarrow |x \sin \pi x| = -x \sin \pi x$

Thus, $\quad |x \sin \pi x| = \begin{cases} x \sin \pi x, & \text{if } -1 < x < 1 \\ -x \sin \pi x, & \text{if } 1 < x < \frac{3}{2} \end{cases}$

$$\therefore \quad I = \int_{-1}^{3/2} |x \sin \pi x|\, dx = \int_{-1}^{1} |x \sin \pi x|\, dx + \int_1^{3/2} |x \sin \pi x|\, dx$$

$$\Rightarrow \quad I = \int_{-1}^{1} x \sin \pi x\, dx + \int_1^{3/2} (-x \sin \pi x)\, dx = \int_{-1}^{1} x \sin \pi x\, dx - \int_1^{3/2} x \sin \pi x\, dx$$

$$\Rightarrow \quad I = 2\int_0^1 x \sin \pi x\, dx - \int_1^{3/2} x \sin \pi x\, dx \qquad [\because x \sin \pi x \text{ is an even function}]$$

Now, $\int \underset{\text{I}}{x} \underset{\text{II}}{\sin \pi x}\, dx = -\frac{x}{\pi}\cos \pi x + \frac{1}{\pi}\int \cos \pi x\, dx = -\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x$

$$\therefore \quad I = \int_{-1}^{3/2} |x \sin \pi x|\, dx$$

$$\Rightarrow \quad I = 2\left[-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right]_0^1 - \left[-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right]_1^{3/2}$$

$$\Rightarrow \quad I = 2\left[\left(-\frac{1}{\pi}\cos \pi + \frac{1}{\pi^2}\sin \pi\right) - (0)\right] - \left[\left(-\frac{3}{2\pi}\cos \frac{3\pi}{2} + \frac{1}{\pi^2}\sin \frac{3\pi}{2}\right) - \left(-\frac{1}{\pi}\cos \pi + \frac{1}{\pi^2}\sin \pi\right)\right]$$

$$\Rightarrow \quad I = 2\left[\left(\frac{1}{\pi} + 0\right)\right] - \left[\left(0 + \frac{1}{\pi^2}(-1)\right) - \left(\frac{1}{\pi} + 0\right)\right] = \frac{2}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi} = \frac{3\pi + 1}{\pi^2}$$

PROPERTY VII *If $f(x)$ is a continuous function defined on $[0, 2a]$, then*

$$\int_0^{2a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$$

PROOF Clearly,

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_a^{2a} f(x)\, dx \qquad \text{...(i)}$$

Consider the integral $\int_a^{2a} f(x)\, dx$

Let $x = 2a - t$. Then, $dx = d(2a - t) \Rightarrow dx = -dt$.

Also, $x = a \Rightarrow t = a$ and $x = 2a \Rightarrow t = 0$

$$\therefore \quad \int_a^{2a} f(x)\, dx = -\int_a^0 f(2a - t)\, dt$$

$$\Rightarrow \quad \int_a^{2a} f(x)\, dx = \int_0^a f(2a - t)\, dt \qquad \text{[Using prop. II]}$$

$$\Rightarrow \quad \int_a^{2a} f(x)\, dx = \int_0^a f(2a - x)\, dx \qquad \text{[Using prop. I]}$$

Substituting $\int_a^{2a} f(x)\, dx = \int_0^a f(2a - x)\, dx$ in (i), we get

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx$$

$$\Rightarrow \quad \int_0^{2a} f(x)\, dx = \int_0^a \{f(x) + f(2a - x)\}\, dx$$

$$\Rightarrow \quad \int_0^{2a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$$

<u>REMARK</u> *If $f(2a-x) = f(x)$, then the graph of $f(x)$ is symmetrical about $x = a$ as shown in Fig. 20.11.*

$$\therefore \quad \int_a^{2a} f(x)\,dx = \int_0^a f(x)\,dx$$

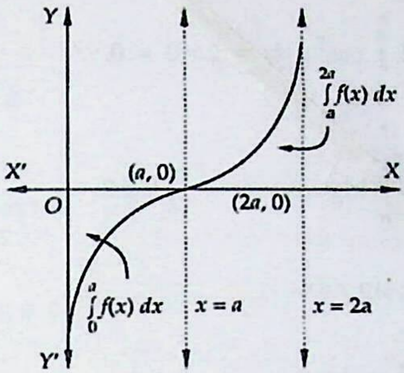

Fig. 20.11

If $f(2a-x) = -f(x)$, then the graph of $f(x)$ is shown in Fig. 20.12.

$$\therefore \quad \int_0^a f(x)\,dx = -\int_a^{2a} f(x)\,dx.$$

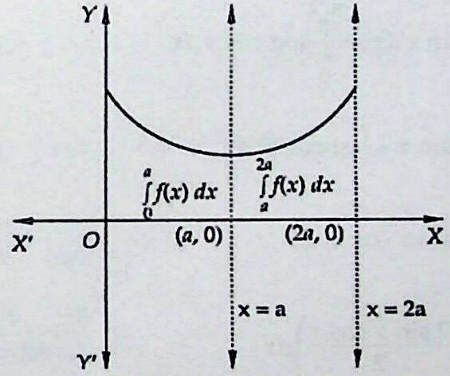

Fig. 20.12

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate :* $\displaystyle\int_0^{2\pi} \cos^5 x\,dx.$ **[NCERT]**

SOLUTION Let $f(x) = \cos^5 x$. Then, $f(2\pi - x) = \{\cos(2\pi - x)\}^5 = \cos^5 x$

$$\therefore \quad \int_0^{2\pi} \cos^5 x\,dx = 2\int_0^{\pi} \cos^5 x\,dx \qquad \text{[Using Prop. VII]}$$

Now,

$$f(\pi - x) = \{\cos(\pi - x)\}^5 = -\cos^5 x = -f(x)$$

$\therefore$ $$\int_0^{\pi} \cos^5 x\,dx = 0.$$ [Using Prop. VII]

Hence, $$\int_0^{2\pi} \cos^5 x\,dx = 2\int_0^{\pi} \cos^5 x\,dx = 2\times 0 = 0.$$

EXAMPLE 2 *Prove that :*

$$\int_0^{\pi/2} \log \sin x\,dx = \int_0^{\pi/2} \log \cos x\,dx = -\frac{\pi}{2}\log 2.$$ **[NCERT, CBSE 2008]**

SOLUTION Let $I = \int_0^{\pi/2} \log \sin x\,dx.$ ...(i)

Then $$I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \cos x\,dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \log \sin x\,dx + \int_0^{\pi/2} \log \cos x\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log(\sin x \cos x)\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log\left(\frac{2\sin x \cos x}{2}\right) dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin 2x\,dx - \int_0^{\pi/2} \log 2\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin 2x\,dx - \frac{\pi}{2}(\log 2)$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin 2x\,dx - \frac{\pi}{2}\log 2. \quad \text{...(iii)}$$

Let $I_1 = \int_0^{\pi/2} \log \sin 2x \, dx.$

Putting $2x = t$, we get

$$I_1 = \int_0^{\pi} \log \sin t \frac{dt}{2}$$

$$\Rightarrow \quad I_1 = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt$$

$$\Rightarrow \quad I_1 = \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin t \, dt$$

$$\Rightarrow \quad I_1 = \int_0^{\pi/2} \log \sin x \, dx = I \qquad \text{[Using Prop. I]}$$

So, from (iii), we get

$$2I = I - \frac{\pi}{2} \log 2 \Rightarrow I = -\frac{\pi}{2} \log 2$$

Hence, $\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2$

EXAMPLE 3 *Prove that* $\int_0^{\pi/2} \log \; | \tan x + \cot x | \; dx = \pi \log_e 2$

SOLUTION Let $I = \int_0^{\pi/2} \log \; | \tan x + \cot x | \; dx$. Then,

$$I = \int_0^{\pi/2} \log \left| \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \right| dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \left| \frac{1}{\sin x \cos x} \right| dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \left(\frac{1}{\sin x \cos x} \right) dx \qquad \left[\because \sin x > 0, \cos x > 0 \text{ for all } x \in \left(0, \frac{\pi}{2}\right) \right]$$

$$\Rightarrow \quad I = -\int_0^{\pi/2} \log (\sin x \cos x) \, dx$$

$$\Rightarrow \quad I = -\int_0^{\pi/2} \log \sin x \, dx - \int_0^{\pi/2} \log \cos x \, dx$$

$$\Rightarrow \quad I = -\left(-\frac{\pi}{2} \log_e 2 \right) - \left(-\frac{\pi}{2} \log_e 2 \right) \qquad \text{[See Example 2]}$$

$$\Rightarrow \quad I = \pi \log_e 2$$

EXAMPLE 4 *Evaluate:* $\int_0^{\pi} \log(1+\cos x)\, dx$ **[NCERT]**

SOLUTION Let $I = \int_0^{\pi} \log(1+\cos x)\, dx$. ...(i)

Then,

$$\Rightarrow \quad I = \int_0^{\pi} \log\{1+\cos(\pi - x)\}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \log(1-\cos x)\, dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \{\log(1+\cos x) + \log(1-\cos x)\}\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi} \log(1-\cos^2 x)\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi} \log \sin^2 x\, dx$$

$$\Rightarrow \quad 2I = 2\int_0^{\pi} \log \sin x\, dx$$

$$\Rightarrow \quad I = 2\int_0^{\pi/2} \log \sin x\, dx \qquad \left[\because \log \sin x = \log \sin(\pi - x) \text{ and } \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx, \text{ if } f(2a-x) = f(x)\right]$$

$$\Rightarrow \quad I = 2 \times -\frac{\pi}{2} \log_e 2 \qquad \left[\because \int_0^{\pi/2} \log \sin x\, dx = -\frac{\pi}{2} \log_e 2\right]$$

$$\Rightarrow \quad I = -\pi \log_e 2$$

EXAMPLE 5 *For* $x > 0$, *let* $f(x) = \int_1^x \frac{\log_e t}{1+t}\, dt$. *Find the function* $f(x) + f\left(\frac{1}{x}\right)$ *and show that* $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$.

SOLUTION We have,

$$f(x) = \int_1^x \frac{\log_e t}{1+t}\, dt \qquad ...(i)$$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\log_e t}{1+t}\, dt.$$

Let $t = \frac{1}{u}$. Then, $dt = -\frac{1}{u^2}\, du$.

Also, $t = 1 \Rightarrow u = 1$ and $t = \frac{1}{x} \Rightarrow u = x$

$$\therefore \quad f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e(1/u)}{1+\frac{1}{u}} \times \frac{-1}{u^2}\, du$$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e u}{(1+u)\,u}\, du$$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e t}{(1+t)\,t}\, dt \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \left\{\frac{\log_e t}{1+t} + \frac{\log_e t}{(1+t)\,t}\right\} dt$$

$$\Rightarrow \quad f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e t}{1+t}\left(\frac{1+t}{t}\right) dt$$

$$\Rightarrow \quad f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e t}{t}\, dt$$

$$\Rightarrow \quad f(x) + f\left(\frac{1}{x}\right) = \int_0^{\log_e x} v\, dv, \text{ where } v = \log_e t \text{ and } \frac{1}{t}\, dt = dv$$

$$\Rightarrow \quad f(x) + f\left(\frac{1}{x}\right) = \left[\frac{v^2}{2}\right]_0^{\log_e x} = \frac{1}{2}\left(\log_e x\right)^2$$

$$\therefore \quad f(e) + f\left(\frac{1}{e}\right) = \frac{(\log_e e)^2}{2} = \frac{1}{2}$$

EXAMPLE 6 *Show that:*

$$\int_0^{\pi/2} f(\sin 2x)\sin x\, dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx$$

SOLUTION Let $I = \int_0^{\pi/2} f(\sin 2x)\sin x\, dx$ ...(i)

Then, $I = \int_0^{\pi/2} f\left\{\sin 2\left(\frac{\pi}{2} - x\right)\right\}\sin\left(\frac{\pi}{2} - x\right) dx$ [Using Prop. IV]

$$\Rightarrow \quad I = \int_0^{\pi/2} f\{\sin(\pi - 2x)\}\cos x\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} f\{\sin 2x\}\cos x\, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\sin 2x) \,.\, (\sin x + \cos x)\, dx$$

$$\Rightarrow \quad 2I = 2\int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)\, dx \qquad \text{[Using Prop. VII]}$$

$$\Rightarrow \quad 2I = 2\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) dx$$

$$\Rightarrow \quad 2I = 2\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin\left(x + \frac{\pi}{4}\right) dx$$

$$\Rightarrow \quad 2I = 2\sqrt{2}\int_0^{\pi/4} f\left\{\sin 2\left(\frac{\pi}{4} - x\right)\right\}\sin\left(\frac{\pi}{4} - x + \frac{\pi}{4}\right) dx \qquad \text{[Using Prop. IV]}$$

$$\Rightarrow \quad 2I = 2\sqrt{2}\int_0^{\pi/4} f\left\{\sin\left(\frac{\pi}{2} - 2x\right)\right\}\sin\left(\frac{\pi}{2} - x\right) dx$$

$$\Rightarrow \quad 2I = 2\sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx$$

$$\therefore \quad I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx$$

Hence, $\displaystyle\int_0^{\pi/2} f(\sin 2x)\sin x\, dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx.$

EXAMPLE 7 *Prove that* : $\displaystyle\int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx = \pi^2$

SOLUTION Let $\displaystyle I = \int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$ ...(i)

Then, $\displaystyle I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}(2\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)}\, dx$

$$\Rightarrow \quad I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} + \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$$

$$\Rightarrow \quad 2I = \int_0^{2\pi} \frac{2\pi\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$$

$$\Rightarrow \quad I = \pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$$

$$\Rightarrow \quad I = 2\pi \int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx \qquad \text{[Using Prop. VII]}$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx \qquad \text{[Using Prop. VII]} \qquad \text{... (iii)}$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n} (\pi/2 - x)}{\sin^{2n} (\pi/2 - x) + \cos^{2n} (\pi/2 - x)}\, dx$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x}\, dx \qquad \text{...(iv)}$$

Adding (iii) and (iv), we get

$$2I = 4\pi \int_0^{\pi/2} 1 \,.\, dx = 4\pi \times \frac{\pi}{2} \Rightarrow I = \pi^2$$

EXERCISE 20.3

Evaluate the following integrals:

1. $\int_1^4 f(x)\, dx$, where $f(x) = \begin{cases} 4x+3, & \text{if } 1 \le x \le 2 \\ 3x+5, & \text{if } 2 \le x \le 4 \end{cases}$

2. $\int_0^9 f(x)\, dx$, where $f(x) \begin{cases} \sin x, & 0 \le x \le \pi/2 \\ 1, & \pi/2 \le x \le 3 \\ e^{x-3}, & 3 \le x \le 9 \end{cases}$

3. $\int_1^4 f(x)\, dx$, where $f(x) = \begin{cases} 7x+3, & \text{if } 1 \le x \le 3 \\ 8x, & \text{if } 3 \le x \le 4 \end{cases}$

4. $\int_{-4}^4 |x+2|\, dx$

5. $\int_{-3}^3 |x+1|\, dx$

6. $\int_{-1}^1 |2x+1|\, dx$

7. (i) $\int_{-2}^2 |2x+3|\, dx$ (ii) $\int_0^2 |x^2 - 3x + 2|\, dx$

8. $\int_0^3 |3x-1|\, dx$

9. $\int_{-6}^6 |x+2|\, dx$

10. $\int_{-2}^2 |x+1|\, dx$

11. $\int_1^2 |x-3|\, dx$

12. $\int_0^{\pi/2} |\cos 2x|\, dx$

13. $\int_0^{2\pi} |\sin x|\, dx$

14. $\int_{-\pi/4}^{\pi/4} |\sin x|\, dx$

15. $\int_0^{\pi/2} \frac{dx}{1+\tan x}$

16. $\int_0^{\pi/2} \frac{1}{1+\cot x}\, dx$

17. $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}\, dx$

18. $\int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x+\cos^{3/2} x}\, dx$ [NCERT]

19. $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}\, dx$ [CBSE 2004]

20. $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\, dx$

21. $\int_0^{a} \frac{1}{x+\sqrt{a^2-x^2}}\, dx$

22. $\int_0^{\infty} \frac{\log x}{1+x^2}\, dx$

23. $\int_0^{1} \frac{\log(1+x)}{1+x^2}\, dx$ [CBSE 2008]

24. $\int_0^{\infty} \frac{x}{(1+x)(1+x^2)}\, dx$

25. $\int_0^{\pi} \frac{x\tan x}{\sec x \operatorname{cosec} x}\, dx$ [CBSE 2001, 2007]

26. $\int_0^{\pi} x\sin x\cos^4 x\, dx$

27. $\int_0^{\pi} x\sin^3 x\, dx$

28. $\int_0^{\pi} x\log\sin x\, dx$

29. $\int_0^{\pi} \frac{x\sin x}{1+\sin x}\, dx$ [NCERT]

30. $\int_0^{\pi} \frac{x}{1+\cos\alpha\sin x}\, dx, \quad 0<\alpha<\pi$

31. $\int_0^{\pi} x\cos^2 x\, dx$

32. $\int_0^{\pi/2} (2\log\cos x-\log\sin 2x)\, dx$

33. $\int_0^{\pi} \sin^2 x\cos^3 x\, dx$

34. $\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\, dx$

35. $\int_{-\pi/2}^{\pi/2} \sin^3 x\, dx$

36. $\int_{-\pi/2}^{\pi/2} \sin^4 x\, dx$

37. $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx$

38. $\int_{-\pi/4}^{\pi/4} \sin^2 x\, dx$

39. $\int_0^{\pi} \log(1-\cos x)\, dx$

40. $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin x}{2+\sin x}\right) dx$

41. $\int_{0}^{a} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\, dx$ **[NCERT]**

42. $\int_{0}^{5} \frac{\sqrt[4]{x+4}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}}\, dx$

43. $\int_{0}^{7} \frac{\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[3]{7-x}}\, dx$

44. $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}}\, dx$ **[CBSE 2007]**

45. $\int_{a}^{b} \frac{f(x)}{f(a+b-x)+f(x)}\, dx$

46. $\int_{2}^{8} |x-5|\, dx$

47. $\int_{-\pi/2}^{\pi/2} \{\sin|x| + \cos|x|\}\, dx$ **[CBSE 2000]**

48. $\int_{0}^{2} x\sqrt{2-x}\, dx$ **[NCERT, CBSE 2007]**

49. $\int_{-5}^{0} f(x)\, dx$, where $f(x) = |x| + |x+2| + |x+5|$ **[CBSE 2005]**

50. $\int_{0}^{4} |x-1|\, dx$ **[NCERT]**

51. If f is an integrable function such that $f(2a-x) = f(x)$, then prove that

$$\int_{0}^{2a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$$

52. If $f(2a-x) = -f(x)$, prove that $\int_{0}^{2a} f(x)\, dx = 0$

53. If f is an integrable function, show that

(i) $\int_{-a}^{a} f(x^2)\, dx = 2\int_{0}^{a} f(x^2)\, dx$ (ii) $\int_{-a}^{a} x f(x^2)\, dx = 0$

54. If $f(x)$ is a continuous function defined on $[0, 2a]$. Then, prove that

$$\int_{0}^{2a} f(x)\, dx = \int_{0}^{a} \{f(x) + f(2a-x)\}\, dx$$

55. If $f(x)$ is a continuous function defined on $[-a, a]$, then prove that

$$\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} \{f(x) + f(-x)\}\, dx$$

ANSWERS

1. 37	2. $3-\frac{\pi}{2}+e^6$	3. 62	4. 20
5. 10	6. $\frac{5}{2}$	7. (i) $\frac{25}{2}$	(ii) 1
8. $\frac{65}{6}$	9. 40	10. 5	11. $\frac{3}{2}$
12. 1	13. 4	14. $(2-\sqrt{2})$	15. $\frac{\pi}{4}$

16. $\frac{\pi}{4}$ 17. $\frac{\pi}{4}$ 18. $\frac{\pi}{4}$ 19. $\frac{\pi}{4}$

20. $\frac{\pi}{4}$ 21. $\frac{\pi}{4}$ 22. 0 23. $\frac{\pi}{8}\log 2$

24. $\frac{\pi}{4}$ 25. $\frac{\pi^2}{4}$ 26. $\frac{\pi}{5}$ 27. $\frac{2\pi}{3}$

28. $-\frac{\pi^2}{2}\log 2$ 29. $\pi\left(\frac{\pi}{2}-1\right)$ 30. $\frac{\pi\alpha}{\sin\alpha}$ 31. $\frac{\pi^2}{4}$

32. $-\frac{\pi}{2}\log 2$ 33. 0 34. $\frac{\pi^2}{16}$ 35. 0

36. $\frac{3\pi}{8}$ 37. 0 38. $\frac{\pi}{4}-\frac{1}{2}$ 39. $-\pi\log 2$

40. 0 41. $\frac{9}{2}$ 42. $\frac{5}{2}$ 43. $\frac{7}{2}$

44. $\frac{\pi}{12}$ 45. $\frac{(b-a)}{2}$ 46. 9 47. 4

48. $\frac{16\sqrt{2}}{15}$ 49. $\frac{63}{2}$ 50. 5

HINTS TO SELECTED PROBLEMS

4. $\int_{-4}^{4} |x+2|\, dx = \int_{-4}^{-2} -(x+2)\, dx + \int_{-2}^{4} (x+2)\, dx$

5. $\int_{-3}^{3} |x+1|\, dx = \int_{-3}^{-1} -(x+1)\, dx + \int_{-1}^{3} (x+1)\, dx$

6. $\int_{-1}^{1} |2x+1|\, dx = \int_{-1}^{-1/2} -(2x+1)\, dx + \int_{-1/2}^{1} (2x+1)\, dx$

7. (i) $\int_{-2}^{2} |2x+3|\, dx = \int_{-2}^{-3/2} -(2x+3)\, dx + \int_{-3/2}^{2} (2x+3)\, dx$

8. $\int_{0}^{3} |3x-1|\, dx = \int_{0}^{1/3} -(3x-1)\, dx + \int_{1/3}^{3} (3x-1)\, dx$

9. $\int_{-6}^{6} |x+2|\, dx = \int_{-6}^{-2} -(x+2)\, dx + \int_{-2}^{6} (x+2)\, dx$

10. $\int_{-2}^{2} |x+1|\, dx = \int_{-2}^{-1} -(x+1)\, dx + \int_{-1}^{2} (x+1)\, dx$

11. $\int_{1}^{2} |x-3|\, dx = \int_{1}^{2} -(x-3)\, dx$ $\quad [\because\ x-3<0 \text{ for } 1 \le x \le 2]$

12. $\int_{0}^{\pi/2} |\cos 2x|\, dx = \int_{0}^{\pi/4} \cos 2x\, dx + \int_{\pi/4}^{\pi/2} (-\cos 2x)\, dx$

13. $\int_0^{2\pi} |\sin x|\, dx = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} (-\sin x)\, dx$

14. $\int_{-\pi/4}^{\pi/4} |\sin x|\, dx = \int_{-\pi/4}^{0} -\sin x\, dx + \int_0^{\pi/4} \sin x\, dx$

21. Put $x = a \sin\theta$ to get $I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta + \cos\theta}\, d\theta$

22. Put $x = \tan\theta$ to get $I = \int_0^{\pi/2} \log \tan\theta\, d\theta$

23. Put $x = \tan\theta$ to get $I = \int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta$

24. Put $x = \tan\theta$ to get $I = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta}\, d\theta$

26. Let $I = \int_0^{\pi} x \sin x \cos^4 x\, dx$. Then,

$$I = \int_0^{\pi} (\pi - x) \sin(\pi - x) \cos^4(\pi - x)\, dx = \int_0^{\pi} (\pi - x) \sin x \cos^4 x\, dx$$

$\therefore \quad 2I = \int_0^{\pi} \pi \cos^4 x \sin x\, dx$. Now put $\cos x = t$

27. Let $I = \int_0^{\pi} x \sin^3 x\, dx$ Then, $I = \int_0^{\pi} (\pi - x) \sin^3(\pi - x)\, dx \;\therefore\; 2I = \pi \int_0^{\pi} \sin^3 x\, dx$

29. $I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx \Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \sin(\pi - x)}\, dx \;\therefore\; 2I = \int_0^{\pi} \frac{\pi \sin x}{1 + \sin x}\, dx$

33. Using $\int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx$, we get $I = -I \Rightarrow 2I = 0 \Rightarrow I = 0$

34. We have,

$$I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx \qquad \text{...(i)}$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x \sin x}{\cos^4 x + \sin^4 x}\, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we have

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_0^1 \frac{1}{(1-t)^2 + t^2}\,dt \text{, where } t = \sin^2 x$$

$$\Rightarrow \quad 2I = \frac{\pi}{8}\int_0^1 \frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\,dt$$

$$\Rightarrow \quad 2I = \frac{\pi}{8}\times 2\left[\tan^{-1}(2t-1)\right]_0^1 \Rightarrow I = \frac{\pi}{8}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi^2}{16}$$

36. $I = 2\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} (1-\cos 2x)\,dx$

38. $I = 2\int_0^{\pi/4} \sin^2 x\,dx = \int_0^{\pi/4} (1-\cos 2x)\,dx$

39. $I = \int_0^{\pi} \log(1-\cos x)\,dx = \int_0^{\pi} \log\left(2\sin^2\frac{x}{2}\right)dx$

$$\Rightarrow \quad I = \int_0^{\pi} \log 2\,dx + 2\int_0^{\pi} \log\sin\frac{x}{2}\,dx$$

$$\Rightarrow \quad I = \pi\log 2 + 4\int_0^{\pi/2} \log\sin t\,dt \text{, where } t = \frac{x}{2}$$

$$\Rightarrow \quad I = \pi\log 2 + 4\times -\frac{\pi}{2}\log 2 \qquad \left[\because \int_0^{\pi/2} \log\sin t\,dt = -\frac{\pi}{2}\log 2\right]$$

$$\Rightarrow \quad I = \pi\log 2 - 2\pi\log 2 = -\pi\log 2$$

44. Use $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ **47.** Use $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

46. Use $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

49. We have,

$$I = \int_{-\pi/2}^{\pi/2} \{\sin|x| + \cos|x|\}\,dx$$

$$\Rightarrow \quad I = 2\int_0^{\pi/2} (\sin x + \cos x)\,dx \qquad [\because \sin|x| + \cos|x| \text{ is an even function}]$$

$$\Rightarrow \quad I = 2\times 2 = 4$$

20.4 INTEGRATION AS THE LIMIT OF A SUM

In this section, we shall consider integration as the limit of the sum of certain number of terms when the number of terms tends to infinity and each term tends to zero. As a matter of fact the summation aspect of definite integral is more fundamental and it was invented far before the differentiation was known.

Let $f(x)$ be a continuous real valued function defined on the closed interval $[a, b]$ which is divided into n equal parts each of width h by inserting $(n-1)$ points $a+h$, $a+2h, a+3h, \ldots, a+(n-1)h$ between a and b as shown in Fig. 20.13. Then,

$$nh = b - a \text{ or, } h = \frac{b-a}{n}$$

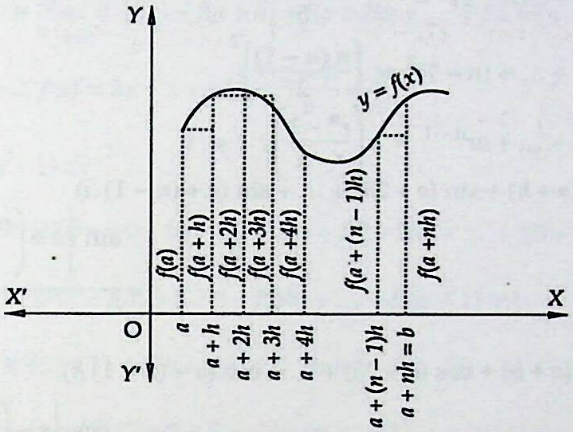

Fig. 20.13

Let S_n denote the sum of the areas of n rectangles shown in Fig. 20.13. Then,

$$S_n = h \cdot f(a) + h \cdot f(a+h) + h \cdot f(a+2h) + \ldots + h \cdot f(a+(n-1)h)$$

$$\Rightarrow \quad S_n = h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)]$$

Clearly, S_n denotes the area which is close to the area of the region bounded the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$. It is evident that if n increases, the number of rectangles will increase and the width of rectangles will decrease. Consequently, S_n gives closer approximation to the area enclosed by the curve $y = f(x)$, x-axis and the ordinates $x = a, x = b$.

Thus, $\lim\limits_{n \to \infty} S_n$ gives the area of the region bounded by the four curves $y = f(x)$, $y = 0$ (x-axis), $x = a$ and $x = b$. It can be proved that this limit exists for all continuous functions defined on closed integral $[a, b]$ and is defined as the definite integral of $f(x)$ over $[a, b]$.

$$\therefore \quad \int_a^b f(x)\,dx = \lim_{n \to \infty} S_n$$

$$\text{or,} \quad \int_a^b f(x)\,dx = \lim_{n \to \infty} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)\right], \text{ where } h = \frac{b-a}{n}$$

$$\text{or,} \quad \int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)\right] \quad \ldots\text{(i)}$$

$$[\because n \to \infty \Leftrightarrow h \to 0]$$

The process of evaluating a definite integral by using the above definition is called *integration from first principles* or *integration by ab-initio method* or *integration as the limit of a sum.*

REMARK *In finding S_n in the above article, we have taken the left end points of the subintervals. We can also take the right end-points of the subintervals throughout to obtain:*

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\,[f(a+h) + f(a+2h) + f(a+3h) + \dots + f(a+nh)],\ h = \frac{b-a}{n}$$

It can be proved that this formula and the formula (i) give the same limit.

Following results will be helpful in evaluating definite integrals as limit of sums.

(i) $1 + 2 + 3 + \dots + (n-1) = \dfrac{n(n-1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \dots (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \dots + (n-1)^3 = \left\{\dfrac{n(n-1)}{2}\right\}^2$

(iv) $a + ar + ar^2 + \dots + ar^{n-1} = a\left(\dfrac{r^n - 1}{r-1}\right),\ r \neq 1$

(v) $\sin a + \sin(a+h) + \sin(a+2h) + \dots + \sin(a+(n-1)h)$

$$= \frac{\sin\left\{a + \left(\frac{n-1}{2}\right)h\right\} \sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}$$

(vi) $\cos a + \cos(a+h) + \cos(a+2h) + \dots + \cos(a+(n-1)h)$

$$= \frac{\cos\left\{a + \left(\frac{n-1}{2}\right)h\right\} \sin\left(\frac{nh}{2}\right)}{\sin\left(\frac{h}{2}\right)}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate the following integrals as limit of sums:*

(i) $\displaystyle\int_0^2 (x+4)\,dx$ (ii) $\displaystyle\int_0^2 (2x+1)\,dx$

SOLUTION (i) We have,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\,[f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)],\ \text{where } h = \frac{b-a}{n}$$

Here, $a = 0, b = 2, f(x) = x + 4$ and $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

$\therefore$ $$I = \int_0^2 (x+4)\,dx$$

$\Rightarrow$ $$I = \lim_{h \to 0} h\,[f(0) + f(0+h) + f(0+2h) + \dots + f(0+(n-1)h)]$$

$\Rightarrow$ $$I = \lim_{h \to 0} h\,[(0+4) + (h+4) + (2h+4) + \dots + ((n-1)h + 4)]$$

$\Rightarrow$ $$I = \lim_{h \to 0} h\,[4n + h\,(1 + 2 + 3 + \dots + (n-1))]$$

$\Rightarrow \quad I = \lim_{h \to 0} h \left[4n + h \dfrac{n(n-1)}{2}\right]$

$\Rightarrow \quad I = \lim_{n \to \infty} \dfrac{2}{n} \left[4n + \dfrac{2}{n} \cdot \dfrac{n(n-1)}{2}\right] \qquad \left[\because h = \dfrac{2}{n} \text{ and } h \to 0 \Rightarrow n \to \infty\right]$

$\Rightarrow \quad I = \lim_{n \to \infty} \left[8 + 2\left(1 - \dfrac{1}{n}\right)\right] = 8 + 2(1 - 0) = 10$

(ii) We have,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a + (n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here $\quad a = 0, b = 2, f(x) = 2x + 1$ and $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

$\therefore \quad I = \displaystyle\int_0^2 (2x+1)\,dx$

$\Rightarrow \quad I = \lim_{h \to 0} h\,[f(0) + f(0+h) + f(0+2h) + f(0+3h) + \ldots + f(0 + (n-1)h)]$

$\Rightarrow \quad I = \lim_{h \to 0} h\,[f(0) + f(h) + f(2h) + f(3h) + \ldots + f((n-1)h)]$

$\Rightarrow \quad I = \lim_{h \to 0} h\,[(0+1) + (2h+1) + (2 \cdot 2h + 1) + (2 \cdot 3h + 1) + \ldots + (2(n-1)h + 1)]$

$\Rightarrow \quad I = \lim_{h \to 0} h\,[n + 2h(1 + 2 + 3 + \ldots + (n-1))]$

$\Rightarrow \quad I = \lim_{h \to 0} h \left[n + 2h \cdot \dfrac{n(n-1)}{2}\right]$

$\Rightarrow \quad I = \lim_{n \to 0} h\,[n + nh(n-1)]$

$\Rightarrow \quad I = \lim_{n \to \infty} \dfrac{2}{n} \left[n + n \cdot \dfrac{2}{n}(n-1)\right] \qquad \left[\because h = \dfrac{2}{n} \text{ and } h \to 0 \Rightarrow n \to \infty\right]$

$\Rightarrow \quad I = \lim_{n \to 0} \left[2 + 4\left(\dfrac{n-1}{n}\right)\right] = \lim_{n \to \infty} \left[2 + 4\left(1 - \dfrac{1}{n}\right)\right] = 2 + 4 = 6$

EXAMPLE 2 *Evaluate the following integrals as limit of sums:*

(i) $\displaystyle\int_1^3 (2x+1)\,dx$ (ii) $\displaystyle\int_2^4 (2x-1)\,dx$

SOLUTION (i) We have,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a) + (n-h)], \text{ where } h = \frac{b-a}{n}$$

Here, $\quad a = 1, b = 3, f(x) = 2x + 1$ and $h = \dfrac{3-1}{n} = \dfrac{2}{n}$

$\therefore \quad I = \displaystyle\int_1^3 (2x+1)\,dx$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\ [f(1) + f(1+h) + f(1+2h) + \ldots + f(1+(n-1)\,h)]$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\ [(2 \times 1 + 1) + \{2\,(1+h)+1\} + \{2\,(1+2h)+1\} + \ldots + \{2\,(1+(n-1)\,h+1\}]$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\ [3 + (3+2h) + (3+2\times 2h) + (3+2\times 3h) + \ldots + (3+2\,(n-1)\,h)]$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\ \{3n + 2h\ (1+2+3+\ldots+(n-1)\}$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\ \left\{3n + 2h \cdot \frac{n\,(n-1)}{2}\right\}$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\ \{3n + hn\,(n-1)\}$

$\Rightarrow \quad I = \lim\limits_{n \to \infty} \frac{2}{n} \left\{3n + \frac{2}{n} \times n\,(n-1)\right\} \qquad \left[\because h = \frac{2}{n} \text{ and } h \to 0 \Rightarrow n \to \infty\right]$

$\Rightarrow \quad I = \lim\limits_{n \to \infty} \left\{6 + 4\left(\frac{n-1}{n}\right)\right\} = \lim\limits_{n \to \infty} \left\{6 + 4\left(1 - \frac{1}{n}\right)\right\} = 6 + 4\ (1-0) = 10$

(ii) We have

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)\,h)], \text{ where } h = \frac{b-a}{n}$$

Here, $\quad a = 2, b = 4, f(x) = 2x - 1$ and $h = \frac{4-2}{n} = \frac{2}{n}$

$\therefore \quad I = \int_2^4 f(x)\,dx = \lim\limits_{h \to 0} h\ [f(2) + f(2+h) + f(2+2h) + \ldots + f(2+(n-1)\,h)]$

$\Rightarrow \quad I = \int_2^4 (2x-1)\,dx$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\ [\{2\,(2)-1)\} + \{2\,(2+h)-1\} + \{2\,(2+2h)-1\} + \ldots + \{2\,(2+(n-1)\,h)-1\}]$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\ [3 + (3+2h) + (3+4h) + (3+6h) + \ldots + (3+2\,(n-1)h)]$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\ [3n + 2h\,(1+2+3+\ldots+(n-1)]$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h \left\{3n + 2h \times \frac{n(n-1)}{2}\right\}$

$\Rightarrow \quad I = \lim\limits_{n \to \infty} \frac{2}{n} \left\{3n + 2 \times \frac{2}{n} \times \frac{n(n-1)}{2}\right\}$

$\Rightarrow \quad I = \lim\limits_{n \to \infty} \left\{6 + 4\left(\frac{n-1}{n}\right)\right\} = \lim\limits_{n \to \infty} \left\{6 + 4\left(1 - \frac{1}{n}\right)\right\} = 6 + 4\,(1-0) = 10$

EXAMPLE 3 *Evaluate the following integrals as limit of sums:*

(i) $\int_0^2 (x^2+3)\,dx$ **[CBSE 2001C]** (ii) $\int_1^3 (2x^2+5)\,dx$ **[CBSE 2010]**

SOLUTION (i) We have,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\ [f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)\,h], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0, b = 2, f(x) = x^2 + 3$ and $h = \frac{2-0}{n} = \frac{2}{n}$

$$\therefore \quad I = \int_0^2 (x^2+3)\,dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[f(0) + f(0+h) + f(0+2h) + \ldots + f(0+(n-1)h)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[f(0) + f(h) + f(2h) + \ldots + f(n-1)\,h)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[(0+3) + (h^2+3) + (2^2h^2+3) + (3^2h^2+3) + \ldots + ((n-1)^2h^2+3)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[3n + h^2\,(1^2+2^2+\ldots+(n-1)^2)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} \left\{3n + h^2\,\frac{n\,(n-1)\,(2n-1)}{6}\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \frac{2}{n}\left\{3n + \frac{4}{n^2}\cdot\frac{n\,(n-1)\,(2n-1)}{6}\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \left\{6 + \frac{8}{6}\,\frac{(n-1)\;(2n-1)}{n^2}\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \left\{6 + \frac{8}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right\} = 6 + \frac{8}{6}\,(1-0)\,(2-0) = 6 + \frac{8}{3} = \frac{26}{3}$$

(ii) We have,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\;[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)\,h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 1, b = 3, f(x) = 2x^2 + 5$ and $h = \frac{3-1}{n} = \frac{2}{n}$

$$\therefore \quad I = \int_1^3 (2x^2+5)\,dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[f(1) + f(1+h) + f(1+2h) + \ldots + f(1+(n-1)h)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\{2\,(1)^2+5\} + \{2\,(1+h)^2+5\} + \{2\,(1+2h)^2+5\} + \ldots + \{2\,(1+(n-1)h^2)+5\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[2\,\{1^2 + (1+h)^2 + (1+2h)^2 + (1+3h)^2 + \ldots + (1+(n-1)h)^2\} + 5n\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[2\,\{n + 2h\,(1+2+3+\ldots+(n-1)) + h^2\,(1^2+2^2+\ldots+(n-1)^2)\} + 5n\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[2\left\{n + 2h\cdot\frac{n\,(n-1)}{2} + h^2\,\frac{n(n-1)\,(2n-1)}{6}\right\} + 5n\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left\{2n + 2h\,n(n-1) + 2h^2\,\frac{n\,(n-1)\,(2n-1)}{6} + 5n\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \frac{2}{n}\left\{7n + 2\times\frac{2}{n}\,n\,(n-1) + 2\left(\frac{4}{n^2}\right)\frac{n\,(n-1)\,(2n-1)}{6}\right\} \qquad \left[\because\ h = \frac{2}{n}\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty}\left[14 + 8\left(\frac{n-1}{n}\right) + \frac{8}{3}\frac{(n-1)(2n-1)}{n^2}\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty}\left[14 + 8\left(1 - \frac{1}{n}\right) + \frac{8}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$\Rightarrow \quad I = 14 + 8(1-0) + \frac{8}{3}(1-0)(2-0) = 14 + 8 + \frac{16}{3} = \frac{82}{3}$$

EXAMPLE 4 *Evaluate the following integrals as limit of sums:*

(i) $\int_1^3 (x^2 + x)\, dx$ **[CBSE 2000C]** (ii) $\int_1^3 (x^2 + 5x)\, dx$ **[CBSE 2010]**

(iii) $\int_a^b x^2\, dx$

SOLUTION (i) We have,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h\, [f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 1, b = 3, f(x) = x^2 + x$ and $h = \frac{3-1}{n} = \frac{2}{n}$

$$\therefore \quad I = \int_1^3 (x^2 + x)\, dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[f(1) + f(1+h) + f(1+2h) + f(1+3h) + \ldots + f(1+(n-1)h)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\{1^2 + 1\} + \{(1+h)^2 + (1+h)\} + \{(1+2h)^2 + (1+2h)\} + \ldots + \{(1+(n-1)h)^2 + (1+(n-1)h)\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\{1^2 + (1+h)^2 + (1+2h)^2 + \ldots + (1+(n-1)h^2\} + \{1 + (1+h) + (1+2h) + \ldots + (1+(n-1)h)\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\{n + 2h(1+2+3+\ldots+(n-1)) + h^2(1^2+2^2+\ldots+(n-1)^2)\} + \{n + h(1+2+3+\ldots+(n-1))\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[n + 2h \times \frac{n(n-1)}{2} + h^2 \times \frac{n(n-1)(2n-1)}{6} + n + h \times \frac{n(n-1)}{2}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[2n + 3h \times \frac{n(n-1)}{2} + h^2 \times \frac{n(n-1)(2n-1)}{6}\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \frac{2}{n}\left[2n + \frac{6}{n} \times \frac{n(n-1)}{2} + \frac{4}{n^2} \times \frac{n(n-1)(2n-1)}{6}\right] \qquad \left[\because h = \frac{2}{n}\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty}\left[4 + 6\left(\frac{n-1}{n}\right) + \frac{4}{3}\frac{(n-1)(2n-1)}{n^2}\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty}\left[4 + 6\left(1 - \frac{1}{n}\right) + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$\Rightarrow \quad I = 4 + 6(1-0) + \frac{4}{3}(1-0)(2-0) = 4 + 6 + \frac{8}{3} = \frac{38}{3}$$

(ii) $\int_a^b f(x)\,dx = \lim_{h \to 0} h\ [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h]$, where $h = \frac{b-a}{n}$

Here $a = 1, b = 3, f(x) = x^2 + 5x$ and $h = \frac{3-1}{n} = \frac{2}{n}$

$\therefore$ $I = \int_1^3 (x^2 + 5x)\,dx$

$\Rightarrow$ $I = \lim_{h \to 0} h\left[f(1) + f(1+h) + f(1+2h) + \dots + f(1+(n-1)h)\right]$

$\Rightarrow$ $I = \lim_{h \to 0} h\left[\{(1^2 + 5 \times 1)\} + \{(1+h)^2 + 5(1+h)\} + \{(1+2h)^2 + 5(1+2h)\} + \dots + \{(1+(n-1)h)^2 + 5(1+(n-1)h)\}\right]$

$\Rightarrow$ $I = \lim_{h \to 0} h\left[\{(1^2 + (1+h)^2 + (1+2h)^2 + \dots + (1+(n-1)h)^2\} + 5\{1 + (1+h) + (1+2h) + \dots + (1+(n-1)h)\}\right]$

$\Rightarrow$ $I = \lim_{h \to 0} h\left[\{n + 2h(1+2+3+\dots+(n-1) + h^2(1^2 + 2^2 + \dots + (n-1)^2)\} + 5\{n + h(1+2+\dots+(n-1))\}\right]$

$\Rightarrow$ $I = \lim_{h \to 0} h\left[6n + 7h(1+2+3+\dots+(n-1) + h^2(1^2 + 2^2 + \dots + \dots + (n-1)^2\right]$

$\Rightarrow$ $I = \lim_{h \to 0} h\left\{6n + 7h \times \frac{n(n-1)}{2} + h^2 \times \frac{n(n-1)(2n-1)}{6}\right\}$

$\Rightarrow$ $I = \lim_{n \to \infty} \frac{2}{n}\left\{6n + \frac{14}{n} \times \frac{n(n-1)}{2} + \frac{4}{n^2} \times \frac{n(n-1)(2n-1)}{6}\right\}$

$\Rightarrow$ $I = \lim_{n \to \infty}\left\{12 + 14\left(\frac{n-1}{n}\right) + \frac{8}{6}\frac{(n-1)(2n-1)}{n^2}\right\}$

$\Rightarrow$ $I = \lim_{n \to \infty}\left\{12 + 14\left(1 - \frac{1}{n}\right) + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right\} = 12 + 14 + \frac{4}{3} \times 2 = 12 + 14 + \frac{8}{3} = \frac{86}{3}$

(iii) $\int_a^b f(x)\,dx = \lim_{h \to 0} h\ [f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h]$, where $h = \frac{b-a}{n}$

Here, $f(x) = x^2$

$\therefore$ $I = \int_a^b x^2\,dx$

$\Rightarrow$ $I = \lim_{h \to 0} h\left[a^2 + (a+h)^2 + (a+2h)^2 + (a+3h)^2 + \dots + (a+(n-1)h)^2\right]$

$\Rightarrow$ $I = \lim_{h \to 0} h\left[na^2 + 2ah(1+2+3+\dots+(n-1) + h^2\left\{1^2 + 2^2 + \dots + (n-1)^2\right\}\right]$

$\Rightarrow$ $I = \lim_{h \to 0} h\left\{na^2 + 2ah \times \frac{n(n-1)}{2} + h^2 \times \frac{n(n-1)(2n-1)}{6}\right\}$

$\Rightarrow$ $I = \lim_{h \to 0}\left\{(nh)a^2 + a(nh)(nh-h) + \frac{1}{6}(nh)(nh-h)(2nh-h)\right\}$

$$\Rightarrow \quad I = \lim_{h\to 0} \left\{(b-a)\,a^2 + a\,(b-a)\,(b-a-h) + \frac{1}{6}(b-a)(b-a-h)\,(2\,(b-a)-h)\right\}$$

$$\left[\because\ h = \frac{b-a}{n} \Rightarrow nh = b-a\right]$$

$$\Rightarrow \quad I = \left\{(b-a)\,a^2 + a\,(b-a)^2 + \frac{2}{6}(b-a)^3\right\}$$

$$\Rightarrow \quad I = \frac{(b-a)}{3}\,[3a^2 + 3a\,(b-a) + (b-a)^2]$$

$$\Rightarrow \quad I = \frac{(b-a)}{3}\,(3a^2 + 3ab - 3a^2 + b^2 - 2ab + a^2)$$

$$\Rightarrow \quad I = \frac{1}{3}(b-a)\,(a^2 + ab + b^2) = \frac{1}{3}(b^3 - a^3)$$

EXAMPLE 5 *Evaluate the following integrals as limit of sums:*

(i) $\displaystyle\int_0^2 e^x\,dx$ (ii) $\displaystyle\int_{-1}^1 e^x\,dx$ **[NCERT]**

SOLUTION (i) We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)\,h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0,\ b = 2, f(x) = e^x$ and $h = \frac{2-0}{n} = \frac{2}{n}$

$$\therefore \quad I = \int_0^2 e^x\,dx$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[f(0) + f(h) + f(2h) + \ldots + f((n-1)\,h)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[e^0 + e^h + e^{2h} + \ldots + e^{(n-1)\,h}\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[e^0\left\{\frac{(e^h)^n - 1}{e^h - 1}\right\}\right] \qquad \left[\text{Using} : a + ar + \ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r-1}\right)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left\{\frac{e^{nh} - 1}{e^h - 1}\right\} = \lim_{h\to 0} \frac{h}{h}\left\{\frac{e^2 - 1}{\left(\frac{e^h - 1}{h}\right)}\right\} \qquad \left[\because\ h = \frac{2}{n} \Rightarrow nh = 2\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} \frac{e^2 - 1}{\left(\frac{e^h - 1}{h}\right)} = \frac{e^2 - 1}{1} = e^2 - 1 \qquad \left[\lim_{h\to 0} \frac{e^h - 1}{h} = 1\right]$$

(ii) $$\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)\,h)], \text{ where } h = \frac{b-a}{n}.$$

Here, $a = -1, b = 1, f(x) = e^x$ and $h = \frac{1-(-1)}{n} = \frac{2}{n}$

$$\therefore \quad I = \int_{-1}^1 e^x\,dx$$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\left[f(-1) + f(-1+h) + f(-1+2h) + \ldots + f(-1+(n-1)h)\right]$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\left[e^{-1} + e^{-1+h} + e^{-1+2h} + \ldots + e^{-1+(n-1)h}\right]$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\, e^{-1}\left[1 + e^{h} + e^{2h} + \ldots + e^{(n-1)h}\right]$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\, e^{-1}\left[\left\{\dfrac{(e^h)^n - 1}{e^h - 1}\right\}\right] \qquad \left[\text{Using: } a + ar + \ldots + ar^{n-1} = a\left(\dfrac{r^n - 1}{r - 1}\right)\right]$

$\Rightarrow \quad I = \lim\limits_{h \to 0} e^{-1}\left\{\dfrac{e^2 - 1}{\left(\dfrac{e^h - 1}{h}\right)}\right\} \qquad \left[\because\ h = \dfrac{2}{n} \Rightarrow nh = 2\right]$

$\Rightarrow \quad I = e^{-1}\left(\dfrac{e^2 - 1}{1}\right) = e - e^{-1} \qquad \left[\because\ \lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = 1\right]$

EXAMPLE 6 *Evaluate* $\int_a^b \sin x\, dx$ *as limit of sums.*

SOLUTION We have,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h\, [f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}.$$

Here, $f(x) = \sin x$

$\therefore \quad I = \int_a^b \sin x\, dx$

$\Rightarrow \quad I = \lim\limits_{h \to 0} h\left[\sin a + \sin(a+h) + \sin(a+2h) + \ldots + \sin(a+(n-1)h)\right]$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\frac{\sin\left(a + (n-1)\frac{h}{2}\right)\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right] = \lim_{h \to 0} h\left[\frac{\sin\left(a + \frac{nh}{2} - \frac{h}{2}\right)\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\frac{\sin\left(a + \frac{b-a}{2} - \frac{h}{2}\right)\sin\left(\frac{b-a}{2}\right)}{\sin\frac{h}{2}}\right] \qquad [\because\ nh = b-a]$$

$$\Rightarrow \quad I = \lim_{h \to 0}\left[\frac{\frac{h}{2}}{\sin\frac{h}{2}} \times 2\sin\left(\frac{a+b}{2} - \frac{h}{2}\right)\sin\left(\frac{b-a}{2}\right)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0}\left(\frac{\frac{h}{2}}{\sin\frac{h}{2}}\right) \times \lim_{h \to 0} 2\sin\left(\frac{a+b}{2} - \frac{h}{2}\right)\sin\left(\frac{b-a}{2}\right) = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$\Rightarrow \quad I = \cos a - \cos b \qquad [\because\ 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$

EXAMPLE 7 *Evaluate the following integrals as a limit of sums.*

(i) $\int_0^1 e^{2-3x}\,dx$ **[NCERT]** (ii) $\int_2^4 2^x\,dx$

SOLUTION (i) We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\Big[f(a)+f(a+h)+f(a+2h)+\ldots+f(a+(n-1)h)\Big], \text{ where } h=\frac{b-a}{n}$$

Here, $a=0, b=1$ and $f(x)=e^{2-3x}$

$\therefore \quad h=\frac{1}{n} \Rightarrow nh=1$

Thus, we have

$$I=\int_0^1 e^{2-3x}\,dx$$

$$\Rightarrow \quad I=\lim_{h\to 0} h\Big[f(0)+f(h)+f(2h)+\ldots+f((n-1)h)\Big]$$

$$\Rightarrow \quad I=\lim_{h\to 0} h\Big[e^2+e^{2-3h}+e^{2-3(2h)}+\ldots+e^{2-3(n-1)h}\Big]$$

$$\Rightarrow \quad I=\lim_{h\to 0} he^2\Big[1+e^{-3h}+e^{-3(2h)}+e^{-3(3h)}+\ldots+e^{-3(n-1)h}\Big]$$

$$\Rightarrow \quad I=\lim_{h\to 0} he^2\left\{\frac{(e^{-3h})^n-1}{e^{-3h}-1}\right\}$$

$$\Rightarrow \quad I=\lim_{h\to 0} he^2\left[\frac{e^{-3nh}-1}{e^{-3h}-1}\right]=e^2\lim_{h\to 0}\frac{e^{-3}-1}{\left(\frac{e^{-3h}-1}{-3h}\right)}\times-\frac{1}{3} \qquad [\because nh=1]$$

$$\Rightarrow \quad I=e^2(e^{-3}-1)\times-\frac{1}{3}=-\frac{1}{3}(e^{-1}-e^2)=\frac{1}{3}(e^2-e^{-1})$$

(ii) Here, $a=2, b=4$ and $f(x)=2^x$

$\therefore \quad h=\frac{4-2}{n} \Rightarrow nh=2$

Thus, we have

$$\int_2^4 2^x\,dx=\lim_{h\to 0} h\Big[f(2)+f(2+h)+f(2+2h)+\ldots+f\{2+(n-1)h\}\Big]$$

$$\Rightarrow \quad \int_2^4 2^x\,dx=\lim_{h\to 0} h\Big[2^2+2^{2+h}+2^{2+2h}+\ldots+2^{2+(n-1)h}\Big]$$

$$\Rightarrow \quad \int_2^4 2^x\,dx=\lim_{h\to 0} 4h\Big[1+2^h+2^{2h}+\ldots+2^{(n-1)h}\Big]$$

$$\Rightarrow \quad \int_2^4 2^x\,dx=\lim_{h\to 0} 4h\left[\frac{(2^h)^n-1}{2^h-1}\right]=4\lim_{h\to 0}\left[\frac{2^{nh}-1}{\left(\frac{2^h-1}{h}\right)}\right]$$

$$\Rightarrow \quad \int_2^4 2^x\,dx=4\times\left(\frac{2^2-1}{\log 2}\right)=\frac{12}{\log 2} \qquad \left[\because nh=2 \text{ and } \lim_{h\to 0}\frac{2^h-1}{h}=\log 2\right]$$

EXERCISE 20.4

Evaluate the following integrals as limit of sums:

1. $\int_0^3 (x+4)\,dx$
2. $\int_0^2 (x+3)\,dx$
3. $\int_1^3 (3x-2)\,dx$
4. $\int_{-1}^1 (x+3)\,dx$
5. $\int_0^5 (x+1)\,dx$
6. $\int_1^3 (2x+3)\,dx$
7. $\int_3^5 (2-x)\,dx$
8. $\int_0^2 (x^2+1)\,dx$
9. $\int_1^2 x^2\,dx$
10. $\int_2^3 (2x^2+1)\,dx$
11. $\int_1^2 (x^2-1)\,dx$
12. $\int_0^2 (x^2+4)\,dx$
13. $\int_1^4 (x^2-x)\,dx$ [CBSE 2010]
14. $\int_0^1 (3x^2+5x)\,dx$
15. $\int_0^2 e^x\,dx$ [NCERT]
16. $\int_a^b e^x\,dx$
17. $\int_a^b \cos x\,dx$
18. $\int_0^{\pi/2} \sin x\,dx$
19. $\int_0^{\pi/2} \cos x\,dx$
20. $\int_1^4 (3x^2+2x)\,dx$
21. $\int_0^2 (3x^2-2)\,dx$
22. $\int_0^2 (x^2+2)\,dx$ [CBSE 2000]
23. $\int_0^4 (x+e^{2x})\,dx$ [NCERT]
24. $\int_0^2 (x^2+x)\,dx$ [CBSE 2005]
25. $\int_0^2 (x^2+2x+1)\,dx$ [CBSE 2007]
26. $\int_0^3 (2x^2+3x+5)\,dx$ [CBSE 2007]
27. $\int_a^b x\,dx$ [NCERT]
28. $\int_0^5 (x+1)\,dx$ [NCERT]
29. $\int_2^3 x^2\,dx$ [NCERT]
30. $\int_1^4 (x^2-x)\,dx$ [NCERT]

ANSWERS

1. $\frac{33}{2}$ 2. 8 3. 8 4. 6 5. $\frac{35}{2}$ 6. 14 7. -4 8. $\frac{14}{3}$ 9. $\frac{7}{3}$ 10. $\frac{41}{3}$

11. $\frac{4}{3}$ 12. $\frac{32}{3}$ 13. $\frac{27}{2}$ 14. $\frac{7}{2}$ 15. e^2-1 16. e^b-e^a 17. $\sin b-\sin a$

18. 1 19. 1 20. 78 21. 4 22. $\frac{20}{3}$ 23. $\frac{15+e^8}{2}$ 24. $\frac{14}{3}$ 25. $\frac{26}{3}$ 26. $\frac{93}{2}$

27. $\frac{b^2-a^2}{2}$ 28. $\frac{35}{2}$ 29. $\frac{19}{3}$ 30. $\frac{27}{2}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the value of $\int_0^{\pi/2} \sin^2 x\,dx$.
2. Write the value of $\int_0^{\pi/2} \cos^2 x\,dx$.
3. Write the value of $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$.
4. Write the value of $\int_{-\pi/2}^{\pi/2} \cos^2 x\,dx$.
5. Write the value of $\int_{-\pi/2}^{\pi/2} \sin^3 x\,dx$. [CBSE 2010]
6. Write the value of $\int_{-\pi/2}^{\pi/2} x\cos^2 x\,dx$.
7. Write the value of $\int_0^{\pi/4} \tan^2 x\,dx$.
8. Write the value of $\int_0^1 \frac{1}{x^2+1}\,dx$.
9. Write the value of $\int_{-2}^1 \frac{|x|}{x}\,dx$.
10. Write the value of $\int_0^\infty e^{-x}\,dx$.
11. Write the value of $\int_0^4 \frac{1}{\sqrt{16-x^2}}\,dx$.
12. Write the value of $\int_0^3 \frac{1}{x^2+9}\,dx$.
13. Write the value of $\int_0^{\pi/2} \sqrt{1-\cos 2x}\,dx$
14. Write the value of $\int_0^{\pi/2} \log\tan x\,dx$.
15. Write the value of $\int_0^{\pi/2} \log\left(\frac{3+5\cos x}{3+5\sin x}\right)dx$.
16. Write the value of $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}\,dx$, $n \in N$.
17. Write the value of $\int_0^{\pi} \cos^5 x\,dx$.
18. Write the value of $\int_0^2 [x]\,dx$.
19. Write the value of $\int_0^{1.5} [x]\,dx$.
20. Write the value of $\int_0^1 \{x\}\,dx$, where $\{x\}$ denotes the fractional part of x.
21. Write the value of $\int_0^1 e^{\{x\}}\,dx$.
22. Write the value of $\int_0^2 x[x]\,dx$.
23. Write the value of $\int_0^1 2^{x-[x]}\,dx$.
24. Write the value of $\int_{-\pi/2}^{\pi/2} \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right)d\theta$.

25. Write the value of $\int_{-1}^{1} x\,|x|\,dx$.

26. Write the value of $\int_{1}^{2} \log_e [x]\,dx$.

27. Write the value of $\int_{a}^{b} \frac{f(x)}{f(x)+f(a+b-x)}\,dx$.

28. Write the coefficient a, b, c of which the value of the integral $\int_{-3}^{3} (ax^2+bx+c)\,dx$ is independent.

29. Write the value of the integral $\int_{0}^{\pi/4} \sin\{x\}\,dx$, where { } denotes the fractional part function.

30. Write the value of the integral $\int_{0}^{\sqrt{2}} [x^2]\,dx$.

31. Evaluate: $\int_{0}^{1} \frac{1}{1+x^2}\,dx$ [CBSE 2008]

32. If $\int_{0}^{1} (3x^2+2x+k)\,dx = 0$, find the value of k. [CBSE 2009]

ANSWERS

1. $\frac{\pi}{4}$ 2. $\frac{\pi}{4}$ 3. $\frac{\pi}{2}$ 4. $\frac{\pi}{2}$ 5. 0 6. 0 7. $1-\frac{\pi}{4}$

8. $\frac{\pi}{4}$ 9. -1 10. 1 11. $\frac{\pi}{2}$ 12. $\frac{\pi}{12}$ 13. $\sqrt{2}$ 14. 0 15. 0

16. $\frac{\pi}{4}$ 17. 0 18. 1 19. $\frac{1}{2}$ 20. $\frac{1}{2}$ 21. $e-1$ 22. $\frac{3}{2}$ 23. $\frac{1}{\log_e 2}$

24. 0 25. 0 26. 0 27. $\frac{b-a}{2}$ 28. b 29. $\frac{\sqrt{2}-1}{\sqrt{2}}$ 30. $\sqrt{2}-1$ 31. $\frac{\pi}{4}$

32. -2

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. $\int_{0}^{1} \sqrt{x(1-x)}\,dx =$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{8}$

2. $\int_{0}^{\pi} \frac{1}{1+\sin x}\,dx$ equals

(a) 0 (b) 1/2 (c) 2 (d) 3/2

3. The value of $\int_{0}^{\pi} \frac{x\tan x}{\sec x+\cos x}\,dx$ is

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{2}$ (c) $\frac{3\pi^2}{2}$ (d) $\frac{\pi^2}{3}$

4. The value of $\int_0^{2\pi} \sqrt{1+\sin\frac{x}{2}}\, dx$ is

(a) 0 (b) 2 (c) 8 (d) 4

5. The value of the integral $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx$ is

(a) 0 (b) $\pi/2$ (c) $\pi/4$ (d) none of these

6. $\int_0^{\infty} \frac{1}{1+e^x}\, dx$ equals

(a) $\log 2 - 1$ (b) $\log 2$ (c) $\log 4 - 1$ (d) $-\log 2$

7. $\int_0^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$ equals (a) 2 (b) 1 (c) $\pi/4$ (d) $\pi^2/8$

8. $\int_0^{\pi/2} \frac{\cos x}{(2+\sin x)(1+\sin x)}\, dx$ equals

(a) $\log\left(\frac{2}{3}\right)$ (b) $\log\left(\frac{3}{2}\right)$ (c) $\log\left(\frac{3}{4}\right)$ (d) $\log\left(\frac{4}{3}\right)$

9. $\int_0^{\pi/2} \frac{1}{2+\cos x}\, dx$ equals

(a) $\frac{1}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (b) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (c) $\sqrt{3}\tan^{-1}(\sqrt{3})$ (d) $2\sqrt{3}\tan^{-1}\sqrt{3}$

10. $\int_0^{\pi} \sqrt{\frac{1-x}{1+x}}\, dx =$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{2}-1$ (c) $\frac{\pi}{2}+1$ (d) $\pi+1$

11. $\int_0^{\pi} \frac{1}{a+b\cos x}\, dx =$

(a) $\frac{\pi}{\sqrt{a^2-b^2}}$ (b) $\frac{\pi}{ab}$ (c) $\frac{\pi}{a^2+b^2}$ (d) $(a+b)\pi$

12. $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\, dx$ is

(a) $\pi/3$ (b) $\pi/6$ (c) $\pi/12$ (d) $\pi/2$

13. Given that $\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}\, dx = \frac{\pi}{2(a+b)(b+c)(c+a)}$,

the value of $\int_0^{\infty} \frac{dx}{(x^2+4)(x^2+9)}$, is

(a) $\frac{\pi}{60}$ (b) $\frac{\pi}{20}$ (c) $\frac{\pi}{40}$ (d) $\frac{\pi}{80}$

14. $\int_1^e \log x \, dx =$

(a) 1 (b) $e-1$ (c) $e+1$ (d) 0

15. $\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx$ is equal to

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

16. $\int_0^3 \frac{3x+1}{x^2+9} dx =$

(a) $\frac{\pi}{12}+\log(2\sqrt{2})$ (b) $\frac{\pi}{2}+\log(2\sqrt{2})$ (c) $\frac{\pi}{6}+\log(2\sqrt{2})$ (d) $\frac{\pi}{3}+\log(2\sqrt{2})$

17. The value of the integral $\int_0^\infty \frac{x}{(1+x)(1+x^2)} dx$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) none of these

18. $\int_{-\pi/2}^{\pi/2} \sin|x| \, dx$ is equal to

(a) 1 (b) 2 (c) -1 (d) -2

19. $\int_0^{\pi/2} \frac{1}{1+\tan x} dx$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

20. The value of $\int_0^{\pi/2} \cos x \, e^{\sin x} dx$ is

(a) 1 (b) $e-1$ (c) 0 (d) -1

21. If $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$, then a equals

(a) $\frac{\pi}{2}$ (b) $\frac{1}{2}$ (c) $\frac{\pi}{4}$ (d) 1

22. If $\int_0^1 f(x)\,dx = 1, \int_0^1 x f(x)\,dx = a, \int_0^1 x^2 f(x)\,dx = a^2$, then $\int_0^1 (a-x)^2 f(x)\,dx$ equals

(a) $4a^2$ (b) 0 (c) $2a^2$ (d) none of these

23. The value of $\int_{-\pi}^{\pi} \sin^3 x \cos^2 x \, dx$ is

(a) $\frac{\pi^4}{2}$ (b) $\frac{\pi^4}{4}$ (c) 0 (d) none of these

24. $\int_{\pi/6}^{\pi/3} \frac{1}{\sin 2x} dx$ is equal to

(a) $\log_e 3$ (b) $\log_e \sqrt{3}$ (c) $\frac{1}{2}\log(-1)$ (d) $\log(-1)$

25. $\int_{-1}^{1} |1-x|\, dx$ is equal to

(a) -2 (b) 2 (c) 0 (d) 4

26. The derivative of $f(x) = \int_{x^2}^{x^3} \frac{1}{\log_e t}\, dt, (x > 0)$, is

(a) $\frac{1}{3\ln x}$ (b) $\frac{1}{3\ln x} - \frac{1}{2\ln x}$ (c) $(\ln x)^{-1} x(x-1)$ (d) $\frac{3x^2}{\ln x}$

27. If $I_{10} = \int_{0}^{\pi/2} x^{10} \sin x\, dx$, then the value of $I_{10} + 90I_8$ is

(a) $9\left(\frac{\pi}{2}\right)^9$ (b) $10\left(\frac{\pi}{2}\right)^9$ (c) $\left(\frac{\pi}{2}\right)^9$ (d) $9\left(\frac{\pi}{2}\right)^8$

28. $\int_{0}^{1} \frac{x}{(1-x)^{5/4}}\, dx =$

(a) $\frac{15}{16}$ (b) $\frac{3}{16}$ (c) $-\frac{3}{16}$ (d) $-\frac{16}{3}$

29. $\lim_{n\to\infty} \left\{\frac{1}{2n+1} + \frac{1}{2n+2} + \ldots + \frac{1}{2n+n}\right\}$ is equal to

(a) $\ln\left(\frac{1}{3}\right)$ (b) $\ln\left(\frac{2}{3}\right)$ (c) $\ln\left(\frac{3}{2}\right)$ (d) $\ln\left(\frac{4}{3}\right)$

30. The value of the integral $\int_{-2}^{2} |1-x^2|\, dx$ is

(a) 4 (b) 2 (c) -2 (d) 0

31. $\int_{0}^{\pi/2} \frac{1}{1+\cot^3 x}\, dx$ is equal to

(a) 0 (b) 1 (c) $\pi/2$ (d) $\pi/4$

32. $\int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$ equals to

(a) π (b) $\pi/2$ (c) $\pi/3$ (d) $\pi/4$

33. $\int_{0}^{1} \frac{d}{dx}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\} dx$ is equal to

(a) 0 (b) π (c) $\pi/2$ (d) $\pi/4$

34. $\int_{0}^{\pi/2} x \sin x\, dx$ is equal to

(a) $\pi/4$ (b) $\pi/2$ (c) π (d) 1

35. $\int_0^{\pi/2} \sin 2x \log \tan x \, dx$ is equal to

(a) π (b) $\pi/2$ (c) 0 (d) 2π

36. The value of $\int_0^{\pi} \frac{1}{5 + 3\cos x} \, dx$ is

(a) $\pi/4$ (b) $\pi/8$ (c) $\pi/2$ (d) 0

37. $\int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{1}{1+x^2} \, dx =$

(a) $\pi \ln 2$ (b) $-\pi \ln 2$ (c) 0 (d) $-\frac{\pi}{2} \ln 2$

38. $\int_0^{2a} f(x) \, dx$ is equal to

(a) $2\int_0^a f(x) \, dx$ (b) 0

(c) $\int_0^a f(x) \, dx + \int_0^a t(2a - x) \, dx$ (d) $\int_0^a f(x) \, dx + \int_0^{2a} f(2a - x) \, dx$

39. If $f(a + b - x) = f(x)$, then $\int_a^b x f(x) \, dx$ is equal to

(a) $\frac{a+b}{2} \int_a^b f(b - x) \, dx$ (b) $\frac{a+b}{2} \int_a^b f(b + x) \, dx$

(c) $\frac{b-a}{2} \int_a^b f(x) \, dx$ (d) $\frac{a+b}{2} \int_a^b f(x) \, dx$

40. The value of $\int_0^1 \tan^{-1}\left(\frac{2x - 1}{1 + x - x^2}\right) dx$, is

(a) 1 (b) 0 (c) -1 (d) $\pi/4$

41. The value of $\int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx$ is

(a) 2 (b) $\frac{3}{4}$ (c) 0 (d) -2

42. The value of $\int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1) \, dx$, is

(a) 0 (b) 2 (c) π (d) 1

ANSWERS

1. (d)	2. (c)	3. (a)	4. (d)	5. (c)	6. (b)	7. (a)	8. (d)
9. (b)	10. (b)	11. (a)	12. (c)	13. (a)	14. (a)	15. (a)	16. (a)
17. (b)	18. (b)	19. (a)	20. (b)	21. (b)	22. (b)	23. (c)	24. (b)
25. (b)	26. (c)	27. (b)	28. (d)	29. (c)	30. (a)	31. (d)	32. (d)
33. (d)	34. (c)	35. (c)	36. (a)	37. (b)	38. (c)	39. (d)	40. (b)
41. (c)	42. (c)						

REVISION EXERCISE

Evaluate the following integrals:

1. $\int_0^4 x\sqrt{4-x}\,dx$
2. $\int_1^2 x\sqrt{3x-2}\,dx$
3. $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$
4. $\int_0^1 \cos^{-1} x\,dx$
5. $\int_0^1 \tan^{-1} x\,dx$
6. $\int_0^1 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$
7. $\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$
8. $\int_0^{1/\sqrt{3}} \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$
9. $\int_0^1 \frac{1-x}{1+x}\,dx$
10. $\int_0^{\pi/3} \frac{\cos x}{3+4\sin x}\,dx$
11. $\int_0^{\pi/2} \frac{\sin^2 x}{(1+\cos x)^2}\,dx$
12. $\int_0^{\pi/2} \frac{\sin x}{\sqrt{1+\cos x}}\,dx$
13. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$
14. $\int_0^{\pi} \sin^3 x\,(1+2\cos x)\,(1+\cos x)^2\,dx$
15. $\int_0^{\infty} \frac{x}{(1+x)\,(1+x^2)}\,dx$
16. $\int_0^{\pi/4} \sin 2x \sin 3x\,dx$
17. $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$
18. $\int_1^2 \frac{1}{x^2}\,e^{-1/x}\,dx$
19. $\int_0^{\pi/4} \cos^4 x\,\sin^3 x\,dx$
20. $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx$
21. $\int_0^{\pi/2} x^2 \cos 2x\,dx$
22. $\int_0^1 \log(1+x)\,dx$
23. $\int_2^4 \frac{x^2+x}{\sqrt{2x+1}}\,dx$
24. $\int_0^1 x\,(\tan^{-1} x)^2\,dx$
25. $\int_0^1 (\cos^{-1} x)^2\,dx$
26. $\int_1^2 \frac{x+3}{x\,(x+2)}\,dx$
27. $\int_0^{\pi/4} e^x \sin x\,dx$
28. $\int_0^{\pi/4} \tan^4 x\,dx$
29. $\int_0^1 |\,2x-1\,|\,dx$
30. $\int_1^3 |\,x^2-2x\,|\,dx$

31. $\int_0^{\pi/2} |\sin x - \cos x|\, dx$ 32. $\int_0^1 |\sin 2\pi x|\, dx$ 33. $\int_1^3 |x^2 - 4|\, dx$

34. $\int_{-\pi/2}^{\pi/2} \sin^9 x\, dx$ 35. $\int_{-1/2}^{1/2} \cos x \log\left(\frac{1+x}{1-x}\right) dx$

36. $\int_{-a}^{a} \frac{x e^{x^2}}{1+x^2}\, dx$ 37. $\int_0^{\pi/2} \frac{1}{1+\cot^7 x}\, dx$ 38. $\int_0^{2\pi} \cos^7 x\, dx$

39. $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\, dx$ 40. $\int_0^{\pi/2} \frac{1}{1+\tan^3 x}\, dx$ 41. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$

42. $\int_0^{\pi} x \sin x \cos^4 x\, dx$ 43. $\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$

44. $\int_{-\pi/4}^{\pi/4} |\tan x|\, dx$ 45. $\int_0^{1.5} [x^2]\, dx$ 46. $\int_0^{\pi} \frac{x}{1+\cos\alpha \sin x}\, dx$

47. $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$ 48. $\int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x}\, dx$ 49. $\int_0^{\pi} \cos 2x \log \sin x\, dx$

50. $\int_0^{\pi} \frac{x}{a^2 - \cos^2 x}\, dx,\ a > 1$ 51. $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$ 52. $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\, dx$

53. $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\, dx$ 54. $\int_0^{\pi/2} \frac{x}{\sin^2 x + \cos^2 x}\, dx$ 55. $\int_{-\pi}^{\pi} x^{10} \sin^7 x\, dx$

56. $\int_0^1 \cot^{-1}(1 - x + x^2)\, dx$ 57. $\int_0^{\pi} \frac{dx}{6 - \cos x}$ 58. $\int_0^{\pi/2} \frac{1}{2\cos x + 4\sin x}\, dx$

59. $\int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x \cot x}{1 + \operatorname{cosec}^2 x}\, dx$ 60. $\int_0^{\pi/2} \frac{dx}{4\cos x + 2\sin x}$

Evaluate the following definite integrals as limit of sums: (61- 69)

61. $\int_0^4 x\, dx$ 62.* $\int_0^2 (2x^2 + 3)\, dx$ 63. $\int_1^4 (x^2 + x)\, dx$

64. $\int_{-1}^1 e^{2x}\, dx$ 65. $\int_2^3 e^{-x}\, dx$ 66. $\int_1^3 (2x^2 + 5x)\, dx$

67. $\int_1^3 (x^2 + 3x)\, dx$ 68. $\int_0^2 (x^2 + 2)\, dx$ 69. $\int_0^3 (x^2 + 1)\, dx$

ANSWERS

1. $\frac{128}{15}$	2. $\frac{326}{135}$	3. $\frac{16}{3}$	4. 1
5. $\frac{\pi}{4}-\frac{1}{2}\log 2$	6. $\frac{\pi}{2}-\log 2$	7. $\frac{\pi}{2}-\log 2$	8. $\frac{\pi}{2\sqrt{3}}-\frac{3}{2}\log\frac{4}{3}$
9. $2\log 2-1$	10. $\frac{1}{4}\log\left(\frac{3+2\sqrt{3}}{3}\right)$	11. $2-\frac{\pi}{2}$	12. $2(\sqrt{2}-1)$
13. $\frac{\pi}{4}$	14. $\frac{8}{3}$	15. $\frac{\pi}{4}$	16. $\frac{3}{5\sqrt{2}}$
17. $\frac{\pi}{2}-1$	18. $\frac{\sqrt{e}-1}{e}$	19. $\frac{2}{35}$	20. $\frac{3}{2}$
21. $-\frac{\pi}{4}$	22. $\log\left(\frac{4}{e}\right)$	23. $\frac{57}{5}-\sqrt{5}$	24. $\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log$
25. $\pi-2$	26. $\frac{1}{2}\log 6$	27. $\frac{1}{2}$	28. $\frac{\pi}{4}-\frac{2}{3}$
29. $\frac{1}{2}$	30. 2	31. $2(\sqrt{2}-1)$	32. $\frac{2}{\pi}$
33. 4	34. 0	35. 0	36. 0
37. $\frac{\pi}{4}$	38. 0	39. $\frac{a}{2}$	40. $\frac{\pi}{4}$
41. $\frac{\pi^2}{2}$	42. $\frac{\pi}{5}$	43. $\frac{\pi^2}{2ab}$	44. $\log 2$
45. $2-\sqrt{2}$	46. $\frac{\pi\alpha}{\sin\alpha}$	47. $\frac{\pi^2}{16}$	48. $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$
49. $-\frac{\pi}{2}$	50. $\frac{\pi^2}{2a\sqrt{a^2-1}}$	51. $\frac{\pi}{2}(\pi-2)$	52. $\frac{1}{2}$
53. $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$	54. $\frac{\pi^2}{8}$	55. 0	56. $\frac{\pi}{2}-\log 2$
57. $\frac{\pi}{\sqrt{35}}$	58. $\frac{1}{2\sqrt{5}}\log\left(\frac{\sqrt{5}+1}{2(\sqrt{5}-2)}\right)$		59. $\tan^{-1}\left(\frac{1}{3}\right)$
60. $\frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5}+1}{(\sqrt{5}-1)}\right)$	61. 8	62. $\frac{34}{3}$	63. $\frac{27}{2}$
64. $\frac{1}{2}(e^2-e^{-2})$	65. $e^{-2}-e^{-3}$	66. $\frac{112}{3}$	67. $\frac{62}{3}$
68. $\frac{20}{3}$	69. 12		

SUMMARY

1. Let $\phi(x)$ be the primitive or anti-derivative of a function $f(x)$ defined on $[a, b]$ i.e., $\frac{d}{dx}(\phi(x)) = f(x)$. Then the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)\,dx$ and is defined as $[\phi(b) - \phi(a)]$.

i.e., $$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \qquad \text{...(i)}$$

The numbers a and b are called the limits of integration, 'a' is called the lower limit and 'b' the upper limit. The interval $[a, b]$ is called the interval of integration.

2. Following are some fundamental properties of definite integrals which are very useful in evaluating integrals.

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ i.e., integration is independent of the change of variable.

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

i.e., if the limits of a definite integral are interchanged then its value changes by minus sign only.

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, *where* $a < c < b$.

The above property can be generalized into the following form

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^b f(x)\,dx$$

where $a < c_1 < c_2 < c_3 \ldots < c_{n-1} < c_n < b$.

(iv) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(v) $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{, if } f(x) \text{ is an even function} \\ 0 & \text{, if } f(x) \text{ is an odd function} \end{cases}$$

(vii) $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{, if } f(2a-x) = f(x) \\ 0 & \text{, if } f(2a-x) = -f(x) \end{cases}$$

(viii) $\int_0^{2a} f(x)\,dx = \int_0^{2a} \{f(x) + f(2a-x)\}\,dx$ (ix) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

(x) $\int_a^b f(x)\,dx = (b-a)\int_0^1 f\{(b-a)x+a\}\,dx$

3. **If $f(x)$ is a real valued continuous function defined on $[a, b]$ which is divided into n equal parts each of width h by inserting $(n-1)$ points $a+h, a+2h, \ldots, a+(n-1)h$ between a and b. Then,**

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\{(a+(n-1)h\}\right], \text{ where } h = \frac{b-a}{h}.$$

21

AREAS OF BOUNDED REGIONS

21.1 INTRODUCTION

Integration has a large number of applications in science engineering. In this chapter, we shall use integration for finding the areas of bounded regions. The first step in finding the areas of bounded regions is to identify the region whose area is to be computed. For this, we first draw the rough sketches of the various curves which enclose the region. In order to draw the rough sketches of the curves, readers are advised to go through the appendix prior to this chapter.

21.2 AREA OF BOUNDED REGIONS

THEOREM *Let $f(x)$ be a continuous function defined on $[a, b]$. Then, the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by*

$$\int_a^b f(x)\,dx \text{ or, } \int_a^b y\,dx$$

PROOF Let AD be the curve $y = f(x)$ between the ordinates BA $(x = a)$ and CD $(x = b)$. Then, the required area is the area of region $ABCD$.

Let $P(x, y)$ be any point on the curve and $Q(x + \Delta x,\ y + \Delta y)$ be a neighbouring point on it. Draw ordinates PL and QM. Then,

$PL = y$, $QM = y + \Delta y$ and $LM = \Delta x$.

Let A denote the area $BLPA$, and let $A + \Delta A$ be the area $BMQA$. Then,

ΔA = area $LMQP$.

Also, area $LMNP = y\,\Delta x$, and area $LMQR = (y + \Delta y)\,\Delta x$.

Now, Area of rectangle LMNP $\leq$ Area $LMQP \leq$ Area of rectangle $LMQR$

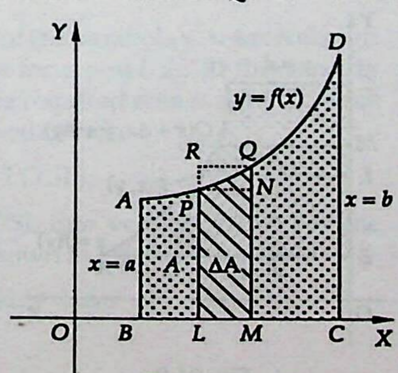

Fig. 21.1

$\Rightarrow \quad y\,\Delta x \le \Delta A \le (y + \Delta y)\,\Delta x$

$\Rightarrow \quad y \le \dfrac{\Delta A}{\Delta x} \le y + \Delta y$

$\Rightarrow \quad \lim_{\Delta x \to 0} y \le \lim_{\Delta x \to 0} \dfrac{\Delta A}{\Delta x} \le \lim_{\Delta y \to 0} y + \Delta y$

$\Rightarrow \quad y \le \dfrac{dA}{dx} \le y$

$\Rightarrow \quad \dfrac{dA}{dx} = y$

$\Rightarrow \quad \displaystyle\int_a^b \dfrac{dA}{dx}\,dx = \int_a^b y\,dx$ [Integrating between the limits a and b]

$\Rightarrow \quad \Big[A\Big]_{x=a}^{x=b} = \displaystyle\int_a^b y\,dx$

$\Rightarrow \quad$ (Area A when $x = b$) $-$ (Area A when $x = a$) $= \displaystyle\int_a^b y\,dx$

$\Rightarrow \quad$ Area $ABCD - 0 = \displaystyle\int_a^b y\,dx$ $\left[\begin{array}{r}\text{When } x = a,\ PL \text{ coincides with } AB \\ \text{So, area } ABLP = 0 \text{ when } x = a\end{array}\right]$

$\Rightarrow \quad$ Area $ABCD = \displaystyle\int_a^b y\,dx = \int_a^b f(x)\,dx$

<u>REMARK 1</u> *If the curve $y = f(x)$ lies below x-axis, then the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is negative. So, area is given by*

$$\left| \int_a^b y\,dx \right|.$$

<u>REMARK 2</u> *The area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = c$ and $y = d$ is given by*

$$\int_c^d f(y)\,dy \text{ or, } \int_c^d x\,dy$$

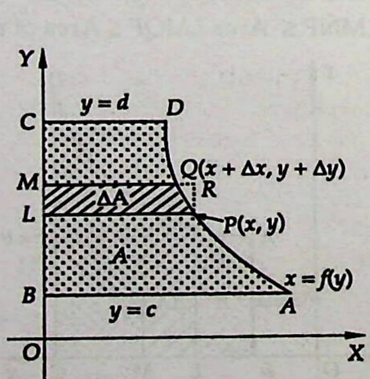

Fig. 21.2

In order to find the area of bounded regions, we may use the following algorithm.

ALGORITHM

STEP I *Make a rough sketch showing the area to be found*

STEP II *Slice the area into horizontal or vertical strips as the case may be*

STEP III *Consider a representative strip and the corresponding approximating rectangle.*

STEP IV *Find the area of the approximating rectangle. If the representative strip is parallel to y-axis, then its width is taken as Δx and if it is parallel to x-axis, then its width is taken as Δy. In Fig. 21.3, RLMQ is the approximating rectangle of area $y\Delta x$ and in Fig. 21.4, the area of the approximating rectangle RLM is $x\,\Delta y$.*

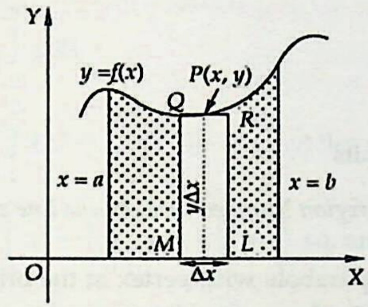

Fig. 21.3

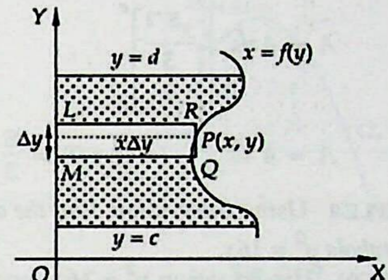

Fig. 21.4

STEP V *Find the limits within which the approximating rectangle can move.*
In Fig. 21.3, the approximating rectangle of area $y\,\Delta x$ can move between $x = a$ and $x = b$, therefore the area bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$ is given by $\int_a^b y\,dx$.

In Fig. 21.4, the approximating rectangle of area $x\,\Delta y$ can move between $y = c$ and $y = d$, therefore area bounded by $x = \phi(y)$, $x = 0$, $y = c$ and $y = d$ is given by $\int_c^d x\,dy$.

The above procedure is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.* **[NCERT]**

SOLUTION A rough sketch of the parabola $y^2 = 4ax$ is shown in Fig. 21.5. Let $S(a, 0)$ be the focus and $L\,SL'$ be the directrix of the parabola $y^2 = 4ax$. The required area is $LOL'L$. Since the curve is symmetrical about x-axis.

So, required area = 2 (Area $LOSL$).

Here, we slice the area $LOSL$ into vertical strips. For the approximating rectangle shown in Fig. 21.5, we have

$$\text{Length} = y, \text{ Width} = \Delta x$$

$$\therefore \quad \text{Area} = y\,\Delta x$$

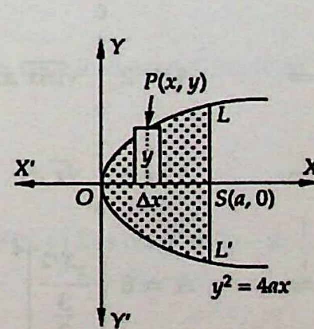

Fig. 21.5

$\Rightarrow$ Area $= \sqrt{4ax}\,\Delta x$ $\left[\because P(x, y) \text{ lies on } y^2 = 4ax \therefore y = \sqrt{4ax}\right]$

Since the approximating rectangle can move between $x = 0$ and $x = a$.

$\therefore$ Required area A is given by

$A = 2$ (Area $LOSL$)

$$\Rightarrow \quad A = 2\int_0^a \sqrt{4ax}\, dx$$

$$\Rightarrow \quad A = 4\sqrt{a}\int_0^a \sqrt{x}\, dx$$

$$\Rightarrow \quad A = 4\sqrt{a}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^a$$

$$\Rightarrow \quad A = 4\sqrt{a} \times \frac{2}{3}\left(a^{3/2} - 0\right) = \frac{8}{3}a^2 \text{ sq. units}$$

EXAMPLE 2 *Using integration, find the area of the region bounded between the line $x = 4$ and the parabola $y^2 = 16x$.*

SOLUTION The equation $y^2 = 16x$ represents a parabola with vertex at the origin and axis of symmetry along the positive direction of x-axis as shown in Fig. 21.6. Clearly, $x = 4$ is a line parallel to y-axis. The region is the shaded portion shown in Fig. 21.6. Since $y^2 = 16x$ is symmetrical about x-axis.

$\therefore$ Required area $= 2$ (Area $OCAO$).

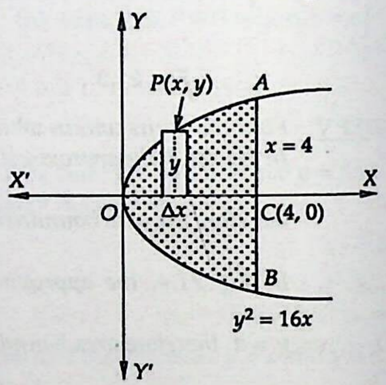

Fig. 21.6

Here, we slice the area above x-axis into vertical strips. For the approximating rectangle shown in Fig. 21.6, we have

Length $= y$, Width $= \Delta x$ and Area $= y\Delta x$.

The approximating rectangle can move between $x = 0$ and $x = 4$.

$\therefore$ Required area A is given by

$A = 2$ (Area $OCAO$)

$$\Rightarrow \quad A = 2\int_0^4 y\, dx$$

$$\Rightarrow \quad A = 2\int_0^4 \sqrt{16x}\, dx \qquad \left[\because P(x, y) \text{ lies on } y^2 = 16x \therefore y = \sqrt{16x}\right]$$

$$\Rightarrow \quad A = 8\int_0^4 \sqrt{x}\, dx$$

$$\Rightarrow \quad A = 8\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^4 = \frac{16}{3}\left(4^{3/2} - 0^{3/2}\right) = \frac{16}{3} \times 8 = \frac{128}{3} \text{ sq. units}$$

EXAMPLE 3 *Sketch the region bounded by $y = 2x - x^2$ and x-axis and find its area using integration.*

SOLUTION We have, $y = 2x - x^2$

Clearly, it represents a parabola opening downward which cuts x-axis at (0, 0) and (2, 0). The rough sketch of the curve is as shown in Fig. 21.7. The required region is the shaded region in Fig. 21.7. Here, we slice this region into vertical strips. For the approximating rectangle shown in Fig. 21.7, we have

Length $= y$, Width $= \Delta x$ and, Area $= y\,\Delta x$

The approximating rectangle can move from $x = 0$ to $x = 2$.

So, required area A is given by

$$\therefore \quad A = \int_0^2 y\,dx$$

$$\Rightarrow \quad A = \int_0^2 (2x - x^2)\,dx \quad [\because P(x, y) \text{ lies on } y = 2x - x^2]$$

$$\Rightarrow \quad A = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \text{ sq. units.}$$

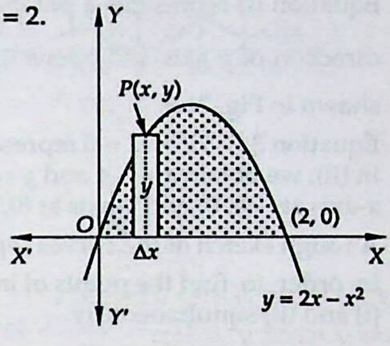

Fig. 21.7

EXAMPLE 4 Find the area of the region bounded by the curve $y^2 = 2y - x$ and the *y-axis*.

SOLUTION The equation of the given curve is

$$y^2 = 2y - x$$
$$\Rightarrow \quad y^2 - 2y = -x$$
$$\Rightarrow \quad y^2 - 2y + 1 = -x + 1$$
$$\Rightarrow \quad (y - 1)^2 = -(x - 1).$$

Clearly, this equation represents a parabola with vertex at (1, 1) and opens on the left. Putting $x = 0$ in $y^2 = 2y - x$, we get

$$y^2 - 2y = 0 \Rightarrow y = 0, 2.$$

So, the curve meets y-axis at (0, 0) and (0, 2). A rough sketch of the curve is as shown in Fig. 21.8 and the required area is the shaded area. Here we slice this region into horizontal strips. For the approximating rectangle shown in Fig. 21.8, we have

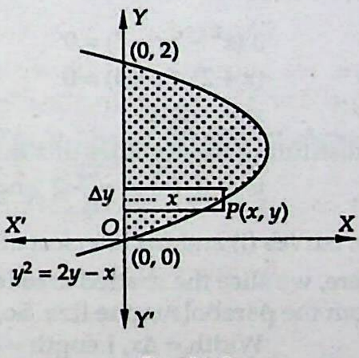

Fig. 21.8

Width $= \Delta y$, Length $= x$, and Area $= x\Delta y$.

The approximating rectangle can move from $y = 0$ to $y = 2$. So, required area A is given by

$$\therefore \quad A = \int_0^2 x\,dy$$

$$\Rightarrow \quad A = \int_0^2 (2y - y^2)\,dy \qquad \left[\begin{array}{l} \because P(x, y) \text{ lies on } y^2 = 2y - x \\ \qquad \therefore x = 2y - y^2 \end{array}\right]$$

$$\Rightarrow \quad A = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \text{ sq. units.}$$

EXAMPLE 5 *Find the area of the region included between the parabola* $y = \frac{3x^2}{4}$ *and the line* $3x - 2y + 12 = 0$. **[NCERT]**

SOLUTION The given quations are

$$y = \frac{3x^2}{4} \quad ...(i)$$

and, $\quad 3x - 2y + 12 = 0 \quad ...(ii)$

Equation (i) represents a parabola having vertex at the origin, axis along the positive direction of y-axis and opens upwards. A free hand sketch of the parabola $y = \frac{3x^2}{4}$ is shown in Fig. 21.9.

Equation $3x - 2y + 12 = 0$ represents a straight line. Putting $y = 0$ and $x = 0$ respectively in (ii), we obtain $x = -4$ and $y = 6$ respectively. So, the straight line given by (ii) meets x-axis at $(-4, 0)$ and y-axis at $(0, 6)$.

A rough sketch of the curves represented by (i) and (ii) is shown in Fig. 21.9.

In order to find the points of intersection of the given parabola and the line, we solve (i) and (ii) simultaneously.

From (ii), we get $y = \frac{3x + 12}{2}$

Putting this value of y in (i), we get

$$\frac{3x + 12}{2} = \frac{3x^2}{4}$$

$\Rightarrow \quad 3(x^2 - 2x - 8) = 0$

$\Rightarrow \quad (x + 2)(x - 4) = 0$

$\Rightarrow \quad x = -2, 4.$

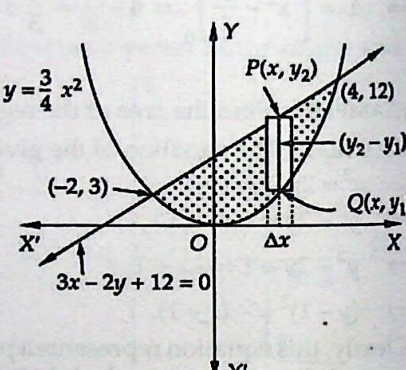

Fig. 21.9

Substituting these values of x in (i) or (ii), we have

$y = 3$ for $x = -2$ and $y = 12$ for $x = 4$

So, curves (i) and (ii) intersect at the points $(-2, 3)$ and $(4, 12)$.

Here, we slice the shaded area into vertical strips. We find that each vertical strip runs from the parabola to the line. So, the approximating rectangle shown in Fig. 21.9 has

Width $= \Delta x$, Length $= (y_2 - y_1)$ and the Area $= (y_2 - y_1)\,\Delta x$.

Since the approximating rectangle can move from $x = -2$ to $x = 4$.

So, required area A is given by

$$A = \int_{-2}^{4} (y_2 - y_1)\, dx$$

$$\Rightarrow \quad A = \int_{-2}^{4} \left(\frac{3x + 12}{2} - \frac{3}{4}x^2 \right) dx \quad \left[\begin{array}{l} \because (x, y_1) \text{ and } (x, y_2) \text{ lie on (i) and (ii) respec.} \\ \therefore y_1 = \frac{3x + 12}{2} \text{ and } y_2 = \frac{3x^2}{4} \end{array} \right]$$

$$\Rightarrow \quad A = \left[\frac{3}{4}x^2 + 6x - \frac{x^3}{4} \right]_{-2}^{4}$$

$$\Rightarrow \quad A = \left(\frac{3}{4} \times 16 + 6 \times 4 - \frac{64}{4} \right) - (3 - 12 + 2) = 27 \text{ sq. units}$$

EXAMPLE 6 *Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.*
[NCERT, CBSE 2004, 2005, 2010, HPSB 2001C]

SOLUTION The equations of the given curves are

$$x^2 = 4y \qquad \text{...(i)}$$

and,
$$x = 4y - 2 \qquad \text{...(ii)}$$

Equation (i) represents a parabola with vertex at the origin and axis along positive direction of y-axis. Equation (ii) represents a straight line which meets the coordinate axes at $(-2, 0)$ and $(0, 1/2)$ respectively.

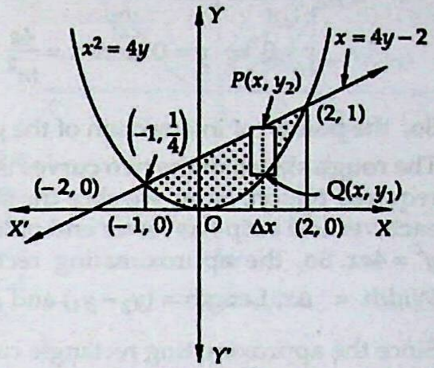

Fig. 21.10

To find the points of intersection of the given parabola and the line, we solve (i) and (ii) simultaneously. Solving the two equations simultaneously

We obtain that the points of intersection of the given parabola and the line are (2, 1) and $(-1, 1/4)$.

The region whose area is to be found out is shaded in Fig. 21.10.

Let us slice the shaded region into vertical strips. We find that each vertical strip runs from parabola (i) to the line (ii). So, the approximating rectangle shown in Fig. 21.10 has

Width = Δx, Length = $(y_2 - y_1)$, and the Area = $(y_2 - y_1)\,\Delta x$.

Since the approximating rectangle can move from $x = -1$ to $x = 2$.

$\therefore$ Required area A is given by

$$A = \int_{-1}^{2} (y_2 - y_1)\, dx$$

$$\Rightarrow \quad A = \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx \qquad \left[\begin{array}{l} \because P(x, y_2) \text{ and } Q(x, y_1) \text{ lie on (ii) and (i) respec.} \\ \therefore y_2 = \dfrac{x+2}{4} \text{ and } y_1 = \dfrac{x^2}{4} \end{array} \right]$$

$$\Rightarrow \quad A = \left[\frac{x^2}{8} + \frac{1}{2}x - \frac{x^3}{12} \right]_{-1}^{2}$$

$$\Rightarrow \quad A = \left(\frac{4}{8} + \frac{2}{2} - \frac{8}{12} \right) - \left(\frac{1}{8} - \frac{1}{2} + \frac{1}{12} \right) = \frac{9}{8} \text{ sq. units}$$

EXAMPLE 7 *Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$.*
[NCERT]

SOLUTION The equations of the given curves are

$$y^2 = 4ax \qquad \text{...(i)}$$

and,
$$y = mx \qquad \text{...(ii)}$$

The equation $y^2 = 4ax$ represents a parabola in standard form and the equation $y = mx$ represents a line passing through the origin having slope m.

In order to find the points of intersection of (i) and (ii), we solve them simultaneously.

Putting $y = mx$ from (ii) in (i), we get

$$m^2 x^2 = 4ax \Rightarrow x(m^2 x - 4a) = 0 \Rightarrow x = 0,\ x = \frac{4a}{m^2}$$

From (ii), we find that

$$x = 0 \Rightarrow y = 0 \text{ and } x = \frac{4a}{m^2} \Rightarrow y = \frac{4a}{m}$$

So, the points of intersection of the given curves are (0, 0) and $\left(4a/m^2,\ 4a/m\right)$.

The rough sketch of the two curves is shown in Fig. 21.11 and the shaded portion is the required region. Now, we slice the shaded region into vertical strips. We observe that each vertical strip has lower end on the line $y = mx$ and the upper end on the parabola $y^2 = 4ax$. So, the approximating rectangle shown in Fig. 21.11 has Width = Δx, Length = $(y_2 - y_1)$ and Area = $(y_2 - y_1)\,\Delta x$.

Since the approximating rectangle can move from $x = 0$ to $x = 4a/m^2$. So, required area A is given by

$$A = \int_0^{4a/m^2} (y_2 - y_1)\,dx$$

$$\Rightarrow \quad A = \int_0^{4a/m^2} (2\sqrt{ax} - mx)\,dx \qquad \left[\begin{array}{r} \because P(x, y_2) \text{ and } Q(x, y_1) \text{ lie on (i)} \\ \text{and (ii) respectively} \\ \therefore\ y_2^2 = 4ax \text{ and } y_1 = mx \end{array}\right]$$

$$\Rightarrow \quad A = \left[2\sqrt{a}\,\frac{x^{3/2}}{\frac{3}{2}} - m\frac{x^2}{2}\right]_0^{4a/m^2}$$

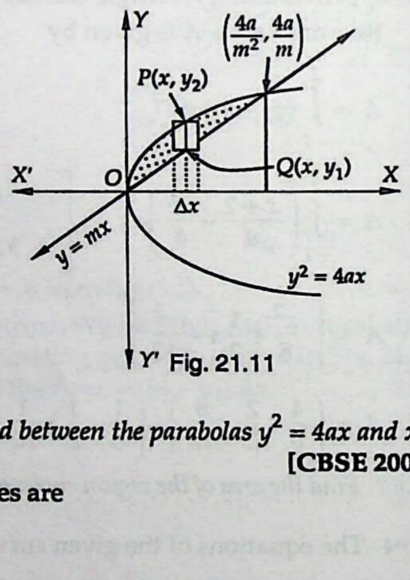

Fig. 21.11

$$\Rightarrow \quad A = \frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2$$

$$\Rightarrow \quad A = \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{3m^3} \text{ sq. units}$$

EXAMPLE 8 *Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$.* **[CBSE 2003, 2004]**

SOLUTION The equations of the given curves are

$$y^2 = 4ax \qquad \text{...(i)}$$

and, $$x^2 = 4ay \qquad \text{...(ii)}$$

Clearly, (i) and (ii) represent parabolas in standard forms. The rough sketch of these parabolas can easily be drawn as shown in Fig. 21.12.

In order to find the points of intersection of the curves (i) and (ii), we solve them simultaneously.

Putting $y = \dfrac{x^2}{4a}$ from (ii) into (i), we have

$$\left(\frac{x^2}{4a}\right)^2 = 4ax \Rightarrow x^4 = 64a^3 x$$

$$\Rightarrow x(x^3 - 64a^3) = 0 \Rightarrow x = 0 \text{ or, } x = 4a$$

From (ii), we observe that

$$x = 0 \Rightarrow y = 0 \text{ and } x = 4a \Rightarrow y = 4a.$$

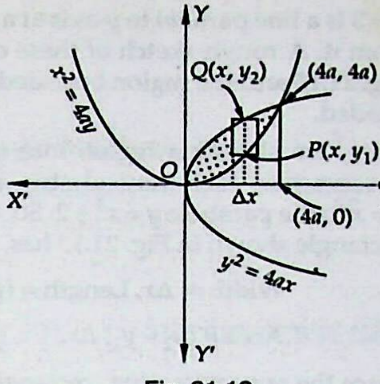

Fig. 21.12

So, the two curves intersect at (0, 0) and $(4a, 4a)$. The region whose area we have to find is the shaded region in Fig. 21.12. Here, we slice this region into vertical strips. We observe that all vertical strips have lower end on the parabola $x^2 = 4ay$ and the upper end on the parabola $y^2 = 4ax$. For the approximating rectangle shown in Fig. 21.12, we have

Width $= \Delta x$, Length $= (y_2 - y_1)$ and the Area $= (y_2 - y_1)\ \Delta x$.

Since the approximating rectangle can move between $x = 0$ and $x = 4a$. So, required area A is given by

$$A = \int_0^{4a} (y_2 - y_1)\, dx$$

$$\Rightarrow A = \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx \qquad \left[\begin{array}{l} \because P(x, y_1) \text{ and } (x, y_2) \text{ lie on (ii) and (i) respec.} \\ \therefore x^2 = 4ay_1 \text{ and } y_2^2 = 4ax \Rightarrow y_2 = \sqrt{4ax} \text{ and } y_1 = \dfrac{x^2}{4a} \end{array}\right]$$

$$\Rightarrow A = \left[\frac{4\sqrt{a}}{3} x^{3/2} - \frac{x^3}{12a}\right]_0^{4a}$$

$$\Rightarrow A = \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3} \text{ sq. units}$$

<u>NOTE</u> *The above area can also be obtained by horizontal slicing. In that case, we have*

$$\textit{Required area} = \int_0^{4a} (x_2 - x_1)\, dy = \int_0^{4a} \left(\frac{y^2}{4a} - \sqrt{4ay}\right) dy$$

EXAMPLE 9 *Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.*

SOLUTION The equations of the given curves are

$$y = x^2 + 2 \quad \text{...(i)}$$

$$y = x \quad \text{...(ii)}$$

$$x = 0 \quad \text{...(iii)}$$

and, $$x = 3 \quad \text{...(iv)}$$

Clearly, $y = x^2 + 2$ represents a parabola with vertex at (0, 2) and axis along positive direction of y-axis. This parabola opens upwards as shown in Fig. 21.13. Clearly, $y = x$ is a line passing through the origin and makes 45° angle with x-axis, $x = 0$ is y-axis and

$x = 3$ is a line parallel to y-axis at a distance of 3 units from it. A rough sketch of these curves is shown in Fig. 21.13 and the region bounded by these curves is shaded.

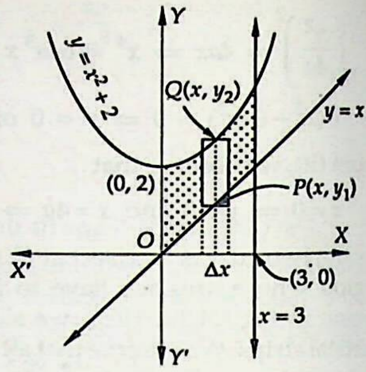

Fig. 21.13

Here, we slice this region into vertical strips. We observe that each vertical strip runs from the line $y = x$ to the parabola $y = x^2 + 2$. So, the approximating rectangle shown in Fig. 21.13 has,

Width = Δx, Length = $(y_2 - y_1)$

Area = $(y_2 - y_1)\,\Delta x$.

Since the approximating rectangle can move from $x = 0$ to $x = 3$. So, required area A is given by

$$A = \int_0^3 (y_2 - y_1)\, dx$$

$$\Rightarrow \quad A = \int_0^3 \left\{(x^2 + 2) - x\right\} dx \qquad \left[\begin{array}{l} \because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on (ii) and (i) respec.} \\ \therefore\ y_1 = x \text{ and } y_2 = x^2 + 2 \end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3 = 9 + 6 - \frac{9}{2} = \frac{21}{2} \text{ sq. units}$$

EXAMPLE 10 *Find the area of the region* $\{(x, y) : x^2 \le y \le x\}$. **[CBSE 1991, 1992, 2005]**

SOLUTION Let $R = \{(x, y) : x^2 \le y \le x\}$ Then,

$R = \{(x, y) : x^2 \le y\} \cap \{x, y) : y \le x\}$

$R = R_1 \cap R_2$, where $R_1 = \{(x, y) : x^2 \le y\}$ and $R_2 = \{(x, y) : y \le x\}$

Region R_1 : Clearly, $x^2 = y$ represents a parabola with vertex at (0, 0), positive direction of y-axis as its axis and it opens upwards. Since $x^2 \le y$, so interior of the parabola is the region R_1.

Region R_2 : Clearly, $y = x$ is a line passing through the origin and making an angle of 45° with the x-axis. Since $y \le x$, so R_2 is the region lying below the line $y = x$.

Hence, the required region R is the shaded region as shown in Fig. 21.14.

Solving $y = x^2$ and $y = x$, we obtain O (0, 0) and A (1, 1) as the points of intersection of $y = x^2$ and $y = x$.

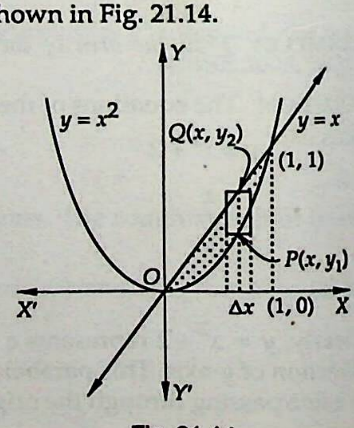

Fig. 21.14

Here, we slice this region R into vertical strips. We observe that each vertical strip has its lower end on the parabola $y = x^2$ and upper end on $y = x$. So, the approximating rectangle shown in Fig. 21.14 has,

Length = $(y_2 - y_1)$,

Width = Δx and Area = $(y_2 - y_1)\,\Delta x$.

Since the approximating rectangle can move from $x = 0$ to $x = 1$.

$\therefore$ Required area A is given by

$$A = \int_0^1 (y_2 - y_1)\, dx$$

$$\Rightarrow \quad A = \int_0^1 (x - x^2)\, dx \qquad \left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = x^2 \text{ and } y = x \text{ respectively. So, } y_1 = x^2 \text{ and } y_2 = x\right]$$

$$\Rightarrow \quad A = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. unit}$$

EXAMPLE 11 *Find the area of the region* $\{(x, y) : x^2 \le y \le |x|\}$. **[NCERT, CBSE 2002]**

SOLUTION Let $R = \{(x, y) : x^2 \le y \le |x|\}$. Then,

$$R = \{(x, y) : x^2 \le y\} \cap \{(x, y) : y \le |x|\}$$

$$\Rightarrow \quad R = \{(x, y) : x^2 \le y\} \cap [\{(x, y) : y \le x, x \ge 0\} \cup \{(x, y) : y \le -x, x < 0\}]$$

$$\Rightarrow \quad R = R_1 \cap (R_2 \cup R_3), \text{ where}$$

$$R_1 = \{(x, y) : x^2 \le y\}, R_2 = \{(x, y) : y \le x, x \ge 0\} \text{ and } R_3 = \{(x, y) : y \le -x, x < 0\}$$

Region R_1: Clearly, $x^2 = y$ represents a parabola with vertex at (0, 0), positive direction of y-axis as its axis and it opens upwards. The interior of the parabola is the region R_1.

Region R_2: Clearly, $y = x$, $x \ge 0$ is a line passing through the origin and making an angle of 45° with the positive direction of x-axis. So, R_2 is the region lying below the line $y = x$.

Region R_3: Clearly, $y = x$, $x < 0$ is a line passing through the origin and making an angle of 135° with the positive direction of x-axis. So, R_3 is the region lying below the line $y = -x$.

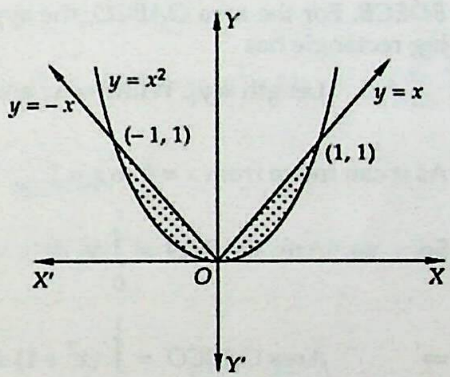

Fig. 21.15

The required region R is the shaded region shown in Fig. 21.15. Since both the curves are symmetrical about y-axis. So, required area A is given by

$$A = 2 \text{ (Shaded area in first quadrant)}$$

$$\Rightarrow \quad A = 2\int_0^1 (x - x^2)\, dx \qquad \text{[See Example 10]}$$

$$\Rightarrow \quad A = 2 \times \frac{1}{6} = \frac{1}{3} \text{ sq. units} \qquad \text{[See Example 10]}$$

EXAMPLE 12 *Find the area of the region*

$\{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1, 0 \le x \le 2\}$. **[NCERT, CBSE 2001C]**

SOLUTION Let $R = \{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1, 0 \le x \le 2\}$. Then,

$$R = \{(x, y) : 0 \le y \le x^2 + 1\} \cap \{(x, y) : 0 \le y \le x + 1\} \cap \{(x, y) : 0 \le x \le 2\}$$

$$\Rightarrow \quad R = R_1 \cap R_2 \cap R_3, \text{ where}$$

$$R_1 = \{(x, y) : 0 \le y \le x^2 + 1\},$$

$$R_2 = \{(x, y) : 0 \le y \le x + 1\} \text{ and } R_3 = \{(x, y) : 0 \le x \le 2\}$$

Region R_1 : Clearly, $y = x^2 + 1$ represents a parabola with vertex at (0, 1), axis along the positive direction of y-axis and it opens upwards.

Since, $0 \le y \le x^2 + 1 \Leftrightarrow y \ge 0$ and $y \le x^2 + 1$.

So, R_1 is the region lying above x-axis and outside the parabola $y = x^2 + 1$.

Region R_2 : Clearly, $y = x + 1$ represents a straight line cutting x-axis at $(-1, 0)$ and y-axis at $(0, 1)$.

Since $0 \le y \le x + 1 \Leftrightarrow y \ge 0$ and $y \le x + 1$.

Therefore, R_2 is the region lying above x-axis and below the line $y = x + 1$.

Region R_3 : Clearly, R_3 is the region lying between the lines $x = 0$ i.e. y-axis and $x = 2$.

Hence, the required region R is the shaded region as shown in Fig. 21.16.

Solving $y = x^2 + 1$ and $y = x + 1$, we find that these two intersect at $(0, 1)$ and $(1, 2)$.

Now, we slice the shaded region into vertical strips. We observe that vertical strips change their character at the point B. Draw a line BC parallel to the y-axis which divides the area $ABDEO$ into two portions $OABCO$ and $BDECB$. For the area $OABCO$, the approximating rectangle has

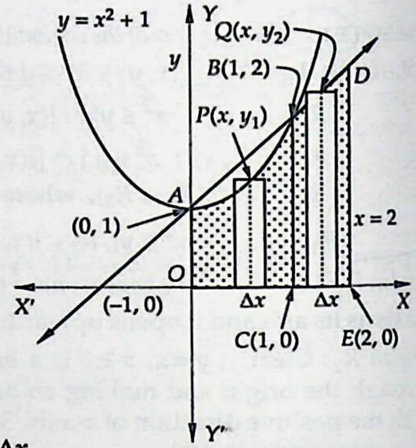

Fig. 21.16

Length $= y_1$, Width $= \Delta x$ and Area $= y_1 \Delta x$

As it can move from $x = 0$ to $x = 1$.

So, $$\text{Area } OABCO = \int_0^1 y_1\, dx$$

$\Rightarrow$ $$\text{Area } OABCO = \int_0^1 (x^2 + 1)\, dx \quad \left[\because P(x, y_1) \text{ lies on } y = x^2 + 1 \therefore y_1 = x^2 + 1\right]$$

For the area $BDECB$, the approximating rectangle shown in Fig. 21.16 has

Length $= y_2$, Width $= \Delta x$ and Area $= y_2 \Delta x$

Since it can move from $x = 1$ to $x = 2$

So, $$\text{Area } BDECB = \int_1^2 y_2\, dx$$

$\Rightarrow$ $$\text{Area } BDECB = \int_1^2 (x + 1)\, dx \quad \left[\because Q(x, y_2) \text{ lies on } y = x + 1 \therefore y_2 = x + 1\right]$$

$\therefore$ Required area = Area $OABCO$ + Area $BDECB$

$\Rightarrow$ $$\text{Required area} = \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx$$

$$\Rightarrow \quad \text{Required area} = \left[\frac{x^3}{3}+x\right]_0^1 + \left[\frac{x^2}{2}+x\right]_1^2 = \frac{4}{3}+\frac{5}{2} = \frac{23}{6} \text{ sq. units}$$

EXAMPLE 13 *Find the area bounded by the curve $y^2 = 4ax$ and the lines $y = 2a$ and y-axis.*

SOLUTION Clearly, the equation $y^2 = 4ax$ represents a parabola with vertex (0, 0) and axis as x-axis. The equation $y = 2a$ represents a straight line parallel to x-axis at a distance $2a$ from it as shown in Fig. 21.17. The required region is the shaded portion in Fig. 21.17.

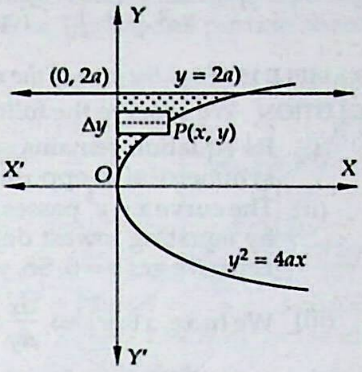

Fig. 21.17

To find the area of the shaded region shown in Fig. 21.17, we slice it into horizontal strips. We observe that each horizontal strip has its left end on y-axis and the right end on the parabola $y^2 = 4ax$.

So, the approximating rectangle shown in Fig. 21.17 has its length = x, width = Δy and area = $x\, \Delta y$.

Since the approximating rectangle can move from $y = 0$ to $y = 2a$. So, required area A is gievn by

$$A = \int_0^{2a} x\, dy$$

$$\Rightarrow \quad A = \int_0^{2a} \frac{y^2}{4a}\, dy \qquad \left[\because P(x, y) \text{ lies on } y^2 = 4ax \;\therefore\; x = \frac{y^2}{4a}\right]$$

$$\Rightarrow \quad A = \frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{2a} = \frac{1}{4a}\left(\frac{8a^3}{3} - 0\right) = \frac{2a^2}{3} \text{ sq. units}$$

EXAMPLE 14 *Find the area bounded by the curve $y^2 = 4a^2 (x - 1)$ and the lines $x = 1$ and $y = 4a$.*

SOLUTION We have,

$$y^2 = 4a^2 (x-1)$$

$$\Rightarrow \quad (y-0)^2 = 4a^2 (x-1)$$

Clearly, this equation represents a parabola with vertex at (1, 0) as shown in Fig. 21.18. The area enclosed by $y^2 = 4a^2 (x-1)$, $x = 1$ and $y = 4a$ is the area of shaded portion in Fig. 21.18.

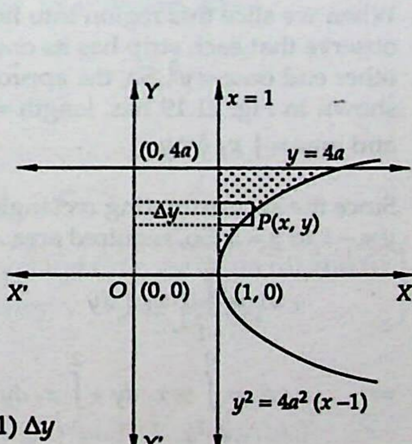

Fig. 21.18

When we slice the area of the shaded portion in horizontal strips, we observe that each strip has its one end on the line $x = 1$ and the other end on the parabola $y^2 = 4a^2 (x - 1)$. So, the approximating rectangle shown in Fig. 21.18 has,

Length = $x - 1$, Width = Δy and Area = $(x - 1)\, \Delta y$

Since, the approximating rectangle can move from $y = 0$ to $y = 4a$. So, required area A is given by

$$A = \int_0^{4a} (x-1)\, dy$$

$$\Rightarrow \quad A = \int_0^{4a} \frac{y^2}{4a^2}\, dy \qquad \left[\because P(x, y) \text{ lies on } y^2 = 4a^2(x-1) \therefore x - 1 = y^2/4a^2\right]$$

$$\Rightarrow \quad A = \frac{1}{4a^2}\left[\frac{y^3}{3}\right]_0^{4a} = \frac{1}{4a^2}\left(\frac{64a^3}{3}\right) = \frac{16a}{3} \text{ sq. units}$$

EXAMPLE 15 *Find the area of the region bounded by $y = -1$, $y = 2$, $x = y^3$ and $x = 0$.*

SOLUTION We observe the following points about the curve $x = y^3$:

(i) Its equation remains same when x is replaced by $-x$ and y by $-y$. So, it is symmetrical in opposite quadrants.

(ii) The curve $x = y^3$ passes through the origin and tangent at the origin is obtained by equating lowest degree term to zero. Equating the lowest degree term to zero, we get $x = 0$. So, y-axis is tangent at the origin.

(iii) We have, $x = y^3 \Rightarrow \frac{dx}{dy} = 3y^2$, $\frac{d^2x}{dy^2} = 6y$ and $\frac{d^3x}{dy^3} = 6 \neq 0$

Now, $\frac{dx}{dy} = 0 \Rightarrow y = 0$. Putting $y = 0$ in $x = y^3$, we obtain $x = 0$.

So, (0, 0) can be a point of local maximum or minimum.

But, $\frac{d^2x}{dy^2} = 0$ at (0, 0) and $\frac{d^3x}{dy^3} \neq 0$.

So, (0, 0) is a point of inflexion.

(iv) Clearly, $\frac{dx}{dy} = 3y^2 > 0$ for all y, so x increases with y.

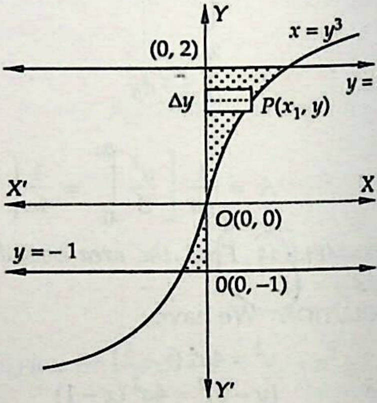

Fig. 21.19

Keeping the above facts in mind, we obtain a rough sketch of the curve $x = y^3$ as shown in Fig. 21.19.

Clearly, $y = -1$ and $y = 2$ are straight lines parallel to x-axis. The required region is shaded in Fig. 21.19. When we slice this region into horizontal strips, we observe that each strip has its one end on y-axis and other end on $x = y^3$. So, the approximating rectangle shown in Fig. 21.19 has, length $= |x_1|$, width $= \Delta y$ and area $= |x_1| \Delta y$.

Since the approximating rectangle can move from $y = -1$ to $y = 2$. So, required area A is given by

$$A = \int_{-1}^{2} |x_1|\, dy$$

$$\Rightarrow \quad A = \int_{-1}^{0} -x_1\, dy + \int_0^2 x_1\, dy \qquad \left[\begin{array}{l}\because x_1 < 0 \text{ for } -1 \le y < 0 \\ \text{and } x_1 > 0 \text{ for } 0 \le y \le 2\end{array}\right]$$

$$\Rightarrow \quad A = \int_{-1}^{0} -y^3\, dy + \int_0^2 y^3\, dy \qquad [\because P(x_1, y) \text{ lies on } x = y^3 \therefore x_1 = y^3]$$

$$\Rightarrow \quad A = -\left[\frac{y^4}{4}\right]_{-1}^{0} + \left[\frac{y^4}{4}\right]_0^2 = \frac{1}{4} + 4 = \frac{17}{4} \text{ sq. units}$$

EXAMPLE 16 *Find the area bounded by the curves $y = x$ and $y = x^3$.*

SOLUTION The equations of the given curves are

$$y = x \quad ...(i)$$

and, $$y = x^3 \quad ...(ii)$$

For the rough sketch of the curve (ii) see Ex. 1 in Appendix. Clearly, $y = x$ is a line passing through the origin and making an angle of 45° with x-axis. The shaded portion shown in Fig. 21.20 is the required region.

Solving $y = x$ and $y = x^3$ simultaneously, find that the two curves intersect at O, $(0, 0)$, $A(1, 1)$ and $B(-1, -1)$

When we slice the shaded region into vertical strips, we observe that the vertical strips change their character at O.

$\therefore$ Required area = Area $BCOB$ + Area $ODAO$

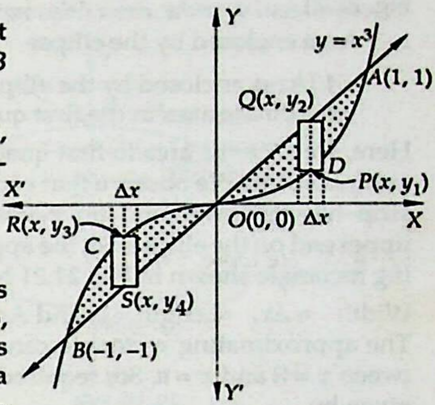

Fig. 21.20

Area BCOB : Each vertical strip in this region has its lower end on $y = x$ and the upper end on $y = x^3$. So, the approximating rectangle shown in this region has Length = $(y_4 - y_3)$, Width $= \Delta x$ and Area $= (y_4 - y_3)\,\Delta x$. Since the approximating rectangle can move from $x = -1$ to $x = 0$.

$$\therefore \quad \text{Area } BCOB = \left| \int_{-1}^{0} (y_4 - y_3)\, dx \right|$$

$$\Rightarrow \quad \text{Area } BCOB = \left| \int_{-1}^{0} (x - x^3)\, dx \right| \qquad \left[\because R(x, y_3) \text{ and } S(x, y_4) \text{ lie on (ii) and (i) respectively } \therefore\ y_3 = x^3 \text{ and } y_4 = x \right]$$

Area ODAO : Each vertical strip in this region has its two ends on (ii) and (i) respectively. So, the approximating rectangle shown in this region has

Length $= y_2 - y_1$, Width $= \Delta x$ and Area $= (y_2 - y_1)\,\Delta x$

Since it can move from $x = 0$ to $x = 1$

$$\therefore \quad \text{Area } ODAO = \int_0^1 (y_2 - y_1)\, dx$$

$$\Rightarrow \quad \text{Area } ODAO = \int_0^1 (x - x^3)\, dx \qquad \left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on (ii) and (i) respectively } \therefore\ y_1 = x^3 \text{ and } y_2 = x \right]$$

Hence,

$$\text{Required area} = \left| \int_{-1}^{0} (x - x^3)\, dx \right| + \int_0^1 (x - x^3)\, dx = \left| \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_{-1}^{0} \right| + \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1$$

$$\Rightarrow \quad \text{Required area} = \left| -\left(\frac{1}{2} - \frac{1}{4} \right) \right| + \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{2} \text{ sq. unit}$$

EXAMPLE 17 *Find the area of the region bounded by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

[NCERT, PSB 2001; HPSB 2001]

SOLUTION The equation of the curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Since all powers of x and y both are even in the equation of the curve. So, it is symmetrical about both the axes as shown in Fig. 21.21.

$\therefore$ Area enclosed by the ellipse

= 4 [Area enclosed by the ellipse and the coordinate axes in the first quadrant].

Here, we slice the area in first quadrant into vertical strips. We observe that each vertical strip has its lower end on x-axis and the upper end on the ellipse. So, the approximating rectangle shown in Fig. 21.21 has

Width $=\Delta x$, Length $= y$, and Area $= y\,\Delta x$.

The approximating rectangle can move between $x = 0$ and $x = a$. So, required area A is given by

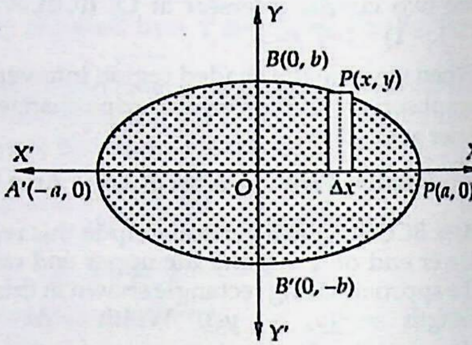

Fig. 21.21

$$A = 4\int_0^a y\,dx$$

$$\Rightarrow \quad A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx \qquad \left[\because P(x,y) \text{ lies on } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \therefore y=\frac{b}{a}\sqrt{a^2-x^2}\right]$$

$$\Rightarrow \quad A = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$\Rightarrow \quad A = \frac{4b}{a}\left[\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_0^a$$

$$\Rightarrow \quad A = \frac{4b}{a}\left\{0+\frac{1}{2}a^2\sin^{-1}(1)\right\} = \frac{4b}{a}\times\frac{1}{2}a^2\left(\frac{\pi}{2}\right) = \pi ab \text{ sq. units}$$

<u>NOTE</u> *If the area enclosed by the ellipse in the first quadrant is sliced into horizontal strips. Then the approximating rectangle has, length = x, width = Δy and area = $x\,\Delta y$ and it moves from $y = 0$ to $y = b$.*

So, $\quad$ Required area $= 4\int_0^b x\,dy = 4\int_0^b \frac{a}{b}\sqrt{b^2-y^2}\,dy$

EXAMPLE 18 *Find the area of the smaller region bounded by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and the straight line* $\frac{x}{a}+\frac{y}{b}=1$.

[NCERT, HPSB 2002C, 2004]

SOLUTION The equations of the given curves are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

and, $\quad \frac{x}{a}+\frac{y}{b} = 1$ ...(ii)

Clearly, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ represents an ellipse as shown in Fig. 21.22 and $\frac{x}{a}+\frac{y}{b}=1$ is the equation of a straight line cutting x and y-axes at $(a, 0)$ and $(0, b)$ respectively. The smaller region bounded by these two curves is shaded in Fig. 21.22.

We slice this region into vertical strips. We observe that each vertical strip has its lower end on the line $\frac{x}{a}+\frac{y}{b}=1$ and upper end on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. So, the approximating rectangle shown in Fig. 21.22 has

Length $= (y_2 - y_1)$,

Width $= \Delta x$ and Area $= (y_2-y_1)\,\Delta x$.

Since the approximating rectangle can move from $x=0$ to $x=a$. So, required area A is given by

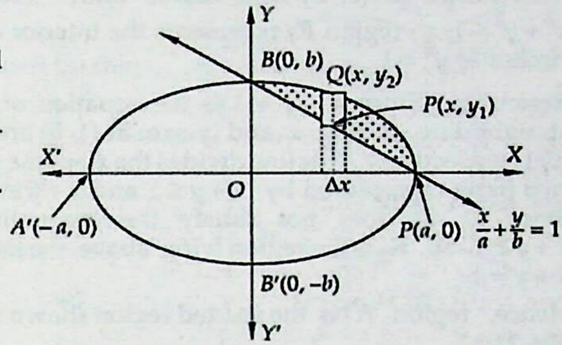

Fig. 21.22

$$A = \int_0^a (y_2 - y_1)\,dx$$

Since point $P(x, y_1)$ lies on $\frac{x}{a}+\frac{y}{b}=1$ and point $Q(x, y_2)$ lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\therefore \quad \frac{x}{a}+\frac{y_1}{b}=1 \text{ and } \frac{x^2}{a^2}+\frac{y_2^2}{b^2}=1$$

$$\Rightarrow \quad y_1 = \frac{b}{a}(a-x) \text{ and } y_2 = \frac{b}{a}\sqrt{a^2-x^2}$$

So, required area A is given by

$$A = \int_0^a \left\{ \frac{b}{a}\sqrt{a^2-x^2} - \frac{b}{a}(a-x) \right\} dx$$

$$\Rightarrow \quad A = \frac{b}{a}\left[\int_0^a \sqrt{a^2-x^2}\,dx - \int_0^a (a-x)\,dx \right]$$

$$\Rightarrow \quad A = \frac{b}{a}\left[\left[\frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) \right]_0^a - \left[ax - \frac{x^2}{2} \right]_0^a \right]$$

$$\Rightarrow \quad A = \frac{b}{a}\left[\left\{ 0 + \frac{1}{2}a^2 \sin^{-1}(1) \right\} - \left\{ a^2 - \frac{a^2}{2} \right\} \right]$$

$$\Rightarrow \quad A = \frac{b}{a}\left\{ \frac{1}{2}a^2 \times \frac{\pi}{2} - \frac{a^2}{2} \right\} = \frac{1}{2}\left(\frac{\pi}{2}-1\right)ab \text{ sq. units}$$

EXAMPLE 19 *Find the area of the region* $\{(x, y) : x^2+y^2 \le 1 \le x+y\}$.

[HSB 2001, 2002, CBSE 2010]

SOLUTION Let $R = \{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$. Then,

$$R = \{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$$

$$\Rightarrow \quad R = \{(x, y) : x^2 + y^2 \leq 1\} \cap \{(x, y) : 1 \leq x + y\}$$

$$\Rightarrow \quad R = R_1 \cap R_2, \text{ where}$$

$$R_1 = \{(x, y) : x^2 + y^2 \leq 1\} \text{ and, } R_2 = \{(x, y) : 1 \leq x + y\}$$

Region R_1 : Clearly $x^2 + y^2 = 1$ represents a circle with centre at (0, 0) and radius unity. Since $x^2 + y^2 \leq 1$, so region R_1 represents the interior of circle $x^2 + y^2 = 1$.

Region R_2 : Since $x + y = 1$ is the equation of a straight line cutting x and y-axes at (1, 0) and (0, 1) respectively. This line divides the xy-plane in two parts represented by $x + y \leq 1$ and $x + y \geq 1$. Since (0, 0) does not satisfy the inequality $x + y \geq 1$. So, R_2 is the region lying above the line $x + y = 1$.

Hence, region R is the shaded region shown in Fig. 21.23.

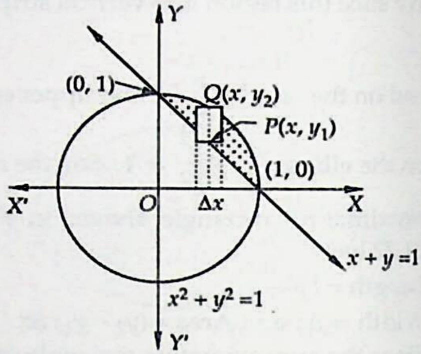

Fig. 21.23

When we slice the shaded region into vertical strips, we observe that each vertical strip has its lower end on the line $x + y = 1$ and the upper end on $x^2 + y^2 = 1$. So, the approximating rectangle shown in Fig. 21.23 has, Length $= (y_2 - y_1)$, Width $= \Delta x$ and Area $= (y_2 - y_1)\,\Delta x$. Since the approximating rectangle can move from $x = 0$ to $x = 1$. So, required area A is given by

$$A = \int_0^1 (y_2 - y_1)\, dx$$

$$\Rightarrow \quad A = \int_0^1 \left\{\sqrt{1 - x^2} - (1 - x)\right\} dx \qquad \left[\begin{array}{l} \because P(x, y_1), \text{ and } Q(x, y_2) \text{ lie on } x + y = 1 \text{ and} \\ x^2 + y^2 = 1 \text{ respectively} \therefore x + y_1 = 1\, a \text{ and} \\ x^2 + y_2^2 = 1. \text{ So, } y_1 = 1 - x \text{ and } y_2 = \sqrt{1 - x^2} \end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{1}{2} x \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1}\left(\frac{x}{1}\right) - x + \frac{x^2}{2}\right]_0^1$$

$$\Rightarrow \quad A = \left\{0 + \frac{1}{2} \sin^{-1} 1 - 1 + \frac{1}{2}\right\} - 0 = \left(\frac{\pi}{4} - \frac{1}{2}\right) \text{ sq. units}$$

EXAMPLE 20 *Find the area of the region $\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$.* **[NCERT]**

or

Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$. **[CBSE 2010]**

SOLUTION Let $R = \{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$

$$\Rightarrow \quad R = \{(x, y) : y^2 \leq 4x\} \cap \{(x, y) : 4x^2 + 4y^2 \leq 9\}$$

$$\Rightarrow \quad R = R_1 \cap R_2,$$

where $\quad R_1 = \{(x, y) : y^2 \leq 4x\}$ and $R_2 = \{(x, y) : 4x^2 + 4y^2 \leq 9\}$.

Region R_1: Clearly, $y^2 = 4x$ is the equation of the parabola with vertex at the origin and axis along x-axis. Since we are given that $y^2 \leq 4x$, so R_1 is the region lying inside the parabola $y^2 = 4x$.

Region R_2: We have,

$$4x^2+4y^2=9 \Rightarrow x^2+y^2=\left(\frac{3}{2}\right)^2$$

Clearly, it represents a circle with centre at the origin and radius $\frac{3}{2}$. It is given that $x^2+y^2 \le \frac{9}{4}$, so R_2 is the region lying inside the circle $x^2+y^2=\left(\frac{3}{2}\right)^2$

Thus, the region R is the region bounded by the parabola $y^2=4x$ and the circle $x^2+y^2=\left(\frac{3}{2}\right)^2$, as shown by the shaded portion in Fig. 21.24.

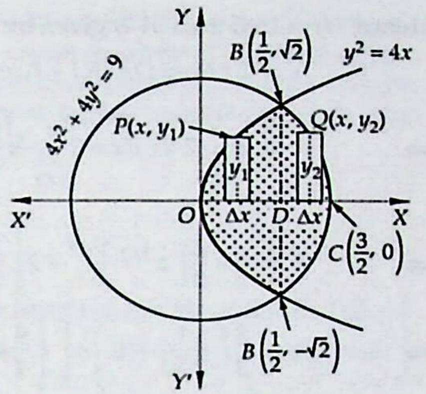

Fig. 21.24

To find the points of intersection of the given curves, we solve their equations simultaneously.

We have, $y^2=4x$ ...(i)

and, $4x^2+4y^2=9$ ..(ii)

Putting $y^2=4x$ from (i) into (ii), we get

$$4x^2+16x=9 \Rightarrow 4x^2+16x-9=0 \Rightarrow (2x+9)(2x-1)=0 \Rightarrow x=\frac{1}{2} \text{ or, } x=-\frac{9}{2}$$

From (i), we find that

$$x=\frac{1}{2} \Rightarrow y=\pm\sqrt{2} \text{ and } x=-\frac{9}{2} \Rightarrow y \text{ is imaginary.}$$

So, the two curves intersect at $(1/2, \sqrt{2})$ and $(1/2, -\sqrt{2})$

Clearly, both the curves are symmetrical about x-axis.

So, Required area = 2 (Area of the shaded region lying above x-axis).

To find this area, we slice it into vertical strips. We observe that vertical strips change their character at the point A. Draw a line AD parallel to y-axis which divides the shaded region lying above x-axis into two portions $OADO$ and $ADCA$.

For the area $OADO$, the approximating rectangle has,

Length = y_1, Width = Δx and Area = $y_1\,\Delta x$.

As it can move from $x=0$ to $x=\frac{1}{2}$

So, $$\text{Area } OADO=\int_0^{1/2} y_1\,dx$$

$\Rightarrow$ $$\text{Area } OADO=\int_0^{1/2} 2\sqrt{x}\,dx \quad \left[\because P(x,y_1) \text{ lies on } y^2=4x \therefore y_1^2=4x \Rightarrow y_1=2\sqrt{x}\right]$$

For the area $ADCA$, the approximating rectangle shown in Fig. 21.24 has,

Length = y_2, Width = Δx and Area = $y_2\,\Delta x$. As it can move from $x=\frac{1}{2}$ to $x=\frac{3}{2}$.

So, Area $ADCA = \int_{1/2}^{3/2} y_2 \, dx$

$\Rightarrow$ Area $ADCA = \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2} \, dx$ $\left[\begin{array}{l} \because Q(x, y_2) \text{ lies on } 4x^2 + 4y^2 = 9 \\ \therefore 4x^2 + 4y_2^2 = 9 \Rightarrow y_2 = \sqrt{\frac{9}{4} - x^2} \end{array}\right]$

Hence, required area A is given by

$$A = 2\,[\text{Area } OADO + \text{Area } ADCA]$$

$$\Rightarrow \quad A = 2\int_0^{1/2} 2\sqrt{x}\, dx + 2\int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\, dx$$

$$\Rightarrow \quad A = 4 \times \frac{2}{3}\left[x^{3/2}\right]_0^{1/2} + 2\left[\frac{1}{2} x \sqrt{\frac{9}{4} - x^2} + \frac{1}{2} \cdot \frac{9}{4} \sin^{-1} \frac{2x}{3}\right]_{1/2}^{3/2}$$

$$\Rightarrow \quad A = \frac{8}{3}\left(\frac{1}{2\sqrt{2}} - 0\right) + \left[\left\{\frac{9}{4} \sin^{-1}(1)\right\} - \left\{\frac{1}{\sqrt{2}} + \frac{9}{4} \sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$\Rightarrow \quad A = \frac{2\sqrt{2}}{3} + \left[\frac{9}{8}\pi - \frac{1}{\sqrt{2}} - \frac{9}{4} \sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$\Rightarrow \quad A = \left[\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4} \sin^{-1}\left(\frac{1}{3}\right)\right] \text{sq. units}$$

EXAMPLE 21 *Find the area of the region* $\{(x, y) : x^2 + y^2 \leq 2ax, y^2 \geq ax, x \geq 0, y \geq 0\}$.

SOLUTION Let $R = \{(x, y) : x^2 + y^2 \leq 2ax, y^2 \geq ax, x \geq 0, y \geq 0\}$

$$\Rightarrow \quad R = \{(x, y) : x^2 + y^2 \leq 2ax\} \cap \{(x, y) : y^2 \geq ax\} \cap \{(x, y) : x \geq 0\ y \geq 0\}$$

$$\Rightarrow \quad R = R_1 \cap R_2 \cap R_3, \text{ where}$$

$R_1 = \{(x, y) : x^2 + y^2 \leq 2ax\}$, $R_2 = \{(x, y) : y^2 \geq ax\}$ and $R_3 = \{(x, y) : x \geq 0, y \geq 0\}$

Region R_1: Clearly, $x^2 + y^2 = 2ax$ or, $(x-a)^2 + (y-0)^2 = a^2$ represents a circle with centre at $(a, 0)$ and radius a as shown in Fig. 21.25.

We have, $(x-a)^2 + (y-0)^2 \leq a^2$. So, R_1 is the region lying inside the circle shown in Fig. 21.25.

Region R_2; Clearly, $y^2 = ax$ represents a parabola with vertex at $(0, 0)$ and its axis along x-axis. It is given that $y^2 \geq ax$. So, region R_2 is the region lying outside the parabola $y^2 = ax$.

Region R_3: It is given that $x \geq 0$ and $y \geq 0$. So, region R_3 is the region in the first quadrant.

Thus, $R = R_1 \cap R_2 \cap R_3$ is the shaded portion shown in Fig. 21.25.

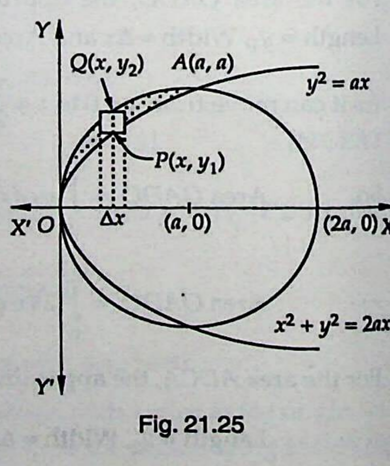

Fig. 21.25

The equations of the given curves are

$$x^2 + y^2 = 2ax \quad \text{...(i)}$$

and, $$y^2 = ax \quad \text{...(ii)}$$

To find the points of intersection of these two curves we solve (i) and (ii) simultaneously. Solving these two equations simultaneously, we find that the two curves intersect at $O(0, 0)$ and $A(a, a)$.

To find the area of the shaded portion shown in Fig. 21.25, we slice it into vertical strips. We observe that each vertical strip has its lower end on the parabola $y^2 = ax$ and upper end on $x^2 + y^2 = 2ax$. So, the approximating rectangle shown in Fig. 21.25 has Length $=(y_2 - y_1)$, Width $= \Delta x$ and Area $= (y_2 - y_1)\,\Delta x$. Since the approximating rectangle can move from $x = 0$ to $x = a$. So, required area A is given by

$$A = \int_0^a (y_2 - y_1)\, dx$$

$$\Rightarrow \quad A = \int_0^a \left(\sqrt{a^2 - (x-a)^2} - \sqrt{ax}\right) dx \quad \left[\begin{array}{l} \because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on (ii) and (i)} \\ \text{respectively } \therefore (x-a)^2 + y_2^2 = a^2 \text{ and } y_1^2 = ax \end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{1}{2}(x-a)\sqrt{a^2 - (x-a)^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x-a}{a}\right) - \frac{2\sqrt{a}}{3}\, x^{3/2}\right]_0^a$$

$$\Rightarrow \quad A = \left[\left\{\frac{1}{2}a^2 \sin^{-1}(0) - \frac{2}{3}a^2\right\} - \left\{\frac{1}{2}a^2 \sin^{-1}(-1)\right\}\right]$$

$$\Rightarrow \quad A = -\frac{2}{3}a^2 - \frac{1}{2}a^2\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \quad A = \left(\frac{\pi a^2}{4} - \frac{2}{3}a^2\right) \text{ sq. units } = \left(\frac{\pi}{4} - \frac{2}{3}\right)a^2 \text{ sq. units}$$

EXAMPLE 22 *Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$.*

[CBSE 2007; HSB 2002; HPSB 2002C]

SOLUTION The equations of the given curves are

$$x^2 + y^2 = 1 \quad \text{...(i)}$$

and, $$(x-1)^2 + (y-0)^2 = 1 \quad \text{...(ii)}$$

Clearly, $x^2 + y^2 = 1$ represents a circle with centre at $(0, 0)$ and radius unity. Also, $(x-1)^2 + (y-0)^2 = 1$ represents a circle with centre at $(1, 0)$ and radius unity. To find the points of intersection of the given curves, we solve (i) and (ii) simultaneously.

Solving (i) and (ii) simultaneously, we find that the two curves intersect at $A(1/2, \sqrt{3}/2)$ and $D(1/2, -\sqrt{3}/2)$.

Since both the curves are symmetrical about x-axis.

So, Required area $= 2$ (Area $OABCO$)

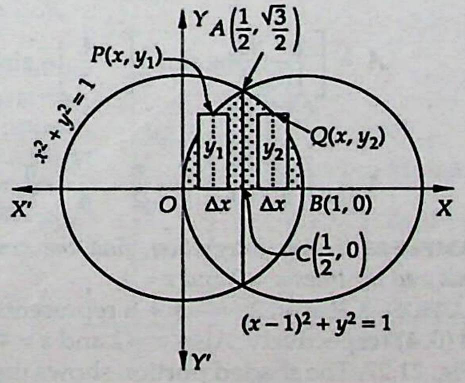

Fig. 21.26

Now, we slice the area $OABCO$ into vertical strips. We observe that the vertical strips change their character at $A\left(1/2, \sqrt{3}/2\right)$. So,

$$\text{Area } OABCO = \text{Area } OACO + \text{Area } CABC.$$

When area $OACO$ is sliced into vertical strips, we find that each strip has its upper end on the circle $(x-1)^2 + (y-0)^2 = 1$ and the lower end on x-axis. So, the approximating rectangle shown in Fig. 21.26 has, Length $= y_1$, Width $= \Delta x$ and Area $= y_1 \, \Delta x$. As it can move from $x = 0$ to $x = 1/2$.

$$\therefore \quad \text{Area } CACO = \int_0^{1/2} y_1 \, dx$$

$$\Rightarrow \quad \text{Area } CACO = \int_0^{1/2} \sqrt{1-(x-1)^2} \, dx \quad \left[\begin{array}{l} \because P(x, y_1) \text{ lies on } (x-1)^2 + y^2 = 1 \\ \therefore (x-1)^2 + y_1^2 = 1 \Rightarrow y_1 = \sqrt{1-(x-1)^2} \end{array}\right]$$

Similarly, approximating rectangle in the region $CABC$ has, Length $= y_2$, Width $= \Delta x$ and Area $= y_2 \, \Delta x$. As it can move from $x = \frac{1}{2}$ to $x = 1$.

$$\therefore \quad \text{Area } CABC = \int_{1/2}^{1} y_2 \, dx$$

$$\Rightarrow \quad \text{Area } CABC = \int_{1/2}^{1} \sqrt{1-x^2} \, dx \quad \left[\begin{array}{l} \because Q(x, y_2) \text{ lies on } x^2 + y^2 = 1 \\ \therefore x^2 + y_2^2 = 1 \Rightarrow y_2 = \sqrt{1-x^2} \end{array}\right]$$

Hence, required area A is given by

$$A = 2\left[\int_0^{1/2} \sqrt{1-(x-1)^2} \, dx + \int_{1/2}^{1} \sqrt{1-x^2} \, dx\right]$$

$$\Rightarrow \quad A = 2\left[\left[\frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_0^{1/2} + \left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{1/2}^{1}\right]$$

$$\Rightarrow \quad A = \left[\left\{-\frac{\sqrt{3}}{4} + \sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right\} + \left\{\sin^{-1}(1) - \frac{\sqrt{3}}{4} - \sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$\Rightarrow \quad A = -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units}$$

EXAMPLE 23 *Using integration, find the area of the region bounded by the line $2y = -x + 8$, x-axis and the lines $x = 2$ and $x = 4$.*

SOLUTION Clearly, $2y = -x + 8$ represents a straight line cutting x and y-axes at $(8, 0)$ and $(0, 4)$ respectively. Also, $x = 2$ and $x = 4$ are straight lines parallel to y-axis as shown in Fig. 21.27. The shaded portion shows the region whose area is to be found. When we slice the shaded region into vertical strips, we find that each vertical strip has its lower

end on x-axis and upper end on the line $2y = -x + 8$. So, the approximating rectangle shown in Fig. 21.27 has, Length $= y$, Width $= \Delta x$ and Area $= y\,\Delta x$. The approximating rectangle can move from $x = 2$ to $x = 4$. So, required area A is given by

$$A = \int_2^4 y\,dx$$

$$\Rightarrow \quad A = \int_2^4 \left(\frac{-x+8}{2}\right) dx \qquad \left[\begin{array}{l} \because P(x, y) \text{ lies on} \\ 2y = -x+8 \ \therefore\ y = \dfrac{-x+8}{2} \end{array}\right]$$

$$\Rightarrow \quad A = \frac{1}{2}\left[-\frac{x^2}{2} + 8x\right]_2^4$$

$$= \frac{1}{2}\left[\left(-\frac{16}{2} + 32\right) - \left(-\frac{4}{2} + 16\right)\right] = 5 \text{ sq. units}$$

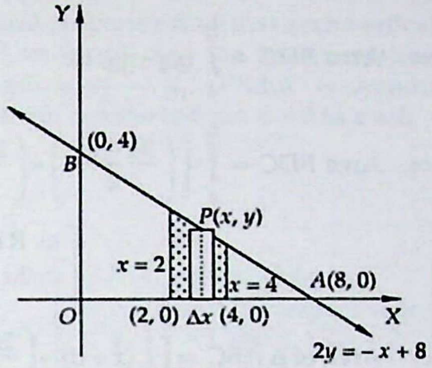

Fig. 21.27

EXAMPLE 24 *Using integration, find the area of the triangle ABC whose vertices have coordinates A (2, 5), B (4,7) and C (6, 2).* **[CBSE 2001, 2010; HPSB 2000C]**

SOLUTION First we find the equations of the sides of triangle ABC by using $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

The equation of AB is

$$y - 5 = \frac{7-5}{4-2}(x-2) \Rightarrow x - y + 3 = 0 \quad \text{...(i)}$$

The equation of BC is

$$y - 7 = \frac{2-7}{6-4}(x-4) \Rightarrow 5x + 2y - 34 = 0 \quad \text{...(ii)}$$

The equation of side AC is

$$y - 5 = \frac{2-5}{6-2}(x-2) \Rightarrow 3x + 4y - 26 = 0 \quad \text{...(iii)}$$

Clearly, Area of $\Delta\, ABC$ = Area ADB + Area BDC

Area ADB : To find area ADB, we slice it into vertical strips. We observe that each vertical strip has its lower end on side AC and the upper end on AB. So, the approximating rectangle has Length $= (y_2 - y_1)$, Width $= \Delta x$ and Area $= (y_2 - y_1)\,\Delta x$. Since the approximating rectangle can move from $x = 2$ to $x = 4$.

$$\therefore \quad \text{Area } ADB = \int_2^4 (y_2 - y_1)\,dx$$

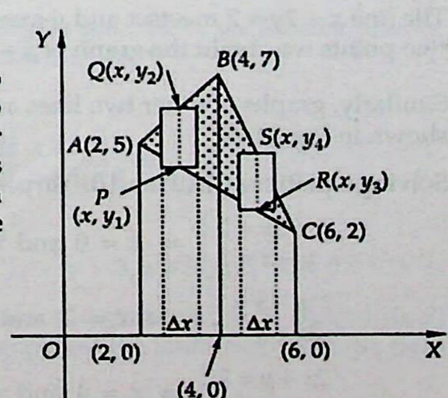

Fig. 21.28

$$\Rightarrow \text{Area } ADB = \int_2^4 \left\{ (x+3) - \left(\frac{26-3x}{4} \right) \right\} dx \quad \left[\begin{array}{l} \because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on (iii) \& (i)} \\ \text{respec. } \therefore 3x + 4y_1 - 26 = 0 \text{ \& } y_2 = x + 3 \end{array} \right]$$

Similarly, we have

$$\Rightarrow \text{Area } BDC = \int_4^6 (y_4 - y_3)\, dx$$

$$\Rightarrow \text{Area } BDC = \int_4^6 \left\{ \left(\frac{34-5x}{2} \right) - \left(\frac{26-3x}{4} \right) \right\} dx$$

$$\left[\begin{array}{l} \because R(x, y_3) \text{ and } S(x, y_4) \text{ lie on (iii) and (ii) respectively} \\ \therefore 3x + 4y_3 - 26 = 0 \text{ and } 5x + 2y_4 - 34 = 0 \end{array} \right]$$

$$\therefore \text{Area of } \Delta ABC = \int_2^4 \left\{ (x+3) - \left(\frac{26-3x}{4} \right) \right\} dx + \int_4^6 \left\{ \left(\frac{34-5x}{2} \right) - \left(\frac{26-3x}{4} \right) \right\} dx$$

$$\Rightarrow \text{Area of } \Delta ABC = \frac{1}{4} \int_2^4 (7x - 14)\, dx + \frac{1}{4} \int_4^6 (42 - 7x)\, dx$$

$$\Rightarrow \text{Area of } \Delta ABC = \frac{1}{4} \left[\left[\frac{7x^2}{2} - 14x \right]_2^4 + \left[42x - \frac{7x^2}{2} \right]_4^6 \right]$$

$$\Rightarrow \text{Area of } \Delta ABC = \frac{1}{4} \left[\{(56 - 56) - (14 - 28)\} + \{(252 - 126) - (168 - 56)\} \right] = 7 \text{ sq. units}$$

EXAMPLE 25 *Compute the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.*

SOLUTION The equations of the given lines are

$$x + 2y = 2 \quad \text{...(i)}$$

$$y - x = 1 \quad \text{...(ii)}$$

$$2x + y = 7 \quad \text{... (iii)}$$

The line $x + 2y = 2$ meets x and y-axes at $(2, 0)$ and $(0, 1)$ respectively. By joining these two points we obtain the graph of $x + 2y = 2$.

Similarly, graphs of other two lines are drawn as shown in Fig. 21.29.

Solving equations (i), (ii) and (iii) in pairs, we have

$$\left.\begin{array}{r} x + 2y = 2 \\ y - x = 1 \end{array}\right\} \Rightarrow x = 0 \text{ and } y = 1;$$

$$\left.\begin{array}{r} y - x = 1 \\ 2x + y = 7 \end{array}\right\} \Rightarrow x = 2 \text{ and } y = 3;$$

$$\left.\begin{array}{r} 2x + y = 7 \\ x + 2y = 2 \end{array}\right\} \Rightarrow x = 4 \text{ and } y = -1.$$

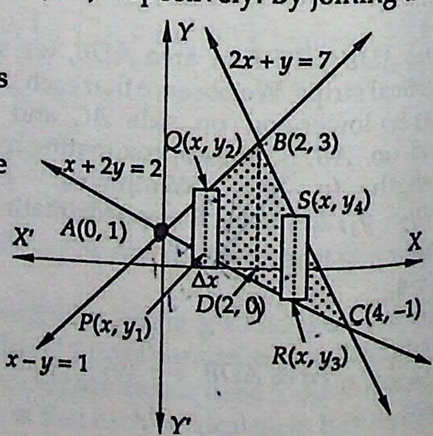

Fig. 21.29

Thus, the points of intersection of the given lines are $A\,(0, 1)$, $B\,(2, 3)$ and $C\,(4, -1)$.

When we slice the shaded region into vertical strips, we observe that the strip change their character at B.

$\therefore$ Required area = (Area ABD) + (Area BDC)

Area ABD : When we slice the area ABD into vertical strips, we find that each vertical strip has its lower end on line (i) and the upper end on line (ii). So, the approximating rectangle shown in Fig. 21.29 has, Length = $y_2 - y_1$, Width = Δx and Area = $(y_2 - y_1)\,\Delta x$. Since the approximating rectangle can move from $x = 0$ to $x = 2$.

$$\therefore \quad \text{Area } ABD = \int_0^2 (y_2 - y_1)\, dy$$

$$\Rightarrow \quad \text{Area } ABD = \int_0^2 \left\{ (x+1) - \left(\frac{2-x}{2}\right) \right\} dx \qquad \left[\begin{array}{l} \because P\,(x, y_1) \text{ and } Q, (x, y_2) \\ \text{lie on (i) and (ii) respectively} \\ \therefore\ y_1 - x = 1 \text{ and } x + 2y_2 = 2 \end{array}\right]$$

Area BDC : When area BDC is sliced into vertical strips, then each strip has its lower end on the line (i) and the upper end on the line (iii). So, the approximating rectangle shown in Fig. 21.29 has, length = $(y_4 - y_3)$, width = Δx and area = $(y_4 - y_3)\,\Delta x$. Since the approximating rectangle can move from $x = 2$ to $x = 4$.

$$\therefore \quad \text{Area } BDC = \int_2^4 (y_4 - y_3)\, dx$$

$$\Rightarrow \quad \text{Area } BDC = \int_2^4 \left\{ (7 - 2x) - \left(\frac{2-x}{2}\right) \right\} dx \qquad \left[\begin{array}{l} \because R\,(x, y_3) \text{ and } S\,(x, y_4) \\ \text{lie on (i) and (ii) respectively} \\ \therefore\ x + 2y_3 = 2 \text{ and } 2x + y_4 = 7 \end{array}\right]$$

$$\therefore \quad \text{Required area} = \text{Area } ABD + \text{Area } BDC$$

$$= \int_0^2 \left\{ (x+1) - \left(\frac{2-x}{2}\right) \right\} dx + \int_2^4 \left\{ (7 - 2x) - \left(\frac{2-x}{2}\right) \right\} dx$$

$$= \int_0^2 \frac{3x}{2}\, dx + \int_2^4 \left(6 - \frac{3x}{2}\right) dx = \left[\frac{3x^2}{4}\right]_0^2 + \left[6x - \frac{3x^2}{4}\right]_2^4 = 6 \text{ sq. units}$$

EXAMPLE 26 *Using integration, find the area of the region bounded by the following curves, after making a rough sketch:* $y = 1 + |x + 1|$, $x = -3$, $x = 3$, $y = 0$.

SOLUTION We have,

$$y = 1 + |\,x + 1\,| \Rightarrow y = \begin{cases} 1 + x + 1\ , & \text{if } x + 1 \geq 0 \\ 1 - (x+1), & \text{if } x + 1 < 0 \end{cases} = \begin{cases} x + 2, & \text{if } x \geq -1 \\ -x\ , & \text{if } x < -1 \end{cases}$$

Thus, the equations of the given curves are

$$y = x + 2 \text{ for } x \geq -1,\ y = -x \text{ for } x < -1,\ x = -3, x = 3 \text{ and } y = 0$$

Clearly, $y = x + 2$ is a straight line cutting x-axes and y-axes at $(-2, 0)$ and $(0, 2)$ respectively. So, $y = x + 2$, for $x > -1$ represents that part of the line which is on the right side of $x = -1$.

The equation $y=-x$ for $x<-1$ represents that part of the line passing through the origin and making an angle of 135° with x-axis which is on the left side of $x=-1$.

Clearly, $x=-3$ and $x=3$ are lines parallel to y-axis.

The region bounded by the given curves is shaded portion shown in Fig. 21.30.

We slice the shaded region into vertical strips. We observe that vertical strips change their character at A.

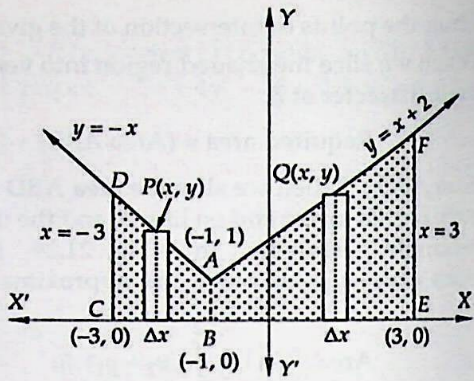

Fig. 21.30

So, required area = Area $CDAB$ + Area $ABEF$.

Area CDAB : In area $CDAB$, each vertical strip has its upper end on $y=-x$ and lower end on x-axis. So, the approximating rectangle shown in Fig. 21.30 has, length $=y$, width $=\Delta x$ and area $=y\,\Delta x$. Since the approximating rectangle can move from $x=-3$ to $x=-1$.

$$\therefore \qquad \text{Area } CDAB = \int_{-3}^{-1} y\,dx = \int_{-3}^{-1} -x\,dx \qquad [\because P(x, y) \text{ lies on } y=-x]$$

Area ABEF : In area $ABEF$, each vertical strip has its lower end on x-axis and the upper end on $y=x+2$. So, the approximating rectangle has,
Length $=y$, Width $=\Delta x$ and Area $=y\,\Delta x$. As it can move from $x=-1$ to $x=3$.

$$\therefore \qquad \text{Area } ABEF = \int_{-1}^{3} y\,dx = \int_{-1}^{3} (x+2)\,dx \qquad [\because Q(x, y) \text{ lies on } y=x+2]$$

Hence, required area = Area $CDAB$ + Area $ABEF$

$$\Rightarrow \qquad \text{Required area} = \int_{-3}^{-1} -x\,dx + \int_{-1}^{3} (x+2)\,dx$$

$$\Rightarrow \qquad \text{Required area} = -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2}+2x\right]_{1}^{3}$$

$$\Rightarrow \qquad \text{Required area} = -\left\{\frac{1}{2}-\frac{9}{2}\right\} + \left\{\left(\frac{9}{2}+6\right)-\left(\frac{1}{2}-2\right)\right\} = 16 \text{ sq. units}$$

EXAMPLE 27 *Sketch the graph* $y=|x+1|$. *Evaluate* $\int_{-3}^{1} |x+1|\,dx$. *What does this value represent on the graph?*

SOLUTION We have,

$$y=|x+1|=\begin{cases} x+1, & \text{if } x+1 \geq 0 \\ -(x+1), & \text{if } x+1<0 \end{cases}$$

$$\Rightarrow \qquad y=|x+1|=\begin{cases} x+1, & \text{if } x \geq -1 \\ -x-1, & \text{if } x<-1 \end{cases}$$

x+1

So, we have $y = x+1$ for $x \geq -1$ and $y = -x-1$ for $x < -1$. Clearly, $y = x+1$ is a straight line cutting x and y-axes at $(-1, 0)$ and $(0, 1)$ respectively. So, $y = x+1, x \geq -1$ represents that portion of the line which lies on the right side of $x = -1$. Similarly, $y = -x-1, x < -1$ represents that part of the line $y = -x-1$ which is on the left side of $x = -1$. A rough sketch of $y = |x+1|$ is shown in Fig. 21.31.

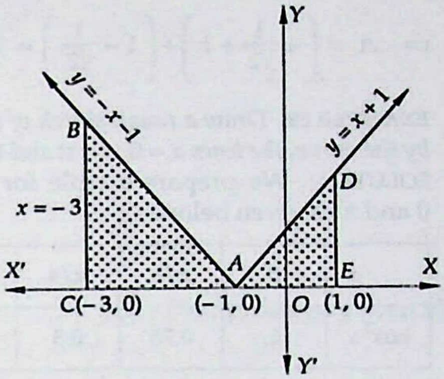

Fig. 21.31

$$\int_{-3}^{1} |x+1|\, dx = \int_{-3}^{-1} |x+1|\, dx + \int_{-1}^{1} |x+1|\, dx$$

$$\Rightarrow \int_{-3}^{1} |x+1|\, dx = \int_{-3}^{-1} -(x+1)\, dx + \int_{-1}^{1} (x+1)\, dx$$

$$\Rightarrow \int_{-3}^{1} |x+1|\, dx = -\left[\frac{(x+1)^2}{2}\right]_{-3}^{-1} + \left[\frac{(x+1)^2}{2}\right]_{-1}^{1}$$

$$\Rightarrow \int_{-3}^{1} |x+1|\, dx = -\left[0 - \frac{4}{2}\right] + \left[\frac{4}{2} - 0\right] = 4$$

This value represents the area of the shaded portion shown in Fig. 21.31.

EXAMPLE 28 *Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$ as x varies from 0 to $\frac{\pi}{2}$ and find the area of the region enclosed by them and x-axis.*

SOLUTION The values of $\sin x$ and $\cos x$ at different points lying between 0 and $\frac{\pi}{2}$ are given as under

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin x$	0	0.5	0.716	0.866	1
$\cos x$	1	0.866	0.716	0.5	0

By plotting these points and joining them with a free hand, we obtain the graphs of $y = \sin x$ and $y = \cos x$ as shown in Fig. 21.32 These two curves intersect at point A. At the point of intersection, we have

$$\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$$

The required region is the shaded region shown in Fig. 21.32. To find the area of this shaded region, we slice it into vertical strips. We observe that vertical strips change their character at A. So, required area A is given by

$$A = \text{Area } OAC + \text{Area } ACB$$

$$\Rightarrow \quad A = \int_0^{\pi/4} y_1\, dx + \int_{\pi/4}^{\pi/2} y_2\, dx$$

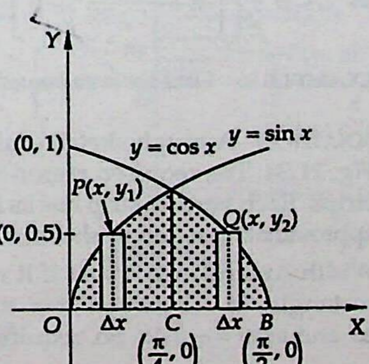

Fig. 21.32

$\Rightarrow \quad A = \int\limits_{0}^{\pi/4} \sin x \, dx + \int\limits_{\pi/4}^{\pi/2} \cos x \, dx \qquad \left[\begin{array}{l} \because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = \sin x \text{ and} \\ y = \cos x \text{ respectively } \therefore y_1 = \sin x \text{ and } y_2 = \cos x \end{array} \right.$

$\Rightarrow \quad A = \Big[-\cos x \Big]_0^{\pi/4} + \Big[\sin x \Big]_{\pi/4}^{\pi/2}$

$\Rightarrow \quad A = \left(-\frac{1}{\sqrt{2}} + 1 \right) + \left(1 - \frac{1}{\sqrt{2}} \right) = 2 - \frac{2}{\sqrt{2}} = (2 - \sqrt{2})$ sq. units

EXAMPLE 29 *Draw a rough sketch of the curve $y = \cos^2 x$ in $[0, \pi]$ and find the area enclosed by the curve, the lines $x = 0$, $x = \pi$ and the x-axis.*

SOLUTION We prepare a table for the values of $\cos^2 x$ at different points between 0 and π as given below.

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\cos^2 x$	1	0.75	0.5	0.25	0	0.25	0.5	0.75	1

By plotting these points and joining them with a free hand, we obtain a rough sketch of $y = \cos^2 x$ as shown in Fig. 21.33. The required region is the shaded region in Fig. 21.33. To find the shaded region, we slice it into vertical strips. Each vertical strip has its lower end on x-axis and the upper end on $y = \cos^2 x$. So, the approximating rectangle shown in Fig. 21.33 has, length $= y$, width $= \Delta x$ and area $= y \, \Delta x$. Since the approximating rectangle can move from $x = 0$ to $x = \pi$. So, required area A is given by

$A = \int\limits_0^{\pi} y \, dx$

$\Rightarrow \quad A = \int\limits_0^{\pi} \cos^2 x \, dx \quad [\because P(x, y) \text{ lies on } y = \cos^2 x]$

$\Rightarrow \quad A = \frac{1}{2} \int\limits_0^{\pi} (1 + \cos 2x) \, dx = \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi}$

$\Rightarrow \quad A = \frac{1}{2} \left[\left(\pi + \frac{\sin 2\pi}{2} \right) - 0 \right] = \frac{\pi}{2}$ sq. units

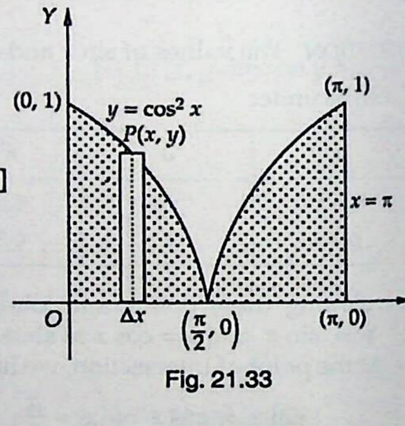

Fig. 21.33

EXAMPLE 30 *Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.* **[NCERT]**

SOLUTION A rough sketch of the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$ is shown in Fig. 21.34. The required region is shaded in Fig. 21.34. We slice this region into vertical strips. Each vertical strip has its lower end on x-axis and the upper end on $y = \sin x$. So, the approximating rectangle shown in Fig. 21.34 has, length $= y$.

Width Δx and area $= y \, \Delta x$ if it moves between $x = 0$ and $x = \pi$. If the approximating rectangle is between $x = \pi$ and $x = 2\pi$ then its length $= -y$, width $= \Delta x$ and area $= -y \, \Delta x$. So, required area A is given by

$A = \text{Area } OAB + \text{Area } BCD$

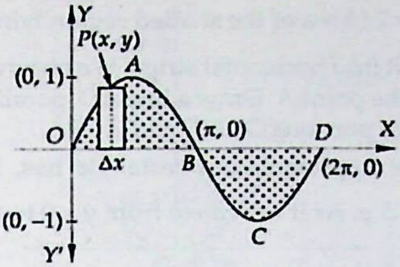

Fig. 21.34

$$\Rightarrow \quad A = \int_0^{\pi} y\,dx + \int_{\pi}^{2\pi} -y\,dx$$

$$\Rightarrow \quad A = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} -\sin x\,dx \qquad [\because P(x, y) \text{ lies on } y = \sin x]$$

$$\Rightarrow \quad A = \Big[-\cos x\Big]_0^{\pi} + \Big[\cos x\Big]_{\pi}^{2\pi} = 2 + 2 = 4 \text{ sq. units}$$

EXAMPLE 31 *Sketch the region common to the circle $x^2 + y^2 = 16$ and the parabola $x^2 = 6y$. Also, find the area of the region using integration.*

OR

Using integration, find the area of the region $\{(x, y) : x^2 + y^2 \leq 16, x^2 \leq 6y\}$. **[CBSE 2010]**

SOLUTION The equations of the given curves are

$$x^2 + y^2 = 16 \qquad ...(i)$$

and, $$x^2 = 6y \qquad ...(ii)$$

Clearly, $x^2 + y^2 = 16$ represents a circle having centre at the origin and radius four units and $x^2 = 6y$ represents a parabola opening upward and having its vertex at the origin. To find the points of intersection of these two curves, we solve (i) and (ii) simultaneously. Putting $y = \frac{x^2}{6}$, obtained from (ii), in (i), we get

$$x^2 + \frac{x^4}{36} = 16$$

$$\Rightarrow \quad x^4 + 36x^2 - 576 = 0$$

$$\Rightarrow \quad (x^2 + 48)(x^2 - 12) = 0$$

$$\Rightarrow \quad x^2 - 12 = 0$$

$$\Rightarrow \quad x = \pm 2\sqrt{3}$$

Putting $x = \pm 2\sqrt{3}$ in (ii), we get $y = 2$.

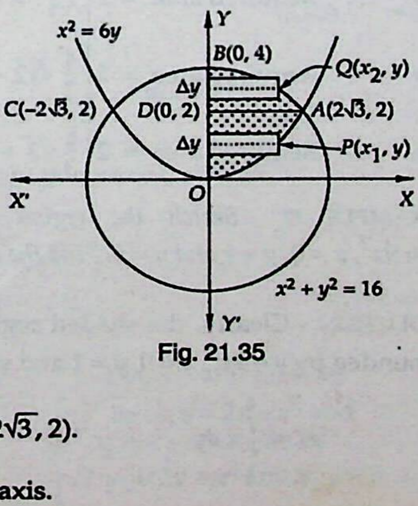

Fig. 21.35

Thus, the two curves intersect at $(2\sqrt{3}, 2)$ and $(-2\sqrt{3}, 2)$.

Clearly, both the curves are symmetrical about y-axis.

Required area = 2 (Area of the shaded region lying in first quadrant)

To find this area, we slice it into horizontal strips. We observe that the horizontal strips change their character at the point A. Draw a line AD parallel to x-axis which divides the region $OABO$ into two portions $OADO$ and $ADBA$.

For the region $OADO$, the approximating rectangle has, lengthd $= x_1$,

Width $= \Delta y$ and area $= x_1 \Delta y$. As it can move from $y = 0$ to $y = 2$.

$$\therefore \quad \text{Area } OADO = \int_0^2 x_1\, dy$$

$$\Rightarrow \quad \text{Area } OADO = \int_0^2 \sqrt{6y}\, dy \qquad \left[\begin{array}{l} \because (x_1, y_1) \text{ lies on } x^2 = 6y \\ \therefore x_1^2 = 6y \Rightarrow x_1 = \sqrt{6y} \end{array}\right]$$

For the area $DABD$, the approximating rectangle as shown in Fig. 21.35 has, Length $= x_2$, Width $= \Delta y$ and Area $= x_2 \Delta y$. As it can move from $y = 2$ to $y = 4$.

$$\text{So,} \quad \text{Area } DABD = \int_2^4 x_2\, dy$$

$$\Rightarrow \quad \text{Area } DABD = \int_2^4 \sqrt{16 - y^2}\, dy \qquad \left[\begin{array}{l} \because (x_2, y) \text{ lies on } x^2 + y^2 = 16 \\ \therefore x_2^2 + y^2 = 16 \Rightarrow x_2 = \sqrt{16 - y^2} \end{array}\right]$$

$$\therefore \quad \text{Required area} = 2\,[\text{Area } OADO + \text{Area } DABD]$$

$$\Rightarrow \quad \text{Required area} = 2\left[\int_0^2 \sqrt{6y}\, dy + \int_2^4 \sqrt{16 - y^2}\, dy\right]$$

$$\Rightarrow \quad \text{Required area} = 2\left[\frac{2}{3}\sqrt{6}\left[y^{3/2}\right]_0^2 + \left[\frac{1}{2} y \sqrt{16 - y^2} + \frac{16}{2} \sin^{-1} \frac{y}{4}\right]_2^4\right]$$

$$\Rightarrow \quad \text{Required area} = 2\left[\frac{2\sqrt{6}}{3} \times (2\sqrt{2} - 0) + 8 \sin^{-1} 1 - \sqrt{16 - 4} - 8 \sin^{-1} \frac{1}{2}\right]$$

$$\Rightarrow \quad \text{Required area} = 2\left[\frac{4}{3}\sqrt{12} + 8 \times \frac{\pi}{2} - \sqrt{12} - 8 \times \frac{\pi}{6}\right]$$

$$\Rightarrow \quad \text{Required area} = 2\left[\frac{8}{3}\sqrt{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right] = \left(\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right) \text{sq.units}$$

EXAMPLE 32 *Sketch the region lying in the first quadrant and bounded by $y = 9x^2$, $x = 0$, $y = 1$ and $y = 4$. Find the area of the region using integration.*

SOLUTION Clearly, the shaded region is the region lying in the first quadrant and bounded by $y = 9x^2$, $x = 0$, $y = 1$ and $y = 4$. So, required area A is given by

$$A = \int_1^4 x\, dy$$

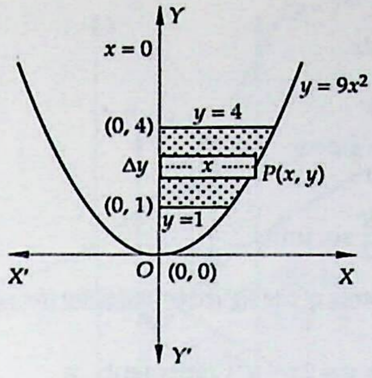

Fig. 21.36

$$\Rightarrow \quad A = \int_1^4 \sqrt{\frac{y}{9}}\, dy \qquad \left[\because P(x, y) \text{ lies on } y = 9x^2 \therefore x = \sqrt{\frac{y}{9}}\right]$$

$$\Rightarrow \quad A = \frac{1}{3}\int_1^4 \sqrt{y}\, dy$$

$$\Rightarrow \quad A = \frac{1}{3} \times \frac{2}{3}\left[y^{3/2}\right]_1^4 = \frac{2}{9}(8-1) = \frac{14}{9} \text{ sq. units}$$

EXAMPLE 33 *Find the area bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$.*

SOLUTION The curve $y = 2x - x^2$ represents a parabola opening downward and crossing x-axis at (0, 0) and (2, 0). Clearly, $y = -x$ represents a line passing through the origin and making 135° with x-axis. A rough sketch of the two curves is shown in Fig. 21.37. The region whose area is to be found is shaded in Fig. 21.37. Here, we slice the shaded region into vertical strips. For the approximating rectangle shown in Fig. 21.37, we have

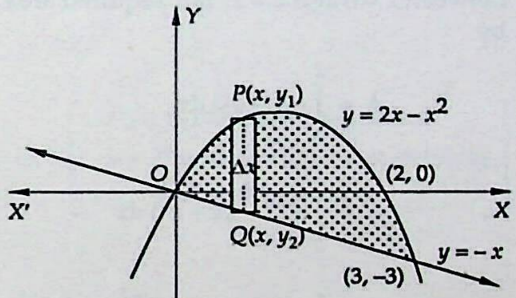

Fig. 21.37

Width $= \Delta x$

Length $= (y_1 - y_2)$

$\therefore$ Area $= (y_1 - y_2)\,\Delta x$

The approximating rectangle can move horizontal between $x = 0$ and $x = 3$.

So, required area A is given by

$$A = \int_0^3 (y_1 - y_2)\, dx$$

$$\Rightarrow \quad A = \int_0^3 \{2x - x^2 - (-x)\}\, dx \qquad \left[\begin{array}{l}\because P(x, y_1) \text{ and } Q(x, y_2) \\ \text{lie on } y = 2x - x^2 \text{ and} \\ y = -x \text{ respectively} \\ \therefore y_1 = 2x - x^2 \text{ and } y_2 = -x\end{array}\right]$$

$$\Rightarrow \quad A = \int_0^3 (3x - x^2)\,dx$$

$$\Rightarrow \quad A = \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3$$

$$\Rightarrow \quad A = \frac{27}{2} - \frac{27}{3} = \frac{9}{2} \text{ sq. units}$$

EXAMPLE 34 *Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$, $y = 2x - x^2$.*

SOLUTION The equation $y = 2x - x^2$ represents a parabola opening downwards having vertex at (1, 1) and crossing x-axis at (0, 0) and (2, 0).

The equation $y = 2^x$ represents the exponential curve as shown in Fig. 21.38. Lines $x = 0$ and $x = 2$ are shown in Fig. 21.38. The area bounded by these curves is shaded in Fig. 21.38. We slice the shaded region into vertical strips. For the approximating rectangle shown in Fig. 21.38, we have,

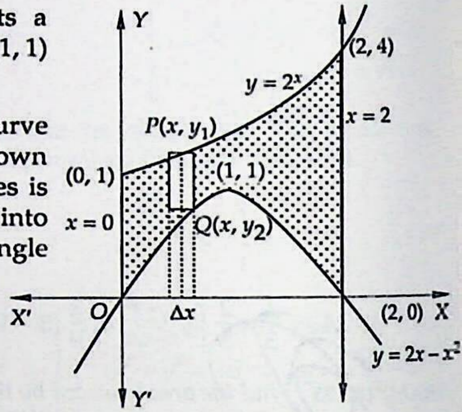

Fig. 21.38

Length $= (y_1 - y_2)$, width $= \Delta x$

$\therefore$ Area $= (y_1 - y_2)\,\Delta x$

The approximating rectangle can move horizontally between $x = 0$ and $x = 2$. So, required area A is given by

$$A = \int_0^2 (y_1 - y_2)\,dx$$

$$\Rightarrow \quad A = \int_0^2 (2^x - 2x + x^2)\,dx$$

$\left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = 2^x \text{ and } y = 2x - x^2 \text{ respectively. } \therefore y_1 = 2^x \text{ and } y_2 = 2x - x^2\right]$

$$\Rightarrow \quad A = \left[\frac{2^x}{\log 2} - x^2 + \frac{x^3}{3}\right]_0^2$$

$$\Rightarrow \quad A = \frac{4}{\log 2} - 4 + \frac{8}{3} - \frac{1}{\log 2} = \frac{3}{\log 2} - \frac{4}{3} \text{ sq. units}$$

EXAMPLE 35 *Sketch the curves and identify the region bounded by the curves $x = \frac{1}{2}$, $x = 2$, $y = \log x$ and $y = 2^x$. Find the area of this region.*

SOLUTION Since the inverse of a logarithmic function is an exponential function and vice-versa and these two curves are on the opposite sides of the line $y = x$. Thus, $y = 2^x$ and $y = \log x$ do not intersect. Their graphs are shown in Fig. 21.39. The shaded region in Fig. 21.39 shows the area bounded by the given curves. Let us slice this region into vertical strips as shown in Fig. 21.39. For the approximating rectangle shown in Fig. 21.39, we have

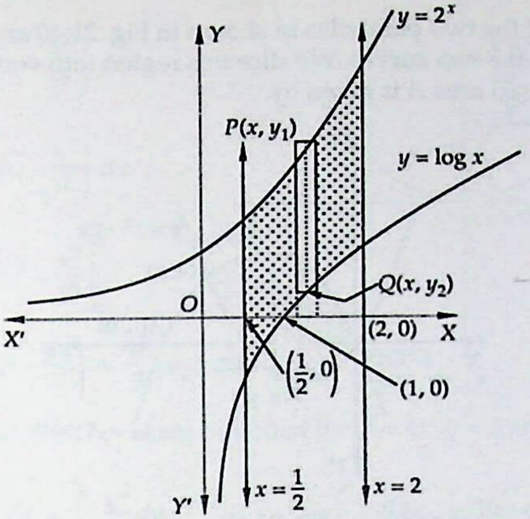

Fig. 21.39

$$\text{Length} = (y_1 - y_2), \text{ Width} = \Delta x$$

$$\text{Area} = (y_1 - y_2)\,dx$$

As the approximating rectangle can move horizontally between $x = \frac{1}{2}$ and $x = 2$.

So, required area A is given by

$$A = \int_{1/2}^{2} (y_1 - y_2)\,dx$$

$$\Rightarrow \quad A = \int_{1/2}^{2} (2^x - \log x)\,dx \qquad \left[\begin{array}{l} \because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on} \\ y = 2^x \text{ and } y = \log x \text{ respectively} \\ \therefore y_1 = 2^x \text{ and } y_2 = \log x \end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{2^x}{\log 2} - x\log x + x\right]_{1/2}^{2}$$

$$\Rightarrow \quad A = \left\{\frac{4}{\log 2} - 2\log 2 + 2\right\} - \left\{\frac{\sqrt{2}}{\log 2} + \frac{1}{2}\log 2 + \frac{1}{2}\right\}$$

$$\Rightarrow \quad A = \frac{(4-\sqrt{2})}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2} \text{ sq. units}$$

EXAMPLE 36 *Find the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$.*

SOLUTION The equation $y = 6x - x^2$ represents a parabola opening downward and cutting x– axis at O (0, 0) and A (6, 0).

Similarly, $y = x^2 - 2x$ also represents a parabola opening upward and crossing x-axis at O (0, 0) and B (2, 0). Solving the equations $y = x^2 - 2x$ and $y = 6x - x^2$, we find that the two parabolas intersect at O (0, 0) and C (4, 8).

The rough sketches of the two parabolas in shown in Fig. 21.40 and the shaded region is the region enclosed the two curves. We slice this region into vertical strips as shown in Fig. 21.40. So, required area A is given by

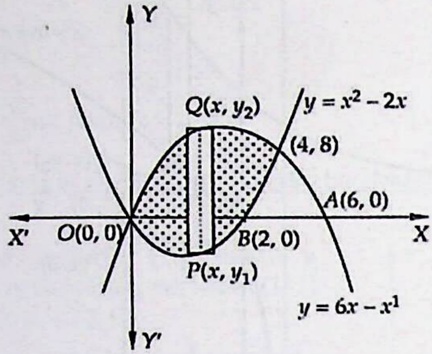

Fig. 21.40

$$A = \int_0^4 (y_2 - y_1)\, dx$$

$$\Rightarrow \quad A = \int_0^4 \{(6x - x^2) - (x^2 - 2x)\}\, dx \qquad \left[\begin{array}{l} \because (x_1, y_1) \text{ lies on } y = x^2 - 2x \ \therefore\ y_1 = x^2 - 2 \\ (x_2, y_2) \text{ lies on } y = 6x - x^2 \ \therefore\ y_2 = 6x - x^2 \end{array}\right]$$

$$\Rightarrow \quad A = \int_0^4 (8x - 2x^2)\, dx$$

$$\Rightarrow \quad A = \left[4x^2 - \frac{2}{3}x^3\right]_0^4 = 64 - \frac{128}{3} = \frac{64}{3} \text{ sq. units}$$

EXAMPLE 37 *Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0, y = 0, x = 4$ and $y = 4$ into three equal parts.* **[NCERT, CBSE 2009]**

SOLUTION Let A_1, A_2 and A_3 denote areas $OAPLO$, $OAPBO$ and $OBPMO$ respectively.

We have to prove that $A_1 = A_2 = A_3$.

$$\text{Now,} \quad A_1 = \int_0^4 y_1\, dx$$

$$\Rightarrow \quad A_1 = \int_0^4 \frac{x^2}{4}\, dx \qquad \left[\because (x_1, y_1) \text{ lies on } x^2 = 4y \ \therefore\ x^2 = 4y_1 \Rightarrow y_1 = \frac{x^2}{4}\right]$$

$$\Rightarrow \quad A_1 = \frac{1}{4}\int_0^4 x^2\, dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{4} \times \frac{64}{3} = \frac{16}{3} \text{ sq. units}$$

$$A_2 = \int_0^4 (y_2 - y_1)\, dx$$

$$\Rightarrow \quad A_2 = \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx$$

$$\Rightarrow \quad A_2 = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

$$\Rightarrow \quad A_2 = \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4$$

$$\Rightarrow \quad A_2 = \left(\frac{4}{3}\times 8 - \frac{64}{12}\right) = \frac{16}{3} \text{ sq. units}$$

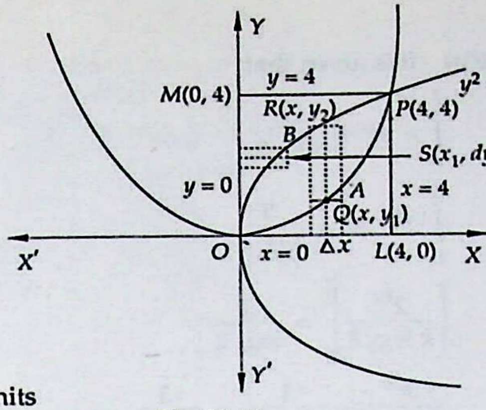

Fig. 21.41

and, A_3 = Area $OBMO$ = Area bounded by $y^2 = 4x$, $y = 0$ and $y = 4$

$$\Rightarrow \quad A_3 = \int_0^4 x_1\, dy = \int_0^4 \frac{y^2}{4} dy \qquad \left[\because S(x_1, y) \text{ lies on } y^2 = 4x \therefore y^2 = 4x_1\right]$$

$$\Rightarrow \quad A_3 = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{1}{4}\times\frac{64}{3} = \frac{16}{3}$$

Clearly, $A_1 = A_2 = A_3 = \dfrac{16}{3}$ sq. units

EXAMPLE 38 *If the area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1 square unit, then find the value of a.*

SOLUTION Clearly, $y = ax^2$ and $x = ay^2$ intersect at $(0, 0)$ and $\left(\dfrac{1}{a}, \dfrac{1}{a}\right)$.

$$\int_0^{1/a} (y_2 - y_1)\, dx = 1$$

$$\Rightarrow \quad \int_0^{1/a} \left\{\sqrt{\frac{x}{a}} - ax^2\right\} dx = 1$$

$$\Rightarrow \quad \left[\frac{2}{3\sqrt{a}}x^{3/2} - \frac{a}{3}x^3\right]_0^{1/a} = 1$$

$$\Rightarrow \quad \frac{2}{3\sqrt{a}}\times\frac{1}{a^{3/2}} - \frac{a}{3}\times\frac{1}{a^3} = 1$$

$$\Rightarrow \quad \frac{2}{3a^2} - \frac{1}{3a^2} = 1$$

$$\Rightarrow \quad \frac{1}{3a^2} = 1 \Rightarrow a^2 = \frac{1}{3} \Rightarrow a = \frac{1}{\sqrt{3}} \qquad [\because a > 0]$$

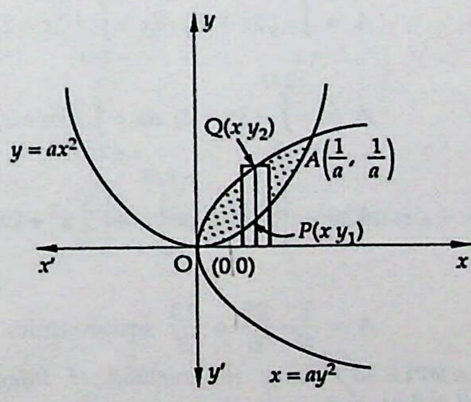

Fig. 21.42

EXAMPLE 39 *If the area above x-axis, bounded by the curves $y = 2^{kx}$ and $x = 0$ and $x = 2$ is $\dfrac{3}{\log_e 2}$, then find the value of k.*

SOLUTION It is given that

$$\int_0^2 y\,dx = \frac{3}{\log_e 2}$$

$$\Rightarrow \quad \int_0^2 2^{kx}\,dx = \frac{3}{\log_e 2}$$

$$\Rightarrow \quad \left[\frac{2^{kx}}{k\log_e 2}\right]_0^2 = \frac{3}{\log_e 2}$$

$$\Rightarrow \quad \frac{2^{2k}}{k\log_e 2} - \frac{1}{k\log_e 2} = \frac{3}{\log_e 2}$$

$$\Rightarrow \quad \frac{4^k - 1}{k} = 3$$

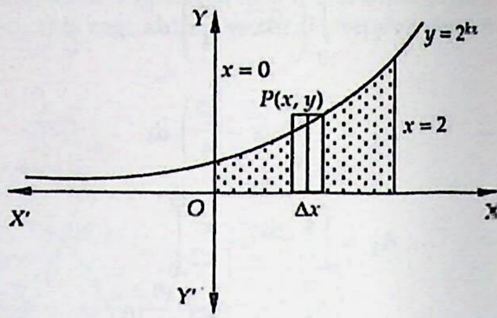

Fig. 21.43

Clearly, $k = 1$ satisfies this equation.

Hence, $k = 1$.

EXAMPLE 40 *Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -1$ and $x = 1$.* **[NCERT]**

SOLUTION Let $P(x, y)$ be any point on the line $y = 3x + 2$. The approximating rectangle shown in Fig. 21.44, has length $|y|$ and width dx. Also, it moves from $(-1, 0)$ to $(1, 0)$.

$\therefore$ Required area A is given by $A = \int_{-1}^{1} |y|\,dx$

$$\Rightarrow \quad A = \int_{-1}^{1} |3x+2|\,dx$$

$$\Rightarrow \quad A = \int_{-1}^{-2/3} |3x+2|\,dx + \int_{-2/3}^{1} |3x+2|\,dx$$

$$\Rightarrow \quad A = -\int_{-1}^{-2/3} (3x+2)\,dx + \int_{-2/3}^{1} (3x+2)\,dx$$

$$\Rightarrow \quad A = -\left[\frac{3}{2}x^2 + 2x\right]_{-1}^{-2/3} + \left[\frac{3}{2}x^2 + 2x\right]_{-2/3}^{1}$$

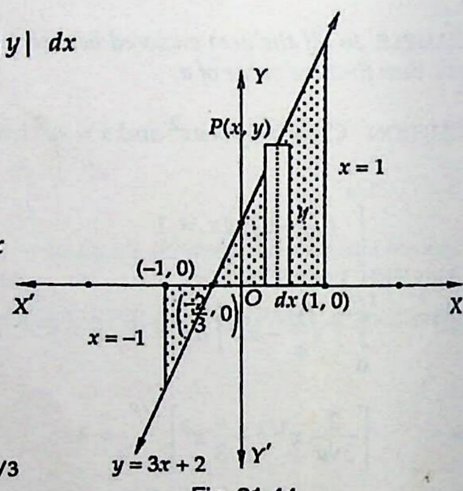

Fig. 21.44

$$\Rightarrow \quad A = \frac{1}{6} + \frac{25}{6} = \frac{13}{3} \text{ square units}$$

EXAMPLE 41 *Using the method of integration find the area-bounded by the curve $|x| + |y| = 1$.* **[NCERT]**

SOLUTION The equation of the curve is

$$|x| + |y| = 1 \Leftrightarrow \begin{cases} x + y = 1 \text{ when } x \ge 0, y \ge 0 \\ -x + y = 1 \text{ when } x < 0, y \ge 0 \\ x - y = 1 \text{ when } x \ge 0, y < 0 \\ -x - y = 1 \text{ when } x < 0, y < 0 \end{cases}$$

Lines $x + y = 1, -x + y = 1, x - y = 1$ and $-x - y = 1$ enclose a square as shown in Fig. 21.45.

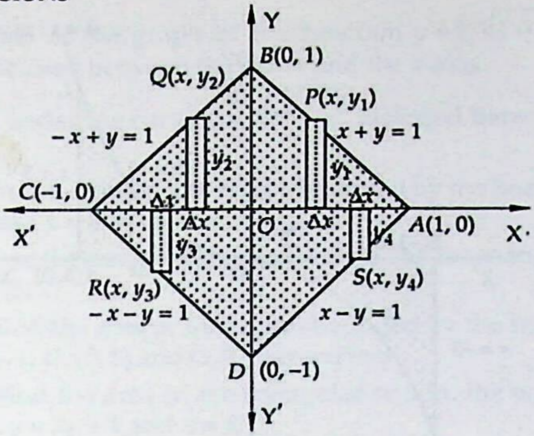

Fig. 21.45

Required area A is given by

$$A = \int_0^1 y_1\, dx + \int_{-1}^0 y_2\, dx + \int_{-1}^0 |y_3|\, dx + \int_0^1 |y_4|\, dx$$

$$\Rightarrow \quad A = \int_0^1 (1-x)\, dx + \int_{-1}^0 (1+x)\, dx + \int_{-1}^0 |-x-1|\, dx + \int_0^1 |x-1|\, dx$$

$$\Rightarrow \quad A = \int_0^1 (1-x)\, dx + \int_{-1}^0 (1+x)\, dx + \int_{-1}^0 (x+1)\, dx + \int_0^1 (1-x)\, dx$$

$$\Rightarrow \quad A = 2\int_{-1}^0 (x+1)\, dx + 2\int_0^1 (1-x)\, dx$$

$$\Rightarrow \quad A = 2\left[\frac{x^2}{2}+x\right]_{-1}^0 + 2\left[x-\frac{x^2}{2}\right]_0^1$$

$$\Rightarrow \quad A = 2\left\{0-\left(\frac{1}{2}-1\right)\right\} + 2\left\{\left(1-\frac{1}{2}\right)-0\right\}$$

$$\Rightarrow \quad A = 1+1 = 2 \text{ square units}$$

EXAMPLE 42 *Find the area bounded by the curve $y = x\,|x|$, x-axis and the ordinates $x = -3$ and $x = 3$.*

SOLUTION The equation of the curve is

$$y = x\,|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

The graph of $y = x\,|x|$ is shown in Fig. 21.46 and the area bounded by $y = x\,|x|$, x-axis and the ordinates $x = -3$ and $x = 3$ is shaded in Fig 21.46.

Clearly, $y = x\,|x|$, being an odd function is symmetric in opposite quadrants.

$\therefore$ Required area = 2 (Area shaded in first quadrant)

$$= 2\int_0^3 x^2\, dx = 2 \times \left[\frac{x^3}{3}\right]_0^3 = 2 \times 9 \text{ sq. units} = 18 \text{ sq. units}$$

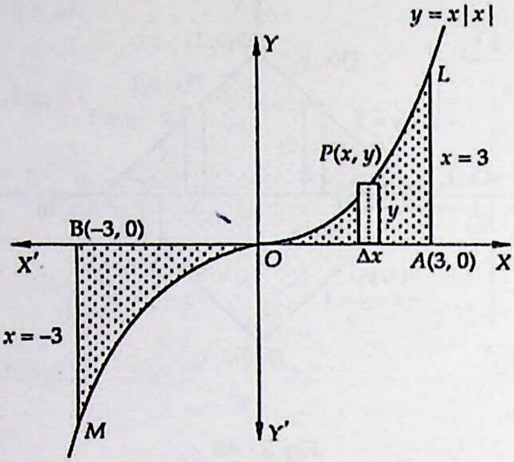

Fig. 21.46

EXERCISE 21.1

1. Using integration, find the area of the region bounded between the line $x = 2$ and the parabola $y^2 = 8x$.
2. Find the area of the region between the curves $y^2 = 4x$ and $x = 3$.
3. Find the area of the region bounded by the parabola $y^2 = 4ax$ and the line $x = a$.
4. Find the area lying above the x-axis and under the parabola $y = 4x - x^2$.
5. Find the area bounded by the parabola $x = 4 - y^2$ and the y-axis.
6. Calculate the area of the region bounded by the parabolas $y^2 = x$ and $x^2 = y$.
7. Find the area of the region common to the parabolas $4y^2 = 9x$ and $3x^2 = 16y$.
8. Draw a rough sketch to indicate the region bounded between the curve $y^2 = 4x$ and the line $x = 3$. Also, find the area of this region.
9. Make a rough sketch of the graph of the function $y = 4 - x^2$, $0 \le x \le 2$ and determine the area enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$.
10. Sketch the graph of $y = \sqrt{x+1}$ in $[0, 4]$ and determine the area of the region enclosed by the curve, the x-axis and the lines $x = 0$, $x = 4$.
11. Find the area under the curve $y = \sqrt{6x+4}$ above x-axis from $x = 0$ to $x = 2$. Draw a sketch of the curve also.
12. Find the area bounded by the curve $y = 4 - x^2$ and the lines $y = 0$, $y = 3$.
13. Draw the rough sketch of $y^2 + 1 = x$, $x \le 2$. Find the area enclosed by the curve and the line $x = 2$.
14. Draw a rough sketch of the graph of the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and evaluate the area of the region under the curve and above the x-axis. [NCERT]
15. Sketch the region $\{(x, y) : 4x^2 + 9y^2 = 36\}$ and find its area, using integration.
16. Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.
17. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.
18. Find the area of the region $\left\{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \le \frac{x}{a} + \frac{y}{b}\right\}$.

19. Draw a rough sketch of the graph of the function $y = 2\sqrt{1-x^2}$, $x \in [0, 1]$ and evaluate the area enclosed between the curve and the x-axis.

20. Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$.

21. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$.

22. Using integration, find the area of the region bounded by the triangle whose vertices are (2, 1), (3, 4) and (5, 2).

23. Using integration, find the area of the region bounded by the triangle ABC whose vertices A, B, C are (–1, 1), (0, 5) and (3, 2) respectively. **[CBSE 2008]**

24. Using integration, find the area of the triangular region, the equations of whose sides are $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

25. Find the area of the region $\{(x, y) : y^2 \le 8x, x^2 + y^2 \le 9\}$

26. Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

27. Find the area of the region between the circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$. **[NCERT]**

28. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

29. Using definite integrals, find the area of the circle $x^2 + y^2 = a^2$.

30. Using integration, find the area of the region bounded by the following curves, after making a rough sketch: $y = 1 + |x+1|$, $x = -2$, $x = 3$, $y = 0$.

31. Sketch the graph $y = |x - 5|$. Evaluate $\int_0^1 |x-5|\, dx$. What does this value of the integral represent on the graph. **[NCERT]**

32. Sketch the graph $y = |x + 3|$. Evaluate $\int_{-6}^{0} |x+3|\, dx$. What does this integral represent on the graph? **[NCERT]**

33. Sketch the graph $y = |x + 1|$. Evaluate $\int_{-4}^{2} |x+1|\, dx$. What does the value of this integral represent on the graph?

34. Find the area of the region bounded by the curve $xy - 3x - 2y - 10 = 0$, x-axis and the lines $x = 3$, $x = 4$.

35. Draw a rough sketch of the curve $y = \frac{\pi}{2} + 2\sin^2 x$ and find the area between x-axis, the curve and the ordinates $x = 0$, $x = \pi$.

36. Draw a rough sketch of the curve $y = \frac{x}{\pi} + 2\sin^2 x$ and find the area between the x-axis, the curve and the ordinates $x = 0$ and $x = \pi$.

37. Find the area bounded by the curve $y = \cos x$, x-axis and the ordinates $x = 0$ and $x = 2\pi$. **[NCERT]**

38. Make a rough sketch of each of the following curves and determine the area of the region bounded by the curve and the axes:

(i) $y = \sin 2x,\ 0 \le x \le \frac{\pi}{2}$ (ii) $y = \cos^2 x,\ 0 \le x \le \frac{\pi}{2}$

(iii) $y = \cos 2x,\ 0 \le x \le \frac{\pi}{4}$ (iv) $y = \cos 3x,\ 0 \le x \le \frac{\pi}{6}$

(v) $y = \sin^2 \frac{x}{2},\ 0 \le x \le \frac{\pi}{2}$ (vi) $y = \tan^2 x,\ 0 \le x \le \frac{\pi}{4}$

(vii) $y = \sin^2 x,\ 0 \le x \le \frac{\pi}{2}$

39. Show that the areas under the curves $y = \sin x$ and $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{3}$ are in the ratio 2 : 3.

40. Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ *between* $x = 0$ *and* $x = \pi$.

41. Draw a rough sketch of the region $\{(x, y) : y^2 \le 3x, 3x^2 + 3y^2 \le 16\}$ and find the area enclosed by the region using method of integration.

42. Draw a rough sketch of the region $\{(x, y) : y^2 \le 5x, 5x^2 + 5y^2 \le 36\}$ and find the area enclosed by the region using method of integration.

43. Draw a rough sketch and find the area of the region bounded by the two parabolas $y^2 = 4x$ and $x^2 = 4y$ by using methods of integration.

44. Find the area included between the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

45. Prove that the area in the first quadrant enclosed by the x-axis, the line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$ is $\pi/3$.

46. Find the area enclosed by the curves $y = |x - 1|$ and $y = -|x - 1| + 1$.

47. Find the area common to the circle $x^2 + y^2 = 16a^2$ and the parabola $y^2 = 6ax$. **[CBSE 2004]**

48. Find the area, lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$. **[CBSE 2008]**

49. Find the area enclosed by the parabolas $y = 5x^2$ and $y = 2x^2 + 9$.

50. Prove that the area common to the two parabolas $y = 2x^2$ and $y = x^2 + 4$ is $\frac{32}{3}$ sq. units.

51. Compute the area of the figure bounded by the straight lines $x = 0, x = 2$ and the curves $y = 2^x, y = 2x - x^2$.

52. Find the area enclosed by the curves $3x^2 + 5y = 32$ and $y = |x - 2|$.

53. Find the area bounded by the curves $4y = |4 - x^2|, x^2 + y^2 = 25$ and $x = 0$ above the x-axis.

54. Find the area in the first quadrant bounded by the parabola $y = 4x^2$ and the lines $x = 0, y = 1$ and $y = 4$.

55. Find the area of the region bounded by the parabola $y^2 = 2x + 1$ and the line $x - y - 1 = 0$.

56. Find the area of the region bounded by the curves $y = x - 1$ and $(y-1)^2 = 4(x+1)$.

57. Find the area bounded by the parabolas $y = 6x - x^2$ and $y = x^2 - 2x$..

58. Find the area bounded by the parabola $y = 2 - x^2$ and the straight line $y + x = 0$.

59. Find the area enclosed by the parabolas $y = 4x - x^2$ and $y = x^2 - x$.

60. Sketch the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 1$. Also, find the area of this region.

61. Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

62. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$?

63. Find the area of the figure bounded by the curves $y = |x - 1|$ and $y = 3 - |x|$.

64. Find the area of the region bounded by $y = |x - 1|$ and $y = 1$.

65. Find the area bounded by the curve $y = x\,|x|$, x-axis and the ordinates $x = 1$ and $x = -1$.

66. Find the area bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the ordinates $x = ae$ and $x = 0$, where $b^2 = a^2(1 - e^2)$ and $e < 1$. **[NCERT]**

67. Find the area of the region bounded by $x^2 = 16y$, $y = 1$, $y = 4$ and the y-axis in the first quadrant.

68. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

69. Find the area of the circle $x^2 + y^2 = 16$ which is exterior to the parabola $y^2 = 6x$. **[CBSE 2007]**

70. Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$. **[NCERT, CBSE 2005]**

71. Make a sketch of the region given below and find its area using integration.

$$\left\{(x, y) : 0 \le y \le x^2 + 3;\ 0 \le y \le 2x + 3;\ 0 \le x \le 3\right\}$$

72. Using integration, find the area of the region bounded by the line $y - 1 = x$, the x-axis and the ordinates $x = -2$ and $x = 3$. **[CBSE 2002]**

73. Find the area of the region bounded by the curve $y = \sqrt{1 - x^2}$, line $y = x$ and the positive x-axis. **[CBSE 2005]**

74. Find the area bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$. **[CBSE 2005]**

75. Find the area of the region enclosed between the two curves $x^2 + y^2 = 9$ and $(x-3)^2 + y^2 = 9$. **[CBSE 2009]**

76. Using integration, find the area of the region enclosed between the circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$. **[CBSE 2010]**

77. Using integration, find the area of the following region:

$$\left\{(x, y) : \frac{x^2}{9} + \frac{y^2}{4} \le 1 \le \frac{x}{3} + \frac{y}{2}\right\}$$ **[CBSE 2010]**

78. Using integration find the area of the region:

$$\{(x, y) : |x-1| \le y \le \sqrt{5-x^2}\}$$ **[CBSE 2010]**

79. Using integration, find the area of the triangle ABC coordinates of whose vertices are $A(4, 1)$, $B(6, 6)$ and $C(8, 4)$.

ANSWERS

1. $\frac{32}{3}$ sq. units
2. $8\sqrt{3}$ sq. units
3. $\frac{8}{3}a^2$ sq. units
4. $\frac{32}{3}$ sq. units
5. $\frac{32}{3}$ sq. units
6. $\frac{1}{3}$ sq. unit
7. 4 sq. units
8. $8\sqrt{3}$ sq. units
9. $\frac{16}{3}$ sq. units
10. $\frac{2}{3}(5^{3/2}-1)$
11. $\frac{56}{9}$ sq. units
12. $\frac{28}{3}$ sq. units
13. $\frac{4}{3}$ sq. units
14. 3π sq. units
15. 6π sq. units
16. $\frac{1}{6}$ sq. unit
17. $\frac{1}{3}$ sq. unit
18. $(\pi-2)\frac{ab}{4}$
19. $\frac{\pi}{2}$ sq. units
21. $\frac{\pi a^2}{4}$ sq. units
21. 96 sq. units
22. 4 sq. units
23. $\frac{15}{2}$ sq. units
24. 8 sq. units
25. $2\left(\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\frac{1}{3}\right)$ sq. units
26. $\frac{4}{3}(4\pi+\sqrt{3})$ sq. units
27. $\left(\frac{8\pi}{3}-2\sqrt{3}\right)$ sq. units
28. $\frac{9}{2}$ sq. units
29. πa^2 sq. units
30. $\frac{27}{2}$ sq. units
31. $\frac{9}{2}$ sq. units
32. 9 sq. units
33. 9 sq. units
34. $3+16\log 2$ sq. units
35. $\frac{\pi}{2}(\pi+2)$ sq. units
36. $\frac{3\pi}{2}$ sq. units
37. 4 sq. units
38. (i) 1 sq. unit (ii) $\frac{\pi}{4}$ sq. units (iii) $\frac{1}{2}$ sq. unit (iv) $\frac{1}{3}$ sq. unit (v) $\left(\frac{\pi}{4}-\frac{1}{2}\right)$ sq. units (vi) $\left(1-\frac{\pi}{4}\right)$ sq. unit (vii) $\frac{\pi}{4}$ sq. unit
40. Each equal to $\frac{\pi}{2}$ sq. unit.
41. $\frac{4}{\sqrt{3}}a^{3/2}+\frac{8\pi}{3}-a\sqrt{16/3-a^2}-\frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)$, where $a=\frac{-9+\sqrt{273}}{6}$
42. $\frac{4\sqrt{5}}{3}a^{3/2}+\frac{18\pi}{5}-a\sqrt{36/5-a^2}-\frac{36}{5}\sin^{-1}\left(\frac{a\sqrt{5}}{6}\right)$, $a=\frac{-25+\sqrt{1345}}{10}$
43. $\frac{16}{3}$ sq. units.
44. $\frac{16}{3}ab$ sq. units
46. $\frac{1}{2}$ sq. units
47. $\frac{4a^2}{3}(4\pi+\sqrt{3})$ sq. units
48. $\left(4\pi-\frac{32}{3}\right)$ sq. units
49. $12\sqrt{3}$ sq. units
51. $\frac{3}{\log 2}-\frac{4}{3}$ sq. units
52. $\frac{33}{2}$ sq. units
53. $25\sin^{-1}\left(\frac{4}{5}\right)+4$ sq. units

54. $\frac{7}{3}$ sq. units
55. $\frac{16}{3}$ sq. units
56. $\frac{64}{3}$ sq. units
57. $\frac{64}{3}$ sq. units
58. $\frac{9}{2}$ sq. units
59. $\frac{125}{24}$ sq. units
60. $\frac{11}{6}$ sq. units
61. 4 sq. units
62. 121:4
63. 4 sq. units
64. 16 sq. units
65. $\frac{2}{3}$ sq.units
66. $ab\left[e\sqrt{1-e^2}+\sin^{-1}e\right]$
67. $\frac{56}{3}$ sq. units.
68. 4π sq. units
69. $\frac{9}{2}$ sq. units
70. $\frac{9}{2}$ sq. units
72. $\frac{17}{2}$ sq. units
73. $\frac{\pi}{8}$ sq. units
74. $\frac{15}{2}$ sq. units
75. $6\pi-\frac{9\sqrt{3}}{2}$ sq. units
76. $\frac{8\pi}{3}-2\sqrt{3}$ sq. units
77. $\left(\frac{3\pi}{2}-3\right)$ sq. units
78. $\left\{\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{5}}\right)-\frac{3}{2}\right\}$ sq. units
79. 7 sq. units

HINTS TO SELECTED PROBLEMS

9. Required area $=\int_0^2 (4-x^2)\,dx$

10. Required area $=\int_0^4 \sqrt{x+1}\,dx$

11. Required area $=\int_0^2 \sqrt{6x+4}\,dx$

12. Required area $=2\int_0^3 \sqrt{4-y}\,dy$

13. Required area $=2\int_1^2 \sqrt{x-1}\,dx$

14. Required area $=\int_{-2}^2 y\,dx=2\int_0^2 \frac{3}{2}\sqrt{4-x^2}\,dx$

15. Given region is the region enclosed by ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.

So, required area $=4\int_0^3 y\,dx=4\int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx$

19. We have, $y=2\sqrt{1-x^2}\Rightarrow y^2=4-4x^2\Rightarrow\frac{x^2}{1}+\frac{y^2}{4}=1$. This is the equation of an ellipse. So, $y=2\sqrt{1-x^2}$ represents the portion of the ellipse lying in the first quadrant.

So, Required area $=\int_0^1 y\,dx=\int_0^1 2\sqrt{1-x^2}\,dx$

20. Required area $=\int_0^a y\,dx=\int_0^a \sqrt{a^2-x^2}\,dx$

23. Required area $=\int_2^8 y\,dx$

29. Required area $= 4\int_0^a y\,dx = 4\int_0^a \sqrt{a^2 - x^2}\,dx$

34. Required area $= \int_3^4 y\,dx = \int_3^4 \left(\frac{3x+10}{x-2}\right) = \int_3^4 \left(3 + \frac{16}{x-2}\right)dx$

35. Required area $= \int_0^\pi y\,dx = \int_0^\pi \left(\frac{\pi}{2} + 2\sin^2 x\right)dx = \int_0^\pi \left(\frac{\pi}{2} + 1 - \cos 2x\right)dx$

36. Required area $= \int_0^\pi y\,dx = \int_0^\pi \left(\frac{x}{\pi} + 2\sin^2 x\right)dx = \int_0^\pi \left(\frac{x}{\pi} + 1 - \cos 2x\right)dx$

37. Required area $= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/2} -\cos x\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$

75. See Example 22.

MULTIPLE CHOICE QUESTIONS (MCQs)

1. If the area above the x-axis, bounded by the curves $y = 2^{kx}$ and $x = 0$, and $x = 2$ is $\frac{3}{\log_e 2}$, then the value of k is
   (a) 1/2 (b) 1 (c) −1 (d) 2
2. The area included between the parabolas $y^2 = 4x$ and $x^2 = 4y$ is (in square units)
   (a) 4/3 (b) 1/3 (c) 16/3 (d) 8/3
3. The area bounded by the curve $y = \log_e x$ and x-axis and the straight line $x = e$ is
   (a) e sq. units (b) 1 sq. units
   (c) $1 - \frac{1}{e}$ sq. units (d) $1 + \frac{1}{e}$ sq.units
4. The area bounded by $y = 2 - x^2$ and $x + y = 0$ is
   (a) $\frac{7}{2}$ sq. units (b) $\frac{9}{2}$ sq. units
   (c) 9 sq. units (d) none of these
5. The area bounded by the parabola $x = 4 - y^2$ and y-axis, in square units, is
   (a) $\frac{3}{32}$ (b) $\frac{32}{3}$ (c) $\frac{33}{2}$ (d) $\frac{16}{3}$
6. If A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \pi/4$, then for $x > 2$
   (a) $A_n + A_{n-2} = \frac{1}{n-1}$ (b) $A_n + A_{n-2} < \frac{1}{n-1}$
   (c) $A_n - A_{n-2} = \frac{1}{n-1}$ (d) none of these

7. The area of the region formed by $x^2 + y^2 - 6x - 4y + 12 \le 0$, $y \le x$ and $x \le 5/2$ is

(a) $\frac{\pi}{6} - \frac{\sqrt{3}+1}{8}$ (b) $\frac{\pi}{6} + \frac{\sqrt{3}+1}{8}$

(c) $\frac{\pi}{6} - \frac{\sqrt{3}-1}{8}$ (d) none of these

8. The area enclosed between the curves $y = \log_e (x + e)$, $x = \log_e \left(\frac{1}{y}\right)$ and the x-axis is

(a) 2 (b) 1 (c) 4 (d) none of these

9. The area of the region bounded by the parabola $(y-2)^2 = x - 1$, the tangent to it at the point with the ordinate 3 and the x-axis is

(a) 3 (b) 6 (c) 7 (d) none of these

10. The area bounded by the curves $y = \sin x$ between the ordinates $x = 0$, $x = \pi$ and the x-axis is

(a) 2 sq. units (b) 4 sq. units

(c) 3 sq. units (d) 1 sq. units

11. The area bounded by the parabola $y^2 = 4ax$ and $x^2 = 4ay$ is

(a) $\frac{8a^3}{3}$ (b) $\frac{16a^2}{3}$ (c) $\frac{32a^2}{3}$ (d) $\frac{64a^2}{3}$

12. The area bounded by the curve $y = x^4 - 2x^3 + x^2 + 3$ with x-axis and ordinates corresponding to the minima of y is

(a) 1 (b) $\frac{91}{30}$ (c) $\frac{30}{9}$ (d) 4

13. The area bounded by the parabola $y^2 = 4ax$, latusrectum and x-axis is

(a) 0 (b) $\frac{4}{3}a^2$ (c) $\frac{2}{3}a^2$ (d) $\frac{a^2}{3}$

14. The area of the region $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$ is

(a) $\frac{\pi}{5}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2} - \frac{1}{2}$ (d) $\frac{\pi^2}{2}$

15. The area common to the parabola $y = 2x^2$ and $y = x^2 + 4$ is

(a) $\frac{2}{3}$ sq. units (b) $\frac{3}{2}$ sq. units

(c) $\frac{32}{3}$ sq. units (d) $\frac{3}{32}$ sq. units

16. The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is given by

(a) $\frac{45}{7}$ (b) $\frac{25}{4}$ (c) $\frac{\pi}{18}$ (d) $\frac{9}{2}$

17. The ratio of the areas between the curves $y = \cos x$ and $y = \cos 2x$ and x-axis from $x = 0$ to $x = \pi/3$ is

(a) $1:2$ (b) $2:1$ (c) $\sqrt{3}:1$ (d) none of these

18. The area between x-axis and curve $y = \cos x$ when $0 \le x \le 2\pi$ is

(a) 0 (b) 2 (c) 3 (d) 4

19. Area bounded by parabola $y^2 = x$ and straight line $2y = x$ is

(a) 4/3 (b) 1 (c) 2/3 (d) 1/3

20. The area bounded by the curve $y = 4x - x^2$ and the x-axis is

(a) $\frac{30}{7}$ sq. units (b) $\frac{31}{7}$ sq. units

(c) $\frac{32}{3}$ sq. units (d) $\frac{34}{3}$ sq. units

21. Area enclosed between the curve $y^2(2a - x) = x^3$ and the line $x = 2a$ above x-axis is

(a) πa^2 (b) $\frac{3}{2}\pi a^2$ (c) $2\pi a^2$ (d) $3\pi a^2$

22. The area of the region (in square units) bounded by the curve $x^2 = 4y$, line $x = 2$ and x-axis is

(a) 1 (b) 2/3 (c) 4/3 (d) 8/3

23. The area bounded by the curve $y = f(x)$, x-axis, and the ordinates $x = 1$ and $x = b$ is $(b-1)\sin(3b+4)$. Then, $f(x)$ is

(a) $(x-1)\cos(3x+4)$ (b) $\sin(3x+4)$

(c) $\sin(3x+4) + 3(x-1)\cos(3x+4)$ (d) none of these

24. The area bounded by the curve $y^2 = 8x$ and $x^2 = 8y$ is

(a) $\frac{16}{3}$ sq. units (b) $\frac{3}{16}$ sq. units

(c) $\frac{14}{3}$ sq. units (d) $\frac{3}{14}$ sq. units

25. The area bounded by the parabola $y^2 = 8x$, the x-axis and the latus rectum is

(a) $\frac{16}{3}$ (b) $\frac{23}{3}$ (c) $\frac{32}{3}$ (d) $\frac{16\sqrt{2}}{3}$

26. Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

(a) -9 (b) $\frac{-15}{4}$ (c) $\frac{15}{4}$ (d) $\frac{17}{4}$

27. The area bounded by the curve $y = x\,|x|$ and the ordinates $x = -1$ and $x = 1$ is given by

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

28. The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$ is

(a) $2(\sqrt{2} - 1)$ (b) $\sqrt{2} - 1$

(c) $\sqrt{2} + 1$ (d) $\sqrt{2}$

29. The area of the circle $x^2 + y^2 = 16$ enterior to the parabola $y^2 = 6x$ is

(a) $\frac{4}{3}(4\pi - \sqrt{3})$ (b) $\frac{4}{3}(4\pi + \sqrt{3})$

(c) $\frac{4}{3}(8\pi - \sqrt{3})$ (d) $\frac{4}{3}(8\pi + \sqrt{3})$

30. Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

(a) $2(\pi - 2)$ (b) $\pi - 2$

(c) $2\pi - 1$ (d) $2(\pi + 2)$

31. Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$

32. Area lying in first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$, is

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

33. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$, is

(a) 2 (b) $\frac{9}{4}$ (c) $\frac{9}{3}$ (d) $\frac{9}{2}$

ANSWERS

1. (b)	2. (c)	3. (b)	4. (b)	5. (b)	6. (a)	7. (c)	8. (a)
9. (c)	10. (a)	11. (b)	12. (b)	13. (b)	14. (c)	15. (c)	16. (d)
17. (b)	18. (d)	19. (a)	20. (c)	21. (b)	22. (b)	23. (c)	24. (a)
25. (c)	26. (d)	27. (c)	28. (b)	29. (c)	30. (b)	31. (b)	32. (a)
33. (b)							

SUMMARY

1. Let $f(x)$ be a continuous function defined on $[a, b]$. Then, the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\, dx \quad or, \int_a^b y\, dx$$

2. If the curve $y = f(x)$ lies below x-axis, then the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is negative. So, area is given by

$$\left| \int_a^b y\, dx \right|$$

3. The area bounded by the curve $x = f(y)$, the y axis and the abscissae $y = c$ *and* $y = d$ is given by

$$\int_c^d f(y)\, dy \quad or, \int_c^d x\, dy$$

22

DIFFERENTIAL EQUATIONS

22.1 SOME DEFINITIONS

DIFFERENTIAL EQUATION *An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a differential equation.*

For instance,

(i) $\frac{dy}{dx} = 2xy$

(ii) $\frac{d^2y}{dx^2} = 4x$

(iii) $\frac{dy}{dx} = \sin x + \cos x$

(iv) $\frac{dy}{dx} + 2xy = x^3$

(v) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

(vi) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$

(vii) $y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

(viii) $(x^2 + y^2)\, dx - 2xy\, dy = 0$

(ix) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(1 + \frac{dy}{dx}\right)^3 = 0$

are examples of differential equations.

ORDER OF A DIFFERENTIAL EQUATION *The order of a differential equation is the order of the highest order derivative appearing in the equation.*

ILLUSTRATION 1 *In the equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$, the order of highest order derivative is 2. So, it is a differential equation of order 2. The equation*

$$\frac{d^3y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0$$

is of order 3, because the order of highest order derivative in it is 3.

NOTE *The order of a differential equation is a positive integer.*

DEGREE OF A DIFFERENTIAL EQUATION *The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions.*

In other words, the degree of a differential equation is the power of the highest order derivative occurring in a differential equation when it is written as a polynomial in differential coefficients.

ILLUSTRATION 2 *Consider the differential equation* $\frac{d^3y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0.$

In this equation the power of highest order derivative is 1. So, it is a differential equation of degree 1.

ILLUSTRATION 3 *Consider the differential equation*

$$x\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx}\right)^4 + y^2 = 0$$

In this equation, the order of the highest order derivative is 3 and its power is 2. So, it is a differential equation of order 3 and degree 2.

ILLUSTRATION 4 *The differential equation*

$$y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

when expressed as a polynomial in derivatives becomes

$$(x^2 - 1)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2 - 1) = 0$$

In this equation, the power of highest order derivative is 2. So, its degree is 2.

ILLUSTRATION 5 *Consider the differential equation*

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\left(\frac{d^2y}{dx^2}\right)$$

The order of highest order differential coefficient is 2. So, its order is 2. To find its degree we express the differential equation as a polynomial in derivatives. When expressed as a polynomial in derivatives it becomes

$$k^2\left(\frac{d^2y}{dx^2}\right)^2 - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0$$

Clearly, the power of the highest order differential coefficient is 2. So, its degree is 2.

ILLUSTRATION 6 *The differential equation* $(x^2 + y^2)\,dx - 2xydy = 0$ *may be written as*

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

So, it is a differential equation of order 1 and degree 1.

ILLUSTRATION 7 *Consider the differential equation*

$$y = px + \sqrt{a^2p^2 + b^2}, \text{ where } p = \frac{dy}{dx}$$

The order of the highest order derivative is 1. So, its order is 1. To determine its degree we express it as a polynomial in differential coefficients as follows:

$$y = px + \sqrt{a^2p^2 + b^2}$$

$$\Rightarrow \quad (y - px)^2 = a^2p^2 + b^2$$

$$\Rightarrow \quad p^2(x^2 - a^2) - 2xyp + y^2 - b^2 = 0$$

$$\Rightarrow \quad (x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + y^2 - b^2 = 0$$

Clearly, the power of highest order differential coefficient is 2. So, its degree is 2.

ILLUSTRATION 8 *Consider the differential equation*

$$\left(\frac{d^2 y}{dx^2}\right)^2 + \sin\left(\frac{dy}{dx}\right) = 0$$

We observe that the highest order derivative present in the differential equation is $\frac{d^2 y}{dx^2}$. So, its order is 2. Since the differential equation cannot be expressed as a polynomial in differential coefficients. So, its degree is not defined.

LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS *A differential equation is a linear differential equation if it is expressible in the form*

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

where $P_0, P_1, P_2, \ldots, P_{n-1}, P_n$ and Q are either constants or functions of independent variable x. Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. Otherwise, it is a non linear differential equation.

It follows from the above definition that a differential equation will be non-linear differential equation if

(i) its degree is more than one.

(ii) any of the differential coefficient has exponent more than one

(iii) exponent of the dependent variable is more than one.

(iv) products containing dependent variable and its differential coefficients are present.

ILLUSTRATION 9 *The differential equation*

$$\left(\frac{d^3 y}{dx^3}\right)^3 - 6\left(\frac{d^2 y}{dx^2}\right)^2 - 4y = 0,$$

is a non-linear differential equation, because its degree is 3, more than one.

ILLUSTRATION 10 *The differential equation*

$$\frac{d^2 y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 9y = x,$$

is a non-linear differential equation, because differential coefficient $\frac{dy}{dx}$ has exponent 2.

ILLUSTRATION 11 *The differential equation $(x^2 + y^2)\,dx - 2xy\,dy = 0$ is a non-linear differential equation, because the exponent of dependent variable y is 2 and it involves the product of y and $\frac{dy}{dx}$.*

ILLUSTRATION 12 *Consider the differential equation*

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$$

This is a linear differential equation of order 2 and degree 1.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Determine the order and degree of each of the following differential equations. State also if they are linear or non-linear.*

(i) $\frac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2} = K$ (ii) $\frac{d^2y}{dx^2} = 1 + \sqrt{\frac{dy}{dx}}$

(iii) $y = \frac{dy}{dx} + \frac{c}{dy/dx}$ (iv) $y + \frac{dy}{dx} = \frac{1}{4}\int y\,dx$

SOLUTION (i) The given differential equation when written as a polynomial in derivatives becomes

$$K^2\left(\frac{d^2y}{dx^2}\right)^2 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ and its power is 2. Therefore, the given differential equation is a non-linear differential equation of second order and second degree.

(ii) The given differential equation when written as a polynomial in derivatives becomes

$$\left(\frac{d^2y}{dx^2} - 1\right)^2 = \frac{dy}{dx} \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 1 = 0$$

Clearly, it is a non-linear differential equation of second order and second degree.

(iii) The given differential equation when written as a polynomial in $\frac{dy}{dx}$ is

$$\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + c = 0$$

Clearly, it is a non-linear differential equation of order 1 and degree 2.

(iv) We have,

$$y + \frac{dy}{dx} = \frac{1}{4}\int y\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{d^2y}{dx^2} = \frac{1}{4}y \qquad \text{[On differentiating w.r. to } x\text{]}$$

Clearly, this is a differential equation of order 2 and degree 1. Also, it is a linear differential equation.

EXAMPLE 2 *In each of the following differential equations indicate its degree, wherever possible. Also, give the order of each of them.*

(i) $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$ (ii) $\frac{d^5y}{dx^5} + e^{dy/dx} + y^2 = 0$

(iii) $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$ [NCERT] (iv) $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

SOLUTION (i) The highest order derivative present in the differential equation is $\frac{dy}{dx}$. So, it is of order 1.

Clearly, LHS of the differential equation cannot be expressed as a polynomial in $\frac{dy}{dx}$. So, its degree is not defined.

(ii) The highest order differential coefficient present in the differential equation is $\frac{d^5y}{dx^5}$. So, it is of order 5.

We observe that the LHS of the differential equation is not expressible as a polynomial in $\frac{dy}{dx}$. So, its degree is not defined.

(iii) The highest order derivative present in the given differention equation is 4, so the order of the given differential equation is 4. As it is not expressible as a polynomial in differential coefficients. So, its degree is not defined.

(iv) The order of the highest order derivative present in the given differential equation is 2. So, its order is 2. The given differential equation is not expressible as a polynomial in differential coefficients. So, its degree is not defined.

EXERCISE 22.1

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

1. $\frac{d^3x}{dt^3}+\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2=e^t$
2. $\frac{d^2y}{dx^2}+4y=0$
3. $\left(\frac{dy}{dx}\right)^2+\frac{1}{dy/dx}=2$
4. $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\left(c\frac{d^2y}{dx^2}\right)^{1/3}$
5. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+xy=0$
6. $\sqrt[3]{\frac{d^2y}{dx^2}}=\sqrt{\frac{dy}{dx}}$
7. $\frac{d^4y}{dx^4}=\left[c+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$
8. $x+\left(\frac{dy}{dx}\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$
9. $y\frac{d^2x}{dy^2}=y^2+1$
10. $s^2\frac{d^2t}{ds^2}+st\frac{dt}{ds}=s$
11. $x^2\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+y^4=0$
12. $\frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}+4y=\sin x$
13. $(xy^2+x)\,dx+(y-x^2y)\,dy=0$
14. $\sqrt{1-y^2}\,dx+\sqrt{1-x^2}\,dy=0$
15. $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^{2/3}$
16. $2\frac{d^2y}{dx^2}+3\sqrt{1-\left(\frac{dy}{dx}\right)^2}-y=0$
17. $5\frac{d^2y}{dx^2}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}$
18. $y=x\frac{dy}{dx}+a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$
19. $y=px+\sqrt{a^2p^2+b^2}$, where $p=\frac{dy}{dx}$
20. $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2=x^2\log\left(\frac{d^2y}{dx^2}\right)$

21. $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$

22. $(y'')^2 + (y')^3 + \sin y = 0$

23. $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$ [NCERT]

24. $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y\sin y = 0$

25. $\frac{dy}{dx} + e^y = 0$

26. $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$ [NCERT]

ANSWERS

	Order	Degree	Linear/Non-linear
1.	3	1	Non-linear
2.	2	1	Linear
3.	1	3	Non-linear
4.	2	2	Non-linear
5.	2	1	Non-linear
6.	2	2	Non-linear
7.	4	2	Non-linear
8.	1	2	Linear
9.	2	1	Linear
10.	2	1	Non-linear
11.	2	3	Non-linear
12.	3	1	Non-linear
13.	1	1	Non-linear
14.	1	1	Non-linear
15.	2	3	Non-linear
16.	2	2	Non-linear
17.	2	2	Non-linear
18.	1	2	Non-linear
19.	1	2	Non-linear
20.	2	Undefined	Non-linear
21.	2	Undefined	Non-linear
22.	3	2	Non-linear
23.	2	1	Non-linear
24.	3	1	Linear
25.	1	1	Non-linear
26.	1	1	Non-linear

22.2 FORMATION OF DIFFERENTIAL EQUATIONS

Consider the family of curves given by $y = A\,e^x$, where A is the parameter. For different values of A, we obtain different members of the family.

Differentiating $y = Ae^x$ with respect to x, we get

$$\frac{dy}{dx} = Ae^x$$

On eliminating the parameter A between $y = A\,e^x$ and $\frac{dy}{dx} = Ae^x$, we get

$$\frac{dy}{dx} = y.$$

This is the differential equation of the family of curves represented by the equation $y = Ae^x$.

Thus, by eliminating one arbitrary constant, a differential equation of first order is obtained. In other words, one parameter family of curves is represented by a first order differential equation.

Now, consider a two parameter family of curves given by

$$y = A \cos 2x + B \sin 2x \qquad \text{...(i)}$$

where A and B are arbitrary constants.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = -2A \sin 2x + 2B \cos 2x \qquad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -4A \cos 2x - 4B \sin 2x \qquad \text{...(iii)}$$

Eliminating A and B from equations (i), (ii) and (iii), we get

$$\frac{d^2y}{dx^2} = -4y \Rightarrow \frac{d^2y}{dx^2} + 4y = 0.$$

Here, we note that by eliminating two arbitrary constants, a differential equation of second order is obtained. In other words, a two parameter family of curves is represented by a second order differential equation.

Similarly one can see that by eliminating three arbitrary constants a differential equation of third order is obtained or, three parameter family of curves is represented by a third order differential equation.

Thus, from the examples cited above it can be concluded that if an equation involves n arbitrary constants, a differential equation of nth order can be obtained by eliminating these n arbitrary constants. In other words, an n-parameter family of curves is represented by an nth order differential equation.

Formulating a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation. If an equation, representing a family of curves, contains n arbitrary constants, then we differentiate the given equation n times to obtain n more equations. Using all these equations, we eliminate the constants. The equation so obtained is the differential equation of order n for the family of given curves.

In order to formulate a differential equation from a given relation containing independent variable (x) dependent variable (y) and some arbitrary constants, we may follow the following algorithm:

ALGORITHM

STEP I *Write the given equation involving independent variable x* (say), *dependent variable y* (say) *and the arbitrary constants.*

STEP II *Obtain the number of arbitrary constants in Step I. Let there be n arbitrary constants.*

STEP III *Differentiate the relation in step I n times with respect to x.*

STEP IV *Eliminate arbitrary constants with the help of n equations involving differential coefficients obtained in step III and an equation in Step I.*
The equation so obtained is the desired differential equation.

The following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Form the differential equation of the family of curves represented by* $c(y+c)^2 = x^3$*, where c is a parameter.*

SOLUTION We have,

$$c(y+c)^2 = x^3 \quad ...(i)$$

Since only one parameter is involved, so we shall get a differential equation of first order.

Differentiating (i) w.r.t. x, we get

$$2c(y+c)\frac{dy}{dx} = 3x^2 \quad ...(ii)$$

Dividing (i) by (ii), we get

$$\frac{c(y+c)^2}{2c(y+c)\left(\frac{dy}{dx}\right)} = \frac{x^3}{3x^2}$$

$$\Rightarrow \quad y+c = \frac{2x}{3}\frac{dy}{dx}$$

$$\Rightarrow \quad c = \frac{2x}{3}\frac{dy}{dx} - y$$

Substituting this value of c in (i), we get

$$\left(\frac{2x}{3}\frac{dy}{dx} - y\right)\left(\frac{2}{3}x\frac{dy}{dx}\right)^2 = x^3$$

$$\Rightarrow \quad \frac{4}{9}\left(\frac{dy}{dx}\right)^2\left(\frac{2}{3}x\frac{dy}{dx} - y\right) = x$$

$$\Rightarrow \quad \frac{8}{27}x\left(\frac{dy}{dx}\right)^3 - \frac{4}{9}\left(\frac{dy}{dx}\right)^2 y = x$$

$$\Rightarrow \quad 8x\left(\frac{dy}{dx}\right)^3 - 12y\left(\frac{dy}{dx}\right)^2 = 27x$$

This is the required differential equation of the family of curves represented by (i).

EXAMPLE 2 *Form the differential equation of the family of curves represented by* $y = c(x-c)^2$, *where* c *is a parameter.*

SOLUTION We have,

$$y = c(x-c)^2 \quad ...(i)$$

Since the given equation contains only one parameter, we differentiate it once, so that

$$\frac{dy}{dx} = 2c(x-c) \quad ...(ii)$$

From (i) and (ii), we get

$$\frac{y}{\frac{dy}{dx}} = \frac{c(x-c)^2}{2c(x-c)}$$

$$\Rightarrow \quad \frac{y}{\frac{dy}{dx}} = \frac{x-c}{2}$$

$$\Rightarrow \quad x - c = \frac{2y}{\frac{dy}{dx}}$$

$$\Rightarrow \quad c = x - \frac{2y}{\frac{dy}{dx}}$$

Substituting this value of c in (i), we get

$$y = \left(x - \frac{2y}{\frac{dy}{dx}}\right)\left(\frac{2y}{\frac{dy}{dx}}\right)^2$$

$$\Rightarrow \quad y\left(\frac{dy}{dx}\right)^3 = 4y^2\left(x\frac{dy}{dx} - 2y\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 = 4y\left(x\frac{dy}{dx} - 2y\right)$$

which is the required differential equation.

EXAMPLE 3 *Form the differential equation representing the family of curves $y = A\cos(x + B)$, where A and B are parameters.* **[CBSE 2007]**

SOLUTION We have,

$$y = A\cos(x + B), \qquad \text{...(i)}$$

Since the given equation contains two arbitrary constants, we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = -A\sin(x + B) \qquad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -A\cos(x + B)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0,$$

which is the required differential equation of the given family of curves.

EXAMPLE 4 *Form the differential equation of the family of curves $y = a\sin(bx + c)$, a and c being parameters.* **[NCERT]**

SOLUTION We have,

$$y = a\sin(bx + c) \qquad \text{...(i)}$$

Since the given equation contains two arbitrary constants, we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = a\,b\cos(bx + c) \qquad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -a\,b^2 \sin(bx+c)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -b^2 y \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + b^2 y = 0$$

This is the required differential equation.

EXAMPLE 5 *Form the differential equation corresponding to $y^2 = a(b-x)(b+x)$ by eliminating parameters a and b.* **[CBSE 2004]**

SOLUTION We have,

$$y^2 = a(b^2 - x^2) \qquad \text{...(i)}$$

Since there are two arbitrary constants in the given equation, so we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = -2ax$$

$$\Rightarrow \quad y\frac{dy}{dx} = -ax \qquad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a \qquad \text{...(iii)}$$

From (ii) and (iii), we get

$$x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$$

This is the required differential equation.

EXAMPLE 6 *Form the differential equation corresponding to $y^2 = m(a^2 - x^2)$ by eliminating parameters m and a.*

SOLUTION We have,

$$y^2 = m(a^2 - x^2) \qquad \text{...(i)}$$

Since the given equation contains two arbitrary constants, so we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating both sides of (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = m(-2x)$$

$$\Rightarrow \quad y\frac{dy}{dx} = -mx \qquad \text{...(ii)}$$

Differentiating both sides of (ii) w.r. to x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -m \qquad \text{...(iii)}$$

From (ii) and (iii), we get

$$x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=y\frac{dy}{dx} \qquad \left[\begin{array}{l}\text{Putting the value of}\\ -m \text{ from(iii) in (ii)}\end{array}\right]$$

This is the required differential equation.

EXAMPLE 7 *Find the differential equation of all circles touching the*

(i) *x-axis at the origin* **[NCERT, CBSE 2005, 2008, 2010]**

(ii) *y-axis at the origin* **[NCERT]**

SOLUTION (i) The equation of the family of circles touching x-axis at the origin is

$$(x-0)^2+(y-a)^2=a^2$$

$$\Rightarrow \quad x^2+y^2-2ay=0 \qquad \text{...(i)}$$

where a is a parameter.

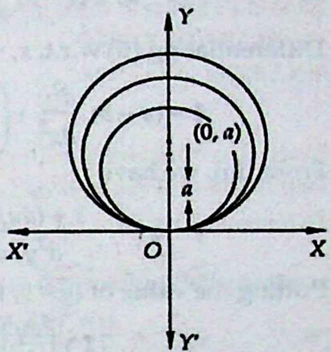

Fig. 22.1

This equation contains only one arbitrary constant, we differentiate it once w.r.t. x, so that

$$2x+2y\frac{dy}{dx}-2a\frac{dy}{dx}=0$$

$$\Rightarrow \quad a=\frac{x+y\,(dy/dx)}{dy/dx} \qquad \text{...(ii)}$$

Putting the value of a from (ii) in (i), we get

$$x^2+y^2=2y\left(\frac{x+y\,(dy/dx)}{dy/dx}\right)$$

$$\Rightarrow \quad (x^2-y^2)\frac{dy}{dx}=2xy$$

This is the required differential equation.

(ii) The equation of the family of circles touching y-axis at the origin is

$$(x-a)^2+(y-0)^2=a^2$$

$$\Rightarrow \quad x^2+y^2-2ax=0 \qquad \text{...(i)}$$

where a is a parameter.

This equation contains only one arbitrary constant, we differentiate it only once w.r.t. x, so that

$$2x+2y\frac{dy}{dx}-2a=0$$

$$\Rightarrow \quad a=x+y\frac{dy}{dx} \qquad \text{...(ii)}$$

Putting the value of a from (ii) in (i), we get

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)=0$$

$$\Rightarrow \quad y^2-x^2=2xy\frac{dy}{dx}$$

This is the required differential equation.

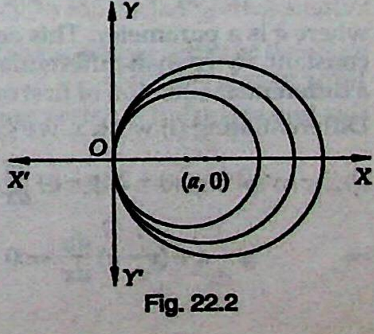

Fig. 22.2

EXAMPLE 8 *Obtain the differential equation of all circles of radius r.* **[CBSE 2010]**

SOLUTION The equation of the family of circles of radius r is

$$(x-a)^2+(y-b)^2 = r^2 \quad \text{...(i)}$$

where a and b are a parameters.

Since equation (i) contains two arbitrary constants, we differentiate it two times w.r.t. x and the differential equation will be of second order.

Differentiating (i) w.r.t. x, we get

$$2(x-a)+2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x-a)+(y-b)\frac{dy}{dx} = 0 \quad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0 \quad \text{...(iii)}$$

From (iii), we have

$$y-b = -\frac{1+(dy/dx)^2}{d^2y/dx^2} \quad \text{...(iv)}$$

Putting the value of $(y-b)$ in (ii), we obtain

$$x-a = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{d^2y/dx^2} \quad \text{...(v)}$$

Substituting the values of $(x-a)$ and $(y-b)$ in (i), we get

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^2\left(\frac{dy}{dx}\right)^2}{(d^2y/dx^2)^2}+\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^2}{(d^2y/dx^2)^2} = r^2$$

$$\Rightarrow \quad \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

This is the required differential equation.

EXAMPLE 9 *Find the differential equation of all the circles in the first quadrant which touch the coordinate axes.* **[NCERT, CBSE 2010]**

SOLUTION The equation of circles in the first quadrant which touch the coordinate axes is

$$(x-a)^2+(y-a)^2 = a^2 \quad \text{...(i)}$$

where a is a parameter. This equation contains one arbitrary constant, so we shall differentiate it once only and we shall get a differential equation of first order.

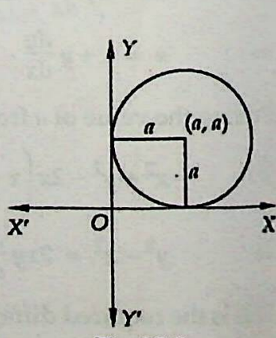

Fig. 22.3

Differentiating (i) w.r.t. x, we get

$$2(x-a)+2(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x-a+(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad a = \frac{x + y\dfrac{dy}{dx}}{1 + \dfrac{dy}{dx}}$$

$$\Rightarrow \qquad a = \frac{x+py}{1+p}, \text{ where } p = \frac{dy}{dx}$$

Substituting the value of a in (i), we get

$$\left(x - \frac{x+py}{1+p}\right)^2 + \left(y - \frac{x+py}{1+p}\right)^2 = \left(\frac{x+py}{1+p}\right)^2$$

$$\Rightarrow \qquad (xp - py)^2 + (y - x)^2 = (x + py)^2$$

$$\Rightarrow \qquad (x-y)^2 p^2 + (x-y)^2 = (x+py)^2$$

$$\Rightarrow \qquad (x-y)^2 (p^2 + 1) = (x + py)^2$$

$$\Rightarrow \qquad (x-y)^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right) = \left(x + y\frac{dy}{dx}\right)^2$$

This is the required differential equation.

EXAMPLE 10 *Show that the differential equation that represents the family of all parabolas having their axis of symmetry coincident with the axis of x is* $yy_2 + y_1^2 = 0$.

SOLUTION The equation that represents a family of parabolas having their axis of symmetry coincident with the axis of x is

$$y^2 = 4a(x-h) \qquad \text{...(i)}$$

This equation contains two parameters a and h, so we will differentiate it twice to obtain a second order differential equation.

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a$$

$$y\frac{dy}{dx} = 2a \qquad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \qquad yy_2 + y_1^2 = 0,$$

which is the required differential equation.

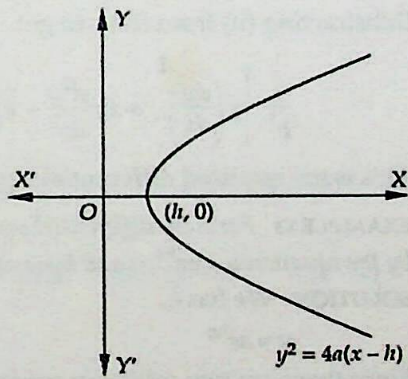

Fig. 22.4

EXAMPLE 11 *Form the differential equation of family of parabolas having vertex at the origin and axis along positive y-axis.* **[NCERT, CBSE 2010]**

SOLUTION The equation of the family of parabolas having vertex at the origin and axis along positive y-axis is

$$x^2 = 4ay, \text{ where } a \text{ is a parameter.} \qquad \text{...(i)}$$

Differentiating w.r. to x, we get

$$2x = 4a\frac{dy}{dx} \Rightarrow a = \frac{x}{2\left(\dfrac{dy}{dx}\right)}$$

Substituting the value of a in (i), we get

$$x^2 = 4 \times \frac{x}{2\left(\frac{dy}{dx}\right)} \times y \Rightarrow x\frac{dy}{dx} = 2y$$

This is the required differential equation.

EXAMPLE 12 *Form the differential equation of the family of ellipses having foci on y-axis and centre at the origin.* **[NCERT]**

SOLUTION The equation of the family of ellipses having centre at the origin and foci on y-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b > a \quad \text{...(i)}$$

Differentiating w.r.to x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0 \quad \text{...(ii)}$$

Differentiating (ii) w.r. to x, we get

$$\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2}\frac{d^2y}{dx^2} = 0 \quad \text{...(iii)}$$

Multiplying throughout by x, we get

$$\frac{x}{a^2} + \frac{x}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{xy}{b^2}\frac{d^2y}{dx^2} = 0 \quad \text{...(iv)}$$

Substracting (ii) from (iv), we get

$$\frac{1}{b^2}\left\{x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\left(\frac{dy}{dx}\right)\right\} = 0 \Rightarrow x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0$$

This is the required differential equation.

EXAMPLE 13 *Form the differential equation not containing the arbitrary constants and satisfied by the equation $y = ae^{bx}$, a and b are arbitrary constants.*

SOLUTION We have,

$$y = ae^{bx} \quad \text{...(i)}$$

Since there are two arbitrary constants in (i), so we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = ae^{bx} \cdot b$$

$$\Rightarrow \quad \frac{dy}{dx} = by \quad \text{[By using (i)]} \quad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = b\frac{dy}{dx} \quad \text{...(iii)}$$

From (ii) and (iii), we obtain

$$\frac{d^2y}{dx^2} = \left(\frac{1}{y}\frac{dy}{dx}\right)\frac{dy}{dx} \qquad \left[\text{From (ii), } b = \frac{1}{y}\cdot\frac{dy}{dx}\right]$$

$$\Rightarrow \quad y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2, \text{ which is the required differential equation}$$

EXAMPLE 14 *Show that the differential equation representing one parameter family of curves* $(x^2 - y^2) = c\,(x^2 + y^2)^2$ *is* $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$ **[NCERT]**

SOLUTION We have,

$$x^2 - y^2 = c\,(x^2 + y^2)^2 \qquad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$2x - 2y\frac{dy}{dx} = 2c\,(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left(x - y\frac{dy}{dx}\right) = 2c\,(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) \qquad \text{...(ii)}$$

[On substituting the value of c from (i) to (ii)]

$$\left(x - y\frac{dy}{dx}\right) = \frac{2\,(x^2 - y^2)}{(x^2 + y^2)^2}(x^2 + y^2)\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (x^2 + y^2)\left(x - y\frac{dy}{dx}\right) = 2\,(x^2 - y^2)\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \{x\,(x^2 + y^2) - 2x\,(x^2 - y^2)\} = \frac{dy}{dx}\{2y\,(x^2 - y^2) + y\,(x^2 + y^2)\}$$

$$\Rightarrow \quad (3xy^2 - x^3) = \frac{dy}{dx}(3x^2y - y^3)$$

$$\Rightarrow \quad (x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy,$$

which is the given differential equation.

EXAMPLE 15 *Represent the following family of curves by forming the corresponding differential equations (a, b are parameters):*

(i) $\frac{x}{a} + \frac{y}{b} = 1$ (ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **[NCERT, CBSE 2007]** (iii) $(y - b)^2 = 4\,(x - a)$

SOLUTION We have,

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{...(i)}$$

where a, b are parameters.

Differentiating (i) with respect to x, we get

$$\frac{1}{a} + \frac{1}{b}\frac{dy}{dx} = 0 \qquad \text{...(ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{1}{b}\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \qquad \frac{d^2 y}{dx^2} = 0$$

This is the required differential equation.

(ii) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

$$\Rightarrow \qquad \frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0 \qquad \text{...(ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{1}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0 \qquad \text{...(iii)}$$

Differentiating (iii) with respect to x, we get

$$\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2}\frac{d^2 y}{dx^2} = 0$$

Multiplying both sides by x, we get

$$\frac{x}{a^2} + \frac{x}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{xy}{b^2}\frac{d^2 y}{dx^2} = 0 \qquad \text{...(iv)}$$

Subtracting (ii) from (iv), we get

$$\frac{1}{b^2}\left\{x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2 y}{dx^2} - y\frac{dy}{dx}\right\} = 0$$

$$\Rightarrow \qquad x\left(\frac{dy}{dx}\right)^2 + xy\frac{x^2 y}{dx^2} - y\frac{dy}{dx} = 0$$

This is the required differential equation.

(iii) $$(y-b)^2 = 4(x-a) \qquad \text{...(i)}$$

Differentiating with respect to x, we get

$$2(y-b)\frac{dy}{dx} = 4$$

$$\Rightarrow \qquad (y-b)\frac{dy}{dx} = 2 \qquad \text{...(ii)}$$

Differentiating with respect to x, we get

$$(y-b)\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \text{...(iii)}$$

Eliminating $(y-b)$ from (ii) and (iii), we get

$$2\frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \qquad 2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3 = 0$$

This is the required differential equation.

EXERCISE 22.2

1. Form the differential equation of the family of curves represented by $y^2 = (x-c)^3$.
2. Form the differential equation corresponding to $y = e^{mx}$ by eliminating m.
3. Form the differential equations from the following primitives where constants are arbitrary:
   (i) $y^2 = 4ax$ (ii) $y = cx + 2c^2 + c^3$
   (iv) $xy = a^2$ (iv) $y = ax^2 + bx + c$
4. Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants.
5. Find the differential equation of the family of curves, $x = A \cos nt + B \sin nt$, where A and B are arbitrary constants. [CBSE 2007]
6. Form the differential equation corresponding to $y^2 = a(b - x^2)$ by eliminating a and b.
7. Form the differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating a. [CBSE 2005]
8. Form the differential equation corresponding to $(x-a)^2 + (y-b)^2 = r^2$ by eliminating a and b.
9. Find the differential equation of all the circles which pass through the origin and whose centres lie on y-axis.
10. Find the differential equation of all the circles which pass through the origin and whose centres lie on x-axis.
11. Assume that a rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop.
12. Find the differential equation of all the parabolas with latus rectum '$4a$' and whose axes are parallel to x-axis.
13. Show that the differential equation of which $y = 2(x^2 - 1) + ce^{-x^2}$ is a solution, is $\frac{dy}{dx} + 2xy = 4x^3$.
14. Find the differential equation of all non-vertical lines in a plane.
15. Form the differential equation of the family of curves represented by the equation (a being the parameter):
    (i) $(2x+a)^2 + y^2 = a^2$
    (ii) $(2x-a)^2 - y^2 = a^2$
    (iii) $(x-a)^2 + 2y^2 = a^2$ [NCERT]
16. Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):
    (i) $x^2 + y^2 = a^2$ (ii) $x^2 - y^2 = a^2$ (iii) $y^2 = 4ax$
    (iv) $x^2 + (y-b)^2 = 1$ (v) $(x-a)^2 - y^2 = 1$ (vi) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(vii) $y^2 = 4a(x-b)$ (viii) $y = ax^3$ (ix) $x^2 + y^2 = ax^3$

(x) $y = e^{ax}$

17. **Form the differential equation representing the family of ellipses having centre at the origin and foci on X-axis** **[NCERT, CBSE 2007]**

18. **Form the differential equation of the family of hyperbolas having foci on X-axis and centre at the origin.** **[NCERT]**

19. **Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.** **[NCERT]**

ANSWERS

1. $8\left(\frac{dy}{dx}\right)^3 = 27y$

2. $x\frac{dy}{dx} = y \log y$

3. (i) $2x\frac{dy}{dx} = y$

   (ii) $y = x\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^3$

   (iii) $y + x\frac{dy}{dx} = 0$

   (iv) $\frac{d^3y}{dx^3} = 0$

4. $\frac{d^2y}{dx^2} = 4y$

5. $\frac{d^2x}{dt^2} + n^2x = 0$

6. $y\frac{dy}{dx} = x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\}$

7. $p^2(x^2 - 2y^2) - 4pxy - x^2 = 0, p = \frac{dy}{dx}$

8. $(1 + p^2)^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2, \; p = \frac{dy}{dx}$

9. $(x^2 - y^2)\frac{dy}{dx} = 2xy$

10. $(x^2 - y^2) + 2xy\frac{dy}{dx} = 0$

11. $\frac{dr}{dt} = -k$

12. $2a\, y_2 + y_1^3 = 0$

14. $\frac{d^2y}{dx^2} = 0$

15. (i) $y^2 - 4x^2 - 2xy\frac{dy}{dx} = 0$

    (ii) $2x\frac{dy}{dx} = 4x^2 + y^2$

    (iii) $2y^2 - x^2 = 4xy\frac{dy}{dx}$

16. (i) $x + y\frac{dy}{dx} = 0$

    (ii) $x - y\frac{dy}{dx} = 0$

    (iii) $y - 2x\frac{dy}{dx} = 0$

    (iv) $x^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = \left(\frac{dy}{dx}\right)^2$

    (v) $y^2\left(\frac{dy}{dx}\right)^2 - y^2 = 1$

    (vi) $x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$

    (vii) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

    (viii) $x\frac{dy}{dx} = 3y$

    (ix) $x^2 + 3y^2 = 2xy\frac{dy}{dx}$

    (x) $x\frac{dy}{dx} = y \log y$

17. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

18. $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$

19. $\left(x+y\frac{dy}{dx}\right)^2=(x+y)^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}$

HINTS TO SELECTED PROBLEMS

1. We have,

$$y^2=(x-c)^3$$
$$\Rightarrow 2y\,y_1=3(x-c)^2$$
$$\Rightarrow (2y\,y_1)^3=27(x-c)^6$$
$$\Rightarrow 8y^3y_1^3=27(y^2)^2 \qquad [\because (x-c)^3=y^2]$$
$$\Rightarrow 8y_1^3=27y,$$ which is the required differential equation.

2. We have,

$$y=e^{mx}$$
$$\Rightarrow y_1=me^{mx}$$
$$\Rightarrow y_1=my.$$

Now, $y=e^{mx} \Rightarrow \log y=mx \Rightarrow m=\frac{\log y}{x}$

Substituting this value of m in $y_1=my$, we get

$xy_1 = y\log y$, which is the required differential equation.

3. (i) We have,

$$y^2=4ax$$
$$\Rightarrow 2y\frac{dy}{dx}=4a$$
$$\Rightarrow a=\frac{1}{2}y\frac{dy}{dx}$$

Substitute this value of a in $y^2=4ax$

(ii) We have,

$$e^x+ce^y=1$$
$$\Rightarrow e^x+ce^y\frac{dy}{dx}=0$$
$$\Rightarrow c=-e^{x-y}\frac{dx}{dy}$$

Substituting this value in $e^x+ce^y=1$, we obtain the required differential equation.

(iii) We have,

$$y=cx+2c^2+c^3 \Rightarrow \frac{dy}{dx}=c$$

Substitute c in $y=cx+2c^2+c^3$, to get the required differential equation.

(iv) We have,

$xy=a^2 \Rightarrow x\frac{dy}{dx}+y=0$, which is the required differential equation.

(v) Differentiate y three times to get $\frac{d^3y}{dx^3}=0$ as the required differential equation.

4. Differentiate y two times to get $\frac{d^2y}{dx^2} = 4y$ as the required differential equation.
5. Differentiate x two times w.r.t. t to obtain

$$\frac{d^2x}{dt^2} = -n^2 x$$

as the required differential equation.
7. We have,

$$y^2 - 2ay + x^2 = a^2 \Rightarrow 2y\frac{dy}{dx} - 2a\frac{dy}{dx} + 2x = 0 \Rightarrow a = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}$$

Substituing the value of a in $y^2 - 2ay + x^2 = a^2$, we obtain the required differential equation.
12. The equation of the family of parabolas is

$(y - k)^2 = 4a\,(x - h)$, where h and k are arbitrary constants.

Differentiate this relation twice and eliminate h and k to get the differential equation.
18. The equation of the family of circles in second quadrant and touching the coordinate axes is

$$(x + a)^2 + (y - a)^2 = a^2,\ a > 0 \ \text{ or, } \ x^2 + y^2 + 2ax - 2ay + a^2 = 0$$

22.3 SOLUTION OF A DIFFERENTIAL EQUATION

DEFINITON *The solution of a differential equation is a relation between the variables involved which satisfies the differential equation. Such a relation and the derivatives obtained therefrom when substituted in the differential equation, makes left hand, and right hand sides identically equal.*

For example, $y = e^x$ is a solution of the differential equation $\frac{dy}{dx} = y$.

Consider the differential equation

$$\frac{d^2y}{dx^2} + y = 0 \qquad ...(i)$$

and, consider $y = A\cos x + B\sin x$ ...(ii)

where A and B are arbitrary constants.

Differentiating (ii), w.r.t. x, we get

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

Differentiating this w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} + y = 0$$

This shows that $y = A \cos x + B \sin x$ satisfies the differential equation (i) and hence it is a solution of (i).

It can be easily verified that $y = 3 \cos x + 2 \sin x$, $y = A \cos x$, $y = B \sin x$ etc., are also solutions of (i).

We find that the solution $y = 3 \cos x + 2 \sin x$ does not contain any arbitrary constant whereas solutions $y = A \cos x$, $y = B \sin x$ contain only one arbitrary constant. The solution $y = A \cos x + B \sin x$ contains two arbitrary constants, so it is known as the general solution of (i) whereas all other solutions are particular solutions.

GENERAL SOLUTION *The solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution of the differential equation.*

For example, $y = A \cos x + B \sin x$ is the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

But, $y = A \cos x$ is not the general solution as it contains one arbitrary constant.

PARTICULAR SOLUTION *Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.*

For example, $y = 3 \cos x + 2 \sin x$ is a particular solution of the differential equation (i).

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Show that* $y = Ax + \frac{B}{x}, x \neq 0$ *is a solution of the differential equation*

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

SOLUTION We have,

$$y = Ax + \frac{B}{x}, x \neq 0 \qquad \text{...(i)}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = A - \frac{B}{x^2} \qquad \text{... (ii)}$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{2B}{x^3} \qquad \text{...(iii)}$$

Substituting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y$, we have

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 \cdot \frac{2B}{x^3} + x\left(A - \frac{B}{x^2}\right) - \left(Ax + \frac{B}{x}\right)$$

$$\Rightarrow \qquad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \frac{2B}{x} + Ax - \frac{B}{x} - Ax - \frac{B}{x} = 0$$

Thus, the function $y = Ax + \frac{B}{x}$ satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

Hence, $y = Ax + \frac{B}{x}$ is a solution of the given differential equation.

EXAMPLE 2 *Show that the function $y = (A + Bx)e^{3x}$ is a solution of the equation*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

SOLUTION We have,

$$y = (A + Bx)\, e^{3x} \qquad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = Be^{3x} + 3e^{3x}(A + Bx) \qquad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 6Be^{3x} + 9e^{3x}(A + Bx) \qquad \text{...(iii)}$$

$$\therefore \quad \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y$$

$$= \{6Be^{3x} + 9e^{3x}(A + Bx)\} - 6\{Be^{3x} + 3e^{3x}(A + Bx)\} + \{9(A + Bx)e^{3x}\}$$

Thus, $y = (A + Bx)e^{3x}$ satisfies the given differential equation. Hence, it is a solution of the given differential equation.

EXAMPLE 3 *Show that $y = ae^{2x} + be^{-x}$ is a solution of the differential equation*

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

SOLUTION We have,

$$y = ae^{2x} + be^{-x} \qquad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 2ae^{2x} - be^{-x} \qquad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$$

$$\therefore \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = (4ae^{2x} + be^{-x}) - (2ae^{2x} - be^{-x}) - 2(ae^{2x} + be^{-x}) = 0$$

So, $y = ae^{2x} + be^{-x}$ satisfies the given differential equation.

Hence, it is a solution of the given differential equation.

EXAMPLE 4 *Show that $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

SOLUTION We have, $y = a\cos(\log x) + b\sin(\log x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$$

Differentiating both sides w.r. to x, we obtain

$$x\frac{d^2y}{dx^2} + 1\cdot\frac{dy}{dx} = -\frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[a\cos(\log x) + b\sin(\log x)]$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0,$$

which is same as the given differential equation.

Hence, $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the given differential equation.

EXAMPLE 5 *Show that $y = cx + \frac{a}{c}$ is a solution of the differential equation*

$$y = x\left(\frac{dy}{dx}\right) + \frac{a}{\left(\frac{dy}{dx}\right)}$$

SOLUTION We have,

$$y = cx + \frac{a}{c} \qquad \text{...(i)}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = c \qquad \text{...(ii)}$$

Now, $$x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}} = xc + \frac{a}{c} \qquad \left[\text{Putting } \frac{dy}{dx} = c\right]$$

$$\Rightarrow \quad x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}} = y \qquad \text{[Using (i)]}$$

This shows that $y = cx + \frac{a}{c}$ is a solution of the given differential equation.

EXAMPLE 6 *Show that $xy = ae^x + be^{-x} + x^2$ is a solution of the differential equation*

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0. \qquad \text{[NCERT]}$$

SOLUTION We have,

$$xy = ae^x + be^{-x} + x^2 \quad \text{...(i)}$$

Differentiating w.r.t. x, we get

$$x\frac{dy}{dx} + y = ae^x - be^{-x} + 2x$$

Differentiating this w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = ae^x + be^{-x} + 2$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = ae^x + be^{-x} + 2 \quad \text{...(ii)}$$

Now, $$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = [ae^x + be^{-x} + 2] - [ae^x + be^{-x} + x^2] + x^2 - 2$$

$$= 0 \quad \text{[Using (i) and (ii)]}$$

Thus, $xy = ae^x + be^{-x} + x^2$ is a solution of the given differential equation.

EXAMPLE 7 *Verify that the function* $y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx$, C_1, C_2 *are arbitrary constants is a solution of the differential equation*

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0 \quad \text{[NCERT]}$$

SOLUTION We have,

$$y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx \quad \text{...(i)}$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = C_1\{a e^{ax} \cos bx - b e^{ax} \sin bx\} + C_2\{a e^{ax} \sin bx + b e^{ax} \cos bx\}$$

$$\Rightarrow \quad \frac{dy}{dx} = a\left\{C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx\right\} + b\left\{-C_1 e^{ax} \sin bx + C_2 e^{ax} \cos bx\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = a\,y + b\left\{-C_1 e^{ax} \sin bx + C_2 e^{ax} \cos bx\right\} \quad \text{...(ii)}$$

Differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = a\frac{dy}{dx} + b\left\{-a\,C_1 e^{ax} \sin bx - b\,C_1 e^{ax} \cos bx + a\,C_2 e^{ax} \cos bx - b\,C_2 e^{ax} \sin bx\right\}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = a\frac{dy}{dx} + ab\left\{-C_1 e^{ax} \sin bx + C_2 e^{ax} \cos bx\right\} - b^2\left\{C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx\right\}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\left\{\frac{dy}{dx} - ay\right\} - b^2 y \quad \text{[Using (ii)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$$

Hence, the given function is a solution of the given differential equation.

EXERCISE 22.3

1. Show that $y = be^x + ce^{2x}$ is solution of the differential equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

2. Verify that $y = 4 \sin 3x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + 9y = 0$$

3. Show that $y = ae^{2x} + be^{-x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$

4. Show that the function $y = A \cos x + B \sin x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ **[NCERT]**

5. Show that the function $y = A \cos 2x - B \sin 2x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 4y = 0$ **[CBSE 2007]**

6. Show that $y = Ae^{Bx}$ is a solution of the differential equation $\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2$

7. Verify that $y = \frac{a}{x} + b$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0$

8. Verify that $y^2 = 4ax$ is a solution of the differential equation $y = x\frac{dy}{dx} + a\frac{dx}{dy}$

9. Show that $Ax^2 + By^2 = 1$ is a solution of the differential equation

$$x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$$

10. Show that $y = ax^3 + bx^2 + c$ is a solution of the differential equation $\frac{d^3y}{dx^3} = 6a$

11. Show that $y = \frac{c - x}{1 + cx}$ is a solution of the differential equation

$$(1 + x^2)\frac{dy}{dx} + (1 + y^2) = 0$$

12. Show that $y = e^x (A \cos x + B \sin x)$ is the solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ **[NCERT]**

13. Verify that $y = cx + 2c^2$ is a solution of the differential equation

$$2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0$$

14. Verify that $y = -x - 1$ is a solution of the differential equation

$$(y - x)\,dy - (y^2 - x^2)\,dx = 0$$

15. Verify that $y^2 = 4a(x+a)$ is a solution of the differntial equations

$$y\left\{1-\left(\frac{dy}{dx}\right)^2\right\} = 2x\frac{dy}{dx}$$

16. Verify that $y = ce^{\tan^{-1} x}$ is a solution of the differential equation

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$

17. Verify that $y = e^{m\cos^{-1} x}$ satisfies the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

18. Verify that $y = \log(x + \sqrt{x^2+a^2})^2$ satisfies the differential equation

$$(a^2+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

19. Show that the differential equation of which $y = 2(x^2-1) + ce^{-x^2}$ is a solution is

$$\frac{dy}{dx} + 2xy = 4x^3$$

20. Show that $y = e^{-x} + ax + b$ is solution of the differential equation $e^x\dfrac{d^2y}{dx^2} = 1$

21. For each of the following differential equations verify that the accompanying function is a solution in the mentioned domain (a, b are parameters).

	Differential equation	*Function*
(i)	$x\dfrac{dy}{dx} = y$	$y = ax, x \in R - \{0\}$
(ii)	$x + y\dfrac{dy}{dx} = 0; x \in R, y \neq 0$	$y = \pm\sqrt{a^2-x^2}, x \in (-a, a)$
(iii)	$x\dfrac{dy}{dx} + y = y^2; x \in R - \{0\}$	$y = \dfrac{a}{x+a}, x \in R - \{a\}$
(iv)	$x^3\dfrac{d^2y}{dx^2} = 1, x \in R - \{0\}$	$y = ax + b + \dfrac{1}{2x}, x \in R - \{0\}$
(v)	$y = \left(\dfrac{dy}{dx}\right)^2, x \in R, y \geq 0$	$y = \dfrac{1}{4}(x \pm a)^2, x \in R$

22.4 INITIAL VALUE PROBLEMS

In section 22.2, we have seen that a first order differential equation represents a one-parameter family of curves, a second order differential equation represents a two-parameter family of curves, and so on. Therefore, if we wish to specify a particular member of such a family of curves, then in addition to the differential equation we require some other conditions for the specification of the parameter(s). These conditions are generally prescribed by assigning values to the unknown function (dependent variable) and its various order derivatives at some point of the domain of definition of independent variable.

For example, the differential equation $\dfrac{dy}{dx} = 4x$ represents one parameter family of curves given by $y = 2x^2 + C$, where C is a parameter. In order to specify a particular member, say $y = 2x^2 + 3$, of this family, we require the differential equation $\dfrac{dy}{dx} = 4x$ and one more condition, namely $y(1) = 5$ i.e. $y = 5$ when $x = 1$.

Similarly, the differential equation

$$\frac{d^2 y}{dx^2} - 6 = 0$$

represents two parameter family of curves given by $y = 3x^2 + ax + b$ where a and b are parameters.

Now, if we want to specify a particular member, say $y = 3x^2 - 2x + 1$, of this family. Then, we require the differential equation $\frac{d^2y}{dx^2} - 6 = 0$ and two conditions, namely $y(0) = 1$ and $y'(0) = -2$.

It follows from the above discussion that to specify a particular member of a family of curves, we require the differential equation representing the given family of curves and the values of dependent variable and its various order derivatives at some point of the domain of definition. These values are generally prescribed at only one point of the domain of definition of independent variable and are generally referred to as *initial values* or *initial conditions.*

The differential equation with these initial values or initial conditions is generally known as an *initial value problem.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Verify that the function defined by* $y = \sin x - \cos x$, $x \in R$ *is a solution of the initial value problem* $\frac{dy}{dx} = \sin x + \cos x$, $y(0) = -1$.

SOLUTION We have,

$$y = \sin x - \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = \cos x + \sin x$$

$\therefore$ $y = \sin x - \cos x$ satisfies the differential given equation and hence it is a solution.

Also, when $x = 0$, $y = \sin 0 - \cos 0 = 0 - 1 = -1$ i.e. $y(0) = -1$.

Hence, $y = \sin x - \cos x$ is a solution of the given initial value problem.

EXAMPLE 2 *Show that* $y = x^2 + 2x + 1$ *is the solution of the initial value problem* $\frac{d^3y}{dx^3} = 0$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 2$.

SOLUTION We have,

$$y = x^2 + 2x + 1$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x + 2, \frac{d^2 y}{dx^2} = 2 \text{ and } \frac{d^3 y}{dx} = 0$$

Thus, $y = x^2 + 2x + 1$ satisfies the differential equation $\frac{d^3 y}{dx^3} = 0$

Also, $y = x^2 + 2x + 1, \frac{dy}{dx} = 2x + 2$ and $\frac{d^2 y}{dx^2} = 2$

$$\Rightarrow \quad y(0) = 0 + 0 + 1 = 1, \left(\frac{dy}{dx}\right)_{x=0} = 2 \text{ and } \left(\frac{d^2 y}{dx^2}\right)_{x=0} = 2$$

$$\Rightarrow \quad y(0) = 1, y'(0) = 2 \text{ and } y''(0) = 2.$$

Hence, $y = x^2 + 2x + 1$ is the solution of the initial value problem.

EXAMPLE 3 *Show that the function ϕ, defined by $\phi(x) = \cos x$ $(x \in R)$; satisfies the initial value problem*

$$\frac{d^2y}{dx^2} + y = 0, y(0) = 1, y'(0) = 0$$

SOLUTION We have,

$$\phi(x) = \cos x$$

$\Rightarrow$ $\phi'(x) = -\sin x$ and $\phi''(x) = -\cos x$

$\Rightarrow$ $\phi'(x) = -\sin x$ and $\phi''(x) = -\phi(x)$

Clearly, if we replace ϕ by y in $\phi''(x) = -\phi(x)$, we obtain

$$\frac{d^2y}{dx^2} = -y \text{ or, } \frac{d^2y}{dx^2} + y = 0$$

Thus, $\phi(x)$ satisfies the given differential equation for all $x \in R$.

Also, $\phi(0) = \cos 0 = 1$ and $\phi'(0) = -\sin 0 = 0$

$\Rightarrow$ $y(0) = 1$ and $y'(0) = 0$

So, ϕ satisfies the initial conditions. Hence, ϕ satisfies the initial value problem.

EXERCISE 22.4

For each of the following initial value problems verify that the accompanying function is a solution:

	Differential equation	*Function*
1.	$x\frac{dy}{dx} = 1, y(1) = 0$	$y = \log x$
2.	$\frac{dy}{dx} = y, y(0) = 1$	$y = e^x$
3.	$\frac{d^2y}{dx^2} + y = 0, y(0) = 0, y'(0) = 1$	$y = \sin x$
4.	$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0, y(0) = 2, y'(0) = 1$	$y = e^x + 1$
5.	$\frac{dy}{dx} + y = 2, y(0) = 3$	$y = e^{-x} + 2$
6.	$\frac{d^2y}{dx^2} + y = 0, y(0) = 1, y'(0) = 1$	$y = \sin x + \cos x$
7.	$\frac{d^2y}{dx^2} - y = 0, y(0) = 2, y'(0) = 0$	$y = e^x + e^{-x}$
8.	$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0, y(0) = 1, y'(0) = 3)$	$y = e^x + e^{2x}$
9.	$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0, y(0) = 1, y'(0) = 2$	$y = xe^x + e^x$

22.5 GENERAL FORM OF A FIRST-ORDER FIRST-DEGREE DIFFERENTIAL EQUATION

A differential equation of first order and first degree involves the independent variable x, dependent variable y and $\frac{dy}{dx}$. So, it can be put in any one of the following forms:

$$\frac{dy}{dx} = f(x, y) \qquad \text{...(i)}$$

or, $$\frac{dy}{dx} = \frac{\phi(x, y)}{\psi(x, y)} \quad \text{...(ii)}$$

or, In general $f\left(x, y, \frac{dy}{dx}\right) = 0$,

where $f(x, y)$ and $g(x, y)$ are obviously the functions of x and y.

In the chapter on differentials and approximations, we have proved that if dx and dy denote differentials of variables x and y, then

$$dy = \frac{dy}{dx}\, dx$$

Therefore, equations (i) and (ii) can be written as

$$dy = f(x, y)\, dx$$

and, $$dy = \frac{\phi(x, y)}{\psi(x, y)}\, dx \text{ or, } \phi(x, y)\, dx = \psi(x, y)\, dy$$

Hence, a first-order first-degree differential equation is expressible in one of the following forms:

$$\frac{dy}{dx} = f(x, y)$$

or, $$\frac{dy}{dx} = \frac{\phi(x, y)}{\psi(x, y)}$$

or, $$dy = f(x, y)\, dx$$

or, $$f(x, y)\, dx + g(x, y)\, dy = 0$$

22.6 GEOMETRICAL INTERPRETATION OF THE DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

The general form of a first order and first degree differential equation is

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \quad \text{...(i)}$$

We know that the tangent of the direction of a curve in Cartesian rectangular coordinates at any point is given by $\frac{dy}{dx}$, so the equation in (i) can be known as an equation which establishes the relationship between the coordinates of a point and the slope of the tangent i.e., $\frac{dy}{dx}$ to the integral curve at that point. Solving the differential equation given by (i) means finding those curves for which the direction of tangent at each point coincides with the direction of the field. All the curves represented by the general solution when taken together will give the locus of the differential equation. Since there is one arbitrary constant in the general solution of the equation of first order, the locus of the equation can be said to be made up of single infinity of curves.

22.7 SOLUTION OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

As discussed earlier a first order and first degree differential equation can be written as

$$f(x, y)\, dx + g(x, y)\, dy = 0$$

or, $$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

or, $$\frac{dy}{dx} = \phi(x, y)$$

where $f(x, y)$ and $g(x, y)$ are obviously the functions of x and y.

It is not always possible to solve this type of equations. The solution of this type of differential equations is possible only when it falls under the category of some standard forms. In the following article we will discuss some of the standard forms and methods of obtaining their solutions.

22.8 METHODS OF SOLVING A FIRST ORDER FIRST DEGREE DIFFERENTIAL EQUATION

In this section, we shall discuss several techniques of obtaining solutions of various types of differential equations.

22.8.1 DIFFERENTIAL EQUATIONS OF THE TYPE $\frac{dy}{dx} = f(x)$

To solve this type of differential equations we integrate both sides to obtain the general solution as discussed below:

$$\frac{dy}{dx} = f(x) \Leftrightarrow dy = f(x)\,dx$$

Integrating both sides, we obtain

$$\int dy = \int f(x)\,dx + C$$

or, $$y = \int f(x)\,dx + C$$

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve :*

(i) $\frac{dy}{dx} = \frac{x}{x^2+1}$ (ii) $(e^x + e^{-x})\frac{dy}{dx} = (e^x - e^{-x})$ [NCERT]

SOLUTION (i) We have,

$$\frac{dy}{dx} = \frac{x}{x^2+1}$$

$$\Rightarrow \quad dy = \frac{x}{x^2+1}\,dx$$

Integrating both sides, we get

$$\int dy = \int \frac{x}{x^2+1}\,dx$$

$$\Rightarrow \quad \int dy = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

$$\Rightarrow \quad y = \frac{1}{2}\log|x^2+1| + C$$

Clearly, $y = \frac{1}{2}\log|x^2+1| + C$ is defined for all $x \in R$.

Hence, $y = \frac{1}{2}\log|x^2+1| + C$, $x \in R$ is a solution of the given differential equation.

(ii) $$(e^x + e^{-x})\frac{dy}{dx} = (e^x - e^{-x})$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\Rightarrow \quad dy = \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

Integrating both sides, we get

$$\int dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

$$\Rightarrow \quad \int dy = \int \frac{dt}{t}, \text{ where } e^x + e^{-x} = t$$

$$\Rightarrow \quad y = \log|t| + C$$

$$\Rightarrow \quad y = \log|e^x + e^{-x}| + C.$$

Clearly, $y = \log|e^x + e^{-x}| + C$ is defined for all $x \in R$.

Hence, $y = \log|e^x + e^{-x}| + C, x \in R$ is a solution of the given differential equation.

EXAMPLE 2 *Solve :*

(i) $(x+2)\frac{dy}{dx} = x^2 + 4x - 9, x \neq -2$

(ii) $\frac{dy}{dx} = \sin^3 x \cos^2 x + x e^x$

SOLUTION We have,

(i) $$(x+2)\frac{dy}{dx} = x^2 + 4x - 9$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 + 4x - 9}{x+2} \qquad [\because x \neq -2]$$

$$\Rightarrow \quad dy = \left(\frac{x^2 + 4x - 9}{x+2}\right)dx$$

Integrating both sides, we get

$$\int dy = \int \frac{x^2 + 4x - 9}{x+2}\,dx$$

$$\Rightarrow \quad \int dy = \int \left(x + 2 - \frac{13}{x+2}\right)dx$$

$$\Rightarrow \quad y = \frac{x^2}{2} + 2x - 13\log|x+2| + C$$

Clearly, it is defined for all $x \in R$, except $x = -2$.

Hence, $y = \frac{x^2}{2} + 2x - 13\log|x+2| + C,\ x \in R - \{2\}$ is a solution of the given differential equation.

(ii) $$\frac{dy}{dx} = \sin^3 x \cos^2 x + x e^x$$

$$\Rightarrow \quad dy = (\sin^3 x \cos^2 x + x e^x)\,dx$$

Integrating both sides, we get

$$\int dy = \int (\sin^3 x \cos^2 x + x e^x)\, dx$$

$$\Rightarrow \quad \int dy = \int \sin^3 x \cos^2 x\, dx + \int x e^x\, dx$$

$$\Rightarrow \quad \int dy = \int \cos^2 x (1 - \cos^2 x) \sin x\, dx + \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^x}\, dx$$

$$\Rightarrow \quad y = -\int t^2 (1 - t^2)\, dt + \{x e^x - \int e^x\, dx\}, \text{ where } t = \cos x$$

$$\Rightarrow \quad y = -\left\{\frac{t^3}{3} - \frac{t^5}{5}\right\} + (xe^x - e^x) + C$$

$$\Rightarrow \quad y = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + x e^x - e^x + C,$$

Clearly, $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + x e^x - e^x + C$ is defined for all $x \in R$.

Hence, $y = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + x e^x - e^x + C, x \in R$ is a solution of the given differentia equation.

EXAMPLE 3 *Solve :*

(i) $\frac{dy}{dx} = \frac{1}{\sin^4 x + \cos^4 x}$ (ii) $\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$

SOLUTION (i) We have,

$$\frac{dy}{dx} = \frac{1}{\sin^4 x + \cos^4 x}$$

$$\Rightarrow \quad dy = \frac{1}{\sin^4 x + \cos^4 x}\, dx$$

Integrating both sides w.r.t. x, we get

$$\int dy = \int \frac{1}{\sin^4 x + \cos^4 x}\, dx$$

$$\Rightarrow \quad \int dy = \int \frac{\sec^4 x}{\tan^4 x + 1}\, dx$$

[Dividing num. and denom. on RHS by $\cos^4 x$ supposition that with the $\cos^4 x \neq 0$ i.e. $\neq (2n+1)\pi/2, n \in Z$

$$\Rightarrow \quad \int dy = \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^4 x + 1}\, dx$$

$$\Rightarrow \quad \int dy = \int \frac{(1 + \tan^2 x) \sec^2 x}{\tan^4 x + 1}\, dx$$

$$\Rightarrow \quad \int dy = \int \frac{1 + t^2}{1 + t^4}\, dt \quad , \text{ where } t = \tan x$$

$$\Rightarrow \quad \int dy = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}}\, dt$$

[Dividing num. and denom. by t^2 with the supposition that $t \neq 0$ i.e. $\tan x \neq 0 \Rightarrow x \neq n\pi, n \in Z$

$$\Rightarrow \quad \int dy = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2}\, dt$$

$$\Rightarrow \quad \int dy = \int \frac{du}{u^2 + (\sqrt{2})^2}, \text{ where } t - \frac{1}{t} = u$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{2}}\right) + C$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{2}}\right) + C, \text{ where } x \neq n\pi, (2n+1)\frac{\pi}{2}, n \in Z$$

This is the required primitive of the given differential equation.

(ii) $$\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3e^{2x}(1 + e^{2x})}{e^x + \frac{1}{e^x}} = \frac{3e^{2x}(1 + e^{2x})}{\frac{e^{2x} + 1}{e^x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3e^{3x}(1 + e^{2x})}{(1 + e^{2x})}$$

$$\Rightarrow \quad \frac{dy}{dx} = 3e^{3x}$$

$$\Rightarrow \quad dy = 3e^{3x}\, dx$$

$$\Rightarrow \quad \int dy = 3\int e^{3x}\, dx$$ [Integrating both sides]

$$\Rightarrow \quad y = 3\left(\frac{e^{3x}}{3}\right) + C$$

$\Rightarrow \quad y = e^{3x} + C$, which is the required solution.

EXAMPLE 4 *Solve the initial value problem* $e^{(dy/dx)} = x + 1;\ y(0) = 5.$

SOLUTION We are given that

$$e^{dy/dx} = x + 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \log(x+1)$$

$$\Rightarrow \quad dy = \log(x+1)\, dx$$

Integrating both sides, we get

$$\int 1 \cdot dy = \int \underset{\text{I}}{\log(x+1)} \cdot \underset{\text{II}}{1}\, dx$$

$$\Rightarrow \quad y = x\log(x+1) - \int \frac{x}{x+1}\, dx$$

$$\Rightarrow \quad y = x\log(x+1) - \int \frac{x+1-1}{x+1}\, dx$$

$$\Rightarrow \quad y = x\log(x+1) - \int \left(1 - \frac{1}{x+1}\right) dx$$

$$\Rightarrow \quad y = x\log(x+1) - x + \log(x+1) + C \quad \text{...(i)}$$

It is given that $y(0) = 5$ i.e., when $x = 0$, we have $y = 5$.

$\therefore \quad 5 = \log 1 - 0 + \log 1 + C$

$\Rightarrow \quad C = 5$ [Substituting $x = 0, y = 5$ in (i)]

Substituting the value of C in (i), we get

$$y = x \log (x+1) - x + \log (x+1) + 5$$

We observe that $x \log (x+1) - x + \log (x+1) + 5$ is defined for all $x \in (-1, \infty)$.

Hence, $y = x \log (x+1) - x + \log (x+1) + 5$, where $x \in (-1, \infty)$ is the solution of the given initial value problem.

EXERCISE 22.5

Solve the following differential equations:

1. $\frac{dy}{dx} = x^2 + x - \frac{1}{x}, x \neq 0$
2. $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}, x \neq 0$
3. $\frac{dy}{dx} + 2x = e^{3x}$
4. $(x^2+1)\frac{dy}{dx} = 1$
5. $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$ [CBSE 2002]
6. $(x+2)\frac{dy}{dx} = x^2 + 3x + 7$
7. $\frac{dy}{dx} = \tan^{-1} x$
8. $\frac{dy}{dx} = \log x$
9. $\frac{1}{x}\frac{dy}{dx} = \tan^{-1} x, \; x \neq 0$
10. $\frac{dy}{dx} = \cos^3 x \sin^2 x + x\sqrt{2x+1}, x \in [-1/2,$
11. $(\sin x + \cos x)\, dy + (\cos x - \sin x)\, dx = 0, x \neq n\pi + \frac{3\pi}{4}, n \in Z$
12. $\frac{dy}{dx} - x \sin^2 x = \frac{1}{x \log x}$
13. $\frac{dy}{dx} = x^5 \tan^{-1}(x^3)$
14. $\sin^4 x \frac{dy}{dx} = \cos x$
15. $\cos x \frac{dy}{dx} - \cos 2x = \cos 3x$
16. $\sqrt{1-x^4}\, dy = x\, dx$
17. $\sqrt{a+x}\, dy + x\, dx = 0$
18. $(1+x^2)\frac{dy}{dx} - x = 2\tan^{-1} x$ [NCERT, CBSE 2007]
19. $\frac{dy}{dx} = x \log x$
20. $\frac{dy}{dx} = x e^x - \frac{5}{2} + \cos^2 x$
21. $(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$ [NCERT, CBSE 2010]

Solve the following initial value problems: (21-24)

22. $\sin\left(\frac{dy}{dx}\right) = k; y(0) = 1$
23. $e^{dy/dx} = x + 1; \; y(0) = 3$
24. $C'(x) = 2 + 0.15x; C(0) = 100$
25. $x\frac{dy}{dx} + 1 = 0; y(-1) = 0$

ANSWERS

1. $y = \frac{x^3}{3} + \frac{x^2}{2} - \log|x| + C, x \neq 0$
2. $y = \frac{x^6}{6} + \frac{x^3}{3} - 2\log|x| + C, x \neq 0$
3. $y + x^2 = \frac{1}{3}e^{3x} + C, x \in R$
4. $y = \tan^{-1} x + C, x \in R$

5. $y = 2\tan\dfrac{x}{2} - x + C, x \neq (2n+1)\pi, n \in Z$

6. $y = \dfrac{x^2}{2} + x + 5\log|x+2| + C, x \neq -2$

7. $y = x\tan^{-1}x - \dfrac{1}{2}\log|1+x^2| + C, x \in R$

8. $y = x(\log x - 1) + C, x \in (0, \infty)$

9. $y = \dfrac{1}{2}(x^2+1)\tan^{-1}x - \dfrac{1}{2}x + C, x \neq 0$

10. $y = \dfrac{1}{3}\sin^3 x - \dfrac{1}{5}\sin^5 x + \dfrac{1}{10}(2x+1)^{5/2} - \dfrac{1}{6}(2x+1)^{3/2} + C, x \in \left[-\dfrac{1}{2}, \infty\right)$

11. $y + \log|\sin x + \cos x| = C, x \neq n\pi + \dfrac{3\pi}{4}, n \in Z$

12. $y = \dfrac{x^2}{4} - \dfrac{x\sin 2x}{4} - \dfrac{\cos 2x}{8} + \log|\log x| + C, x \in (0, 1) \cup (1, \infty)$

13. $y = \dfrac{1}{6}[x^6\tan^{-1}x^3 - x^3 + \tan^{-1}x^3] + C, x \in R$

14. $y = -\dfrac{1}{3}\operatorname{cosec}^3 x + C, x \neq n\pi, n \in Z.$

15. $y = \sin 2x - x + 2\sin x + \log|\sec x + \tan x| + C, x \neq (2n+1)\dfrac{\pi}{2}, n \in Z$

16. $y = \dfrac{1}{2}\sin^{-1}(x^2) + C, x \in (-1, 1)$

17. $y + \dfrac{2}{3}(a+x)^{3/2} - 2a\sqrt{a+x} = C, x \in (-a, \infty)$

18. $y = \dfrac{1}{2}\log|1+x^2| + (\tan^{-1}x)^2 + C$

19. $y = \dfrac{1}{2}x^2\log x - \dfrac{x^2}{4} + C, x \in (0, \infty)$

20. $y = xe^x - e^x - 2x + \dfrac{1}{4}\sin 2x + C$

21. $y = \dfrac{1}{2}\log|x+1| + \dfrac{3}{4}\log(x^2+1) - \dfrac{1}{2}\tan^{-1}x + C$

22. $y - 1 = x\sin^{-1}(k)$, where $k \in (-1, 1)$ and $x \in R$

23. $y = (x+1)\log(x+1) - x + 3, x \in (-1, \infty)$

24. $C(x) = 2x + (0{\cdot}15)\dfrac{x^2}{2} + 100$

25. $y = -\log|x|, x < 0$

HINTS TO SELECTED PROBLEMS

13. Put $x^3 = t$ and integrate by parts

15. We have,

$$\frac{dy}{dx} = \frac{4\cos^3 x - 3\cos x + 2\cos^2 x - 1}{\cos x}, \text{ provided that } \cos x \neq 0$$

$$\Rightarrow \quad \frac{dy}{dx} = 4\cos^2 x - 3 + 2\cos x - \sec x$$

Now, integrate both sides.

16. We have, $\frac{dy}{dx} = \frac{x}{\sqrt{1-x^4}}$, provided that $1 - x^4 < 0$ i.e. $-1 < x < 1$

Put $x^2 = t$ and integrate.

17. We have,

$$dy = -\frac{x}{\sqrt{a+x}}\,dx \text{ , provided that } x > -a.$$

$$\Rightarrow \quad dy = -\frac{(x+a-a)}{\sqrt{a+x}}\,dx. = -\left\{\sqrt{a+x} + \frac{a}{\sqrt{a+x}}\right\} dx$$

Now, integrate both sides.

21. We have,

$$\sin\left(\frac{dy}{dx}\right) = k$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} k$$

$$\Rightarrow dy = \sin^{-1} k \, dx$$

Now, integrate both sides.

22. We have,

$$e^{dy/dx} = x + 1 \Rightarrow \frac{dy}{dx} = \log(x+1), \text{ provided that } x + 1 > 0$$

Now, integrate both sides.

23. We have,

$$\frac{dC}{dx} = 2 + 0.15\,x \quad \Rightarrow C = 2x + \frac{0.15\,x^2}{2} + k$$

For $x = 0$, we have $C = 100$

$\therefore\ k = 100$

Hence, $C = 2x + \frac{0.15}{2}x^2 + 100$

22.8.2 DIFFERENTIAL EQUATIONS OF THE TYPE $\frac{dy}{dx} = f(y)$

To solve this type of differential equations we integrate both sides to obtain the general solution as discussed under:

$$\frac{dy}{dx} = f(y)$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{1}{f(y)}, \text{ provided that } f(y) \neq 0$$

$$\Rightarrow \quad dx = \frac{1}{f(y)}\,dy$$

Integrating both sides, we obtain

$$\int dx = \int \frac{1}{f(y)}\,dy + C \text{ or, } x = \int \frac{1}{f(y)}\,dy + C$$

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve :*

(i) $\frac{dy}{dx}=\frac{1}{y^2+\sin y}, y \neq 0$ (ii) $\frac{dy}{dx}=\sec y, y \neq (2n+1)\frac{\pi}{2}, n \in Z$

SOLUTION (i) We have,

$$\frac{dy}{dx} = \frac{1}{y^2+\sin y}$$

$$\Rightarrow \quad \frac{dx}{dy} = y^2+\sin y$$

$$\Rightarrow \quad dx = (y^2+\sin y)\, dy$$

Integrating both sides, we obtain

$$\int dx = \int (y^2+\sin y)\, dy$$

$$\Rightarrow \quad x = \frac{y^3}{3} - \cos y + C,$$

Hence, $x=\frac{y^3}{3}-\cos y+C$, where $y \neq 0$ is the required solution.

(ii) $$\frac{dy}{dx} = \sec y$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{1}{\sec y} = \cos y$$

$$\Rightarrow \quad dx = \cos y\, dy$$

Integrating both sides, we obtain

$$\int dx = \int \cos y\, dy$$

$$\Rightarrow \quad x = \sin y + C,$$

Hence, $x = \sin y + C$, where $y \neq (2n+1)\frac{\pi}{2}, n \in Z$ is the required solution.

EXAMPLE 2 *Solve :* $\frac{dy}{dx}+y=1$

SOLUTION We have,

$$\frac{dy}{dx}+y = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = 1-y$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{1}{1-y}, \text{ provided that } y \neq 1$$

$$\Rightarrow \quad dx = \frac{1}{1-y}\, dy$$

Integrating both sides, we get

$$\int dx = \int \frac{1}{1-y}\, dy$$

$$\Rightarrow \quad x = -\log|1-y| + C,$$

Hence, $x=-\log|1-y|+C$, where $y \neq 1$, is the general solution of the given differential equation.

EXAMPLE 3 *Solve the initial value problem* $\frac{dy}{dx} + 2y^2 = 0$, $y(1) = 1$ *and find the corresponding solution curve.*

SOLUTION We have,

$$\frac{dy}{dx} + 2y^2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -2y^2$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{1}{2y^2}, \text{ if } y \neq 0$$

Integrating both sides with respect to y, we get

$$\int dx = \int -\frac{1}{2y^2}\, dy$$

$$\Rightarrow \quad x = \frac{1}{2y} + C \quad \text{...(i)}$$

We have, $y(1) = 1$ i.e. $y = 1$ when $x = 1$.

Putting $x = 1$ and $y = 1$ in (i), we get

$$1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

Putting $C = 1/2$ in (i), we get

$$x = \frac{1}{2y} + \frac{1}{2}$$

$$\Rightarrow \quad y = \frac{1}{2x-1}$$

Clearly, it is defined for all $x \in R - \left\{\frac{1}{2}\right\}$.

Hence, the required solution curve is $y = \frac{1}{2x-1}$, $x \neq \frac{1}{2}$

EXERCISE 22

Solve the following differential equations:

1. $\frac{dy}{dx} + \frac{1+y^2}{y} = 0, y \neq 0$
2. $\frac{dy}{dx} = \frac{1+y^2}{y^3}, y \neq 0$
3. $\frac{dy}{dx} = \sin^2 y$
4. $\frac{dy}{dx} = \frac{1-\cos 2y}{1+\cos 2y}$

ANSWERS

1. $x + \frac{1}{2}\log|1+y^2| = C$
2. $x = \frac{y^2}{2} - \frac{1}{2}\log|y^2+1| + C$
3. $x + \cot y = C$
4. $x + \cot y + y = C$

22.8.3 EQUATIONS IN VARIABLE SEPARABLE FORM

If the differential equation can be put in the form $f(x)\, dx = g(y)\, dy$ we say that the variables are separable and such equations can be solved by integrating on both sides. The solution is given by

$$\int f(x)\, dx = \int g(y)\, dy + C,$$

where C is an arbitrary constant.

NOTE *There is no need of introducing arbitrary constants on both sides as they can be combined together to give just one arbitrary constant.*

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve :*

(i) $(x+1)\dfrac{dy}{dx}=2xy$

(ii) $\cos x\,(1+\cos y)\,dx-\sin y\,(1+\sin x)\,dy=0$

SOLUTION (i) We have,

$$(x+1)\frac{dy}{dx} = 2xy$$

$$\Rightarrow \quad (x+1)\,dy = 2xy\,dx$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{2x}{x+1}\,dx, \text{ if } x \neq -1 .$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = 2\int \frac{x}{x+1}\,dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = 2\int \frac{x+1-1}{x+1}\,dx$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = 2\int \left(1-\frac{1}{x+1}\right)dx$$

$$\Rightarrow \quad \log y = 2\,[x-\log|x+1|]+C,$$

Clearly, it is defined for all $x \in R-\{-1\}$.

Hence, $\log y = 2\,[x-\log|x+1|\,]+C$, is the solution of the given differential equation.

(ii) $$\cos x\,(1+\cos y)\,dx-\sin y\,(1+\sin x)\,dy=0$$

$$\Rightarrow \quad \frac{\cos x}{1+\sin x}\,dx-\frac{\sin y}{1+\cos y}\,dy = 0$$

$$\Rightarrow \quad \int \frac{\cos x}{1+\sin x}\,dx+\int \frac{-\sin y}{1+\cos y}\,dy=0 \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log|1+\sin x|+\log|1+\cos y| = \log C$$

$$\Rightarrow \quad \log(|1+\sin x|\cdot|1+\cos y|) = \log C$$

$$\Rightarrow \quad |1+\sin x|\,|1+\cos y| = C$$

$$\Rightarrow \quad (1+\sin x)(1+\cos y) = C,\ x \in R.$$

This is the required solution.

EXAMPLE 2 *Solve :*

(i) $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$ **[NCERT, CBSE 2007]**

(ii) $e^x\sqrt{1-y^2}\,dx+\dfrac{y}{x}\,dy=0$

SOLUTION (i) We have,

$$\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$$

$$\Rightarrow \quad \sec^2 x\tan y\,dx = -\sec^2 y\tan x\,dy$$

$$\Rightarrow \quad \frac{\sec^2 x}{\tan x}\,dx = -\frac{\sec^2 y}{\tan y}\,dy$$

$$\Rightarrow \quad \int \frac{\sec^2 x}{\tan x}\,dx = -\int \frac{\sec^2 y}{\tan y}\,dy \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log|\tan x| = -\log|\tan y| + \log C$$

$$\Rightarrow \quad \log|(\tan x)(\tan y)| = \log C$$

$$\Rightarrow \quad |\tan x \tan y| = C$$

Clearly, it is defined for $x \in R - \{(2n+1)\pi/2 : n \in Z\}$

Hence, $|\tan x \tan y| = C$, where $x \in R - \{(2n+1)\pi/2 : n \in Z\}$ is the solution of the given differential equation.

(ii) We are given that

$$e^x\sqrt{1-y^2}\,dx + \frac{y}{x}\,dy = 0$$

$$\Rightarrow \quad e^x\sqrt{1-y^2}\,dx = -\frac{y}{x}\,dy$$

$$\Rightarrow \quad x\,e^x\,dx = -\frac{y}{\sqrt{1-y^2}}\,dy$$

$$\Rightarrow \quad \int \underset{\text{I}}{x}\,\underset{\text{II}}{e^x}\,dx = -\int \frac{y}{\sqrt{1-y^2}}\,dy \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad xe^x - \int e^x\,dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}, \text{ where } t = 1-y^2$$

$$\Rightarrow \quad xe^x - e^x = \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + C$$

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$$\Rightarrow \quad xe^x - e^x = \sqrt{t} + C$$

$$\Rightarrow \quad xe^x - e^x = \sqrt{1-y^2} + C, \text{ where } x \in R \text{ is the required solution.}$$

EXAMPLE 3 *Solve the differential equation* $(1+e^{2x})\,dy + (1+y^2)\,e^x\,dx = 0$ *given that when* $x = 0, y = 1$. **[NCERT, CBSE 2004, 2005]**

SOLUTION We are given that

$$(1+e^{2x})\,dy + (1+y^2)\,e^x\,dx = 0$$

$$\Rightarrow \quad (1+e^{2x})\,dy = -(1+y^2)\,e^x\,dx$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = -\frac{e^x}{1+e^{2x}}\,dx$$

$$\Rightarrow \quad \int \frac{1}{1+y^2}\,dy = -\int \frac{e^x}{1+e^{2x}}\,dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int \frac{1}{1+y^2}\,dy = -\int \frac{dt}{1+t^2}, \text{ where } t = e^x$$

$$\Rightarrow \quad \tan^{-1} y = -\tan^{-1}(t) + C$$

$$\Rightarrow \quad \tan^{-1} y = -\tan^{-1}(e^x) + C \qquad \text{...(i)}$$

It is given that $y = 1$, when $x = 0$. So, putting $x = 0, y = 1$ in (i), we get

$$\tan^{-1} 1 = -\tan^{-1}(e^0) + C$$

$$\Rightarrow \quad \frac{\pi}{4} = -\frac{\pi}{4} + C \Rightarrow C = \frac{\pi}{2}$$

Putting $C = \frac{\pi}{2}$ in (i), we obtain

$$\tan^{-1} y = -\tan^{-1}(e^x) + \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1} y + \tan^{-1}(e^x) = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1} y = \frac{\pi}{2} - \tan^{-1}(e^x)$$

$$\Rightarrow \quad \tan^{-1} y = \cot^{-1}(e^x)$$

$$\Rightarrow \quad \tan^{-1} y = \tan^{-1}\left(\frac{1}{e^x}\right) \Rightarrow y = \frac{1}{e^x}$$

This is the required solution.

EXAMPLE 4 *Solve the differential equation* $(1 + y^2)(1 + \log x)\, dx + x\, dy = 0$ *given that when* $x = 1, y = 1$.

SOLUTION The given differential equation is

$$(1 + y^2)(1 + \log x)\, dx + x\, dy = 0$$

$$\Rightarrow \quad (1 + \log x)(1 + y^2)\, dx = -x\, dy$$

$$\Rightarrow \quad \frac{(1 + \log x)}{x}\, dx = -\frac{1}{1 + y^2}\, dy$$

$$\Rightarrow \quad \int \frac{1 + \log x}{x}\, dx = -\int \frac{1}{1 + y^2}\, dy \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int t\, dt = -\int \frac{1}{1 + y^2}\, dy, \text{ where } 1 + \log x = t$$

$$\Rightarrow \quad \frac{t^2}{2} = -\tan^{-1} y + C$$

$$\Rightarrow \quad \frac{1}{2} \cdot (1 + \log x)^2 = -\tan^{-1} y + C \qquad \text{...(i)}$$

It is given that when $x = 1, y = 1$. So, putting $x = 1, y = 1$ in (i), we obtain

$$\frac{1}{2}[1 + \log 1]^2 = -\tan^{-1} 1 + C$$

$$\Rightarrow \quad \frac{1}{2} = -\frac{\pi}{4} + C \Rightarrow C = \frac{1}{2} + \frac{\pi}{4}$$

Putting $C = \frac{1}{2} + \frac{\pi}{4}$ in (i), we obtain

$$\frac{1}{2}(1 + \log x)^2 = -\tan^{-1} y + \frac{1}{2} + \frac{\pi}{4}$$

$$\tan^{-1} y = \frac{\pi}{4}+\frac{1}{2}-\frac{1}{2}(1+\log x)^2$$

$$\Rightarrow \quad y = \tan\left\{\frac{\pi}{4}+\frac{1}{2}-\frac{1}{2}(1+\log x)\right\}$$

Clearly, it is defined for all $x > 0$.

Hence, $y = \tan\left\{\frac{\pi}{4}+\frac{1}{2}-\frac{1}{2}(1+\log x)\right\}, x > 0$ is the solution of the given differential equation.

EXAMPLE 5 *Solve the differential equation* $x(1+y^2)\,dx - y(1+x^2)\,dy = 0$, *given that* $y = 0$, *when* $x = 1$.

SOLUTION The given differential equation is

$$x(1+y^2)\,dx - y(1+x^2)\,dy = 0$$

$$\Rightarrow \quad x(1+y^2)\,dx = y(1+x^2)\,dy$$

$$\Rightarrow \quad \frac{x}{1+x^2}\,dx = \frac{y}{1+y^2}\,dy$$

$$\Rightarrow \quad \frac{2x}{1+x^2}\,dx = \frac{2y}{1+y^2}\,dy$$

Integrating both sides, we get

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{2y}{1+y^2}\,dy$$

$$\Rightarrow \quad \log|1+x^2| = \log|1+y^2| + \log C$$

$$\Rightarrow \quad \log\left|\frac{1+x^2}{1+y^2}\right| = \log C$$

$$\Rightarrow \quad \frac{1+x^2}{1+y^2} = C$$

$$\Rightarrow \quad (1+x^2) = (1+y^2)\,C \qquad \text{...(i)}$$

It is given that when $x = 1$, where $y = 0$. So, putting $x = 1$, and $y = 0$ in (i), we get

$$(1+1) = (1+0)\,C \Rightarrow C = 2$$

Putting $C = 2$ in (i), we get

$$(1+x^2) = 2(1+y^2) \Rightarrow y = \pm\sqrt{\frac{x^2-1}{2}}$$

This is defined for $x^2 - 1 \geq 0$ i.e. for $x \in (-\infty, -1] \cup [1, \infty)$.

Thus, we obtain two functions

$f : (-\infty, -1] \cup [1, \infty) \to R$ and $g : (-\infty, -1] \cup [1, \infty) \to R$ given by

$$f(x) = \sqrt{\frac{x^2-1}{2}} \quad \text{and,} \quad g(x) = -\sqrt{\frac{x^2-1}{2}}$$

But, both f and g are not differentiable at $x = \pm 1$.

Hence, $f, g : (-\infty, -1) \cup (1, \infty) \to R$

given by $f(x) = \sqrt{\frac{x^2-1}{2}}$ and $g(x) = -\sqrt{\frac{x^2-1}{2}}$ are solutions of the given differential equation.

EXAMPLE 6 *Solve the following differential equations:*

(i) $\frac{dy}{dx} = 1 + x + y + xy$

(ii) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$ **[CBSE 2002]**

SOLUTION (i) We are given that

$$\frac{dy}{dx} = 1 + x + y + xy$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x) + y(1+x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x)(1+y)$$

$$\Rightarrow \quad \frac{1}{1+y}\,dy = (1+x)\,dx, \text{ if } 1+y \neq 0 \text{ i.e. } y \neq -1$$

$$\Rightarrow \quad \int \frac{1}{1+y}\,dy = \int (1+x)\,dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log|1+y| = x + \frac{x^2}{2} + C,$$

$$\Rightarrow \quad |1+y| = e^{x+\frac{x^2}{2}+C} \Rightarrow y = \pm e^{x+\frac{x^2}{2}+C} - 1$$

Clearly, it is defined for all $x \in R$. Hence, $y = \pm e^{x+\frac{x^2}{2}+C} - 1, x \in R$ is the general solution of the given differential equation.

(ii) The given differential equation is

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad y - ay^2 = \frac{dy}{dx}(a+x)$$

$$\Rightarrow \quad (y - ay^2)\,dx = (a+x)\,dy$$

$$\Rightarrow \quad \frac{dx}{a+x} = \frac{dy}{y - ay^2}, \text{ if } x \neq -a, y \neq 0, \frac{1}{a}$$

$$\Rightarrow \quad \int \frac{1}{a+x}\,dx = \int \frac{1}{y - ay^2}\,dy \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int \frac{1}{a+x}\,dx = \int \left(\frac{1}{y} + \frac{a}{1-ay}\right) dy \qquad \text{[By partial fractions]}$$

$$\Rightarrow \quad \log|x+a| = \log|y| - \log|1-ay| + \log C$$

$$\Rightarrow \quad \log\left|\frac{(x+a)(1-ay)}{y}\right| = \log C$$

$$\Rightarrow \quad \frac{(x+a)(1-ay)}{y} = C$$

$$\Rightarrow \quad (x+a)(1-ay) = Cy$$

Hence, $(x+a)(1-ay) = Cy$, $x \neq -a$ is the general solution of the given differential equation.

EXAMPLE 7 *Solve :*

(i) $(x^2 - yx^2)\,dy + (y^2 + x^2 y^2)\,dx = 0$

(ii) $3e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$ [NCERT]

SOLUTION (i) The given differential equations is

$$x^2(1-y)\,dy + y^2(1+x^2)\,dx = 0$$

$$\Rightarrow \quad x^2(1-y)\,dy = -y^2(1+x^2)\,dx$$

$$\Rightarrow \quad \frac{1-y}{y^2}\,dy = -\left(\frac{1+x^2}{x^2}\right)dx, \text{ if } x, y \neq 0$$

$$\Rightarrow \quad \left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \left(\frac{1}{x^2} + 1\right)dx$$

$$\Rightarrow \quad \int\left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \int\left(\frac{1}{x^2} + 1\right)dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log|y| + \frac{1}{y} = -\frac{1}{x} + x + C$$

Hence, $\log|y| + \frac{1}{y} = -\frac{1}{x} + x + C, x \in R - \{0\}$ gives the general solution of the differential equation.

(ii) We are given that

$$3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$$

$$\Rightarrow \quad 3e^x \tan y\,dx = -(1-e^x)\sec^2 y\,dy$$

$$\Rightarrow \quad \frac{3e^x}{1-e^x}\,dx = -\frac{\sec^2 y}{\tan y}\,dy$$

$$\Rightarrow \quad 3\int \frac{e^x}{1-e^x}\,dx = -\int \frac{\sec^2 y}{\tan y}\,dy, \text{ if } x \neq 0, y \neq 0 \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad 3\int \frac{e^x}{e^x - 1}\,dx = \int \frac{\sec^2 y}{\tan y}\,dy$$

$$\Rightarrow \quad 3\log|e^x - 1| = \log|\tan y| + \log C$$

$$\Rightarrow \quad \log\left(\frac{|e^x - 1|^3}{|\tan y|}\right) = \log C$$

$$\Rightarrow \quad \frac{(e^x - 1)^3}{\tan y} = C$$

$$\Rightarrow \quad (e^x - 1)^3 = C\tan y$$

Hence, $(e^x - 1)^3 = C\tan y, x \in R - \{0\}$ gives the general solution of the given differential equation.

EXAMPLE 8 *Solve :*

(i) $\sin^3 x \dfrac{dx}{dy} = \sin y$ (ii) $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$ [NCERT]

SOLUTION (i) The given differential equation is

$$\sin^3 x \frac{dx}{dy} = \sin y$$

$\Rightarrow \quad \sin^3 x\, dx = \sin y\, dy$

$\Rightarrow \quad \int \sin^3 x\, dx = \int \sin y\, dy$ [Integrating both sides]

$\Rightarrow \quad \int \frac{3 \sin x - \sin 3x}{4}\, dx = \int \sin y\, dy$

$\Rightarrow \quad -\frac{3}{4} \cos x + \frac{1}{12} \cos 3x = -\cos y + C$

$\Rightarrow \quad \cos y - \frac{3}{4} \cos x + \frac{1}{12} \cos 3x = C$

Hence, $\cos y - \frac{3}{4} \cos x + \frac{1}{12} \cos 3x = C, x \in R$ gives the required solution.

(ii) The given differential equation is

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$\Rightarrow \quad \sqrt{1-x^2}\, dy = -\sqrt{1-y^2}\, dx$

$\Rightarrow \quad \frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}}$

$\Rightarrow \quad \int \frac{1}{\sqrt{1-y^2}}\, dy = -\int \frac{1}{\sqrt{1-x^2}}\, dx$ [Integrating both sides]

$\Rightarrow \quad \sin^{-1} y = -\sin^{-1} x + \sin^{-1} C$

$\Rightarrow \quad \sin^{-1} y + \sin^{-1} x = \sin^{-1} C$

$\Rightarrow \quad \sin^{-1} [y\sqrt{1-x^2} + x\sqrt{1-y^2}] = \sin^{-1} C$

$\Rightarrow \quad y\sqrt{1-x^2} + x\sqrt{1-y^2} = C$

This is defined for $1 - x^2 \geq 0$ i.e. $x \in [-1, 1]$.

Hence, $y\sqrt{1-x^2} + x\sqrt{1-y^2} = C, x \in [-1, 1]$ is the general solution of the given differential equation.

EXAMPLE 9 *Solve :*

(i) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ (ii) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ [NCERT]

SOLUTION (i) We have,

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$\Rightarrow \quad dy = (e^{x-y} + x^2 e^{-y})\, dx$

$\Rightarrow \quad e^y\, dy = (e^x + x^2)\, dx$

$\Rightarrow \quad \int e^y\,dy = \int (e^x + x^2)\,dx$ [Integrating both sides]

$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + C$, which is the required solution.

(ii) The given differential equation is

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$\Rightarrow \quad (1+x^2)\,dy = (1+y^2)\,dx$

$\Rightarrow \quad \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

$\Rightarrow \quad \int \frac{1}{1+y^2}\,dy = \int \frac{1}{1+x^2}\,dx$ [Integrating both sides]

$\Rightarrow \quad \tan^{-1} y = \tan^{-1} x + \tan^{-1} C$

$\Rightarrow \quad \tan^{-1} y - \tan^{-1} x = \tan^{-1} C$

$\Rightarrow \quad \tan^{-1}\left(\frac{y-x}{1+xy}\right) = \tan^{-1} C$

$\Rightarrow \quad \frac{y-x}{1+xy} = C$

$\Rightarrow \quad y - x = C(1+xy)$, which is the required solution.

EXAMPLE 10 *Solve :*

(i) $\frac{dy}{dx} = e^{x+y}$ (ii) $\log\left(\frac{dy}{dx}\right) = ax + by$

SOLUTION (i) The given differential equation is

$$\frac{dy}{dx} = e^{x+y}$$

$\Rightarrow \quad \frac{dy}{dx} = e^x e^y$

$\Rightarrow \quad dy = e^x e^y\,dx$

$\Rightarrow \quad e^{-y}\,dy = e^x\,dx$

$\Rightarrow \quad \int e^{-y}\,dx = \int e^x\,dx$ [Integrating both sides]

$\Rightarrow \quad -e^{-y} = e^x + C$, which is the required solution.

(ii) We are given that

$$\log\left(\frac{dy}{dx}\right) = ax + by$$

$\Rightarrow \quad \frac{dy}{dx} = e^{ax+by}$

$\Rightarrow \quad \frac{dy}{dx} = e^{ax} e^{by}$

$\Rightarrow \quad dy = e^{ax} e^{by}\,dx$

$$\Rightarrow \quad e^{-by}\,dy = e^{ax}\,dx$$

$$\Rightarrow \quad \frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + C \qquad \text{[Integrating both sides]}$$

This is the required solution.

EXAMPLE 11 *Solve the initial value problem*

$$y' = y\cot 2x\,,\; y\left(\frac{\pi}{4}\right) = 2$$

SOLUTION We have,

$$y' = y\cot 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = y\cot 2x$$

$$\Rightarrow \quad \frac{1}{y}\,dy = \cot 2x\,dx$$

$$\Rightarrow \quad \log y = \frac{1}{2}\log\sin 2x + \log C \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad 2\log y = \log\sin 2x + 2\log C$$

$$\Rightarrow \quad \log y^2 = \log\sin 2x + \log C^2$$

$$\Rightarrow \quad y^2 = C^2\sin 2x \qquad \text{...(i)}$$

It is given that $y = 2$ when $x = \frac{\pi}{4}$. Putting $x = \pi/4$ and $y = 2$ in (i), we get

$$4 = C^2\sin\frac{\pi}{2} \Rightarrow C^2 = 4$$

Putting $C^2 = 4$ in (i), we get

$$y^2 = 4\sin 2x \Rightarrow y = \pm 2\sqrt{\sin 2x}$$

Clearly, it is defined for $\sin 2x \geq 0$ i.e. $2x \in (2n\pi\,, (2n+1)\pi)$

or, $\quad x \in \left(n\pi\,, \left(n + \frac{1}{2}\right)\pi\right), n \in Z.$

Hence, $y = \pm 2\sqrt{\sin 2x}\,, x \in \left(n\pi, \left(n+\frac{1}{2}\right)\pi\right), n \in Z$ is the solution of the given differential equation.

EXAMPLE 12 *Solve the initial value problem* $x(x\,dy - y\,dx) = y\,dx\,, y(1) = 1.$

SOLUTION We have,

$$x(x\,dy - y\,dx) = y\,dx$$

$$\Rightarrow \quad x^2\,dy = y(x+1)\,dx$$

$$\Rightarrow \quad \frac{1}{y}\,dy = \frac{x+1}{x^2}\,dx\,, \text{ if } x \neq 0, y \neq 0$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = \int\left(\frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$\Rightarrow \quad \log|y| = \log|x| - \frac{1}{x} + C$$

$$\Rightarrow \quad \log|y| - \log|x| = -\frac{1}{x} + C$$

$$\Rightarrow \quad \log\left\{\frac{|y|}{|x|}\right\} = -\frac{1}{x} + C$$

$$\Rightarrow \quad \log\left|\frac{y}{x}\right| = -\frac{1}{x} + C$$

$$\Rightarrow \quad \left|\frac{y}{x}\right| = e^{-1/x + C} \qquad \text{...(i)}$$

It is given that $y = 1$ when $x = 1$.

Putting $x = 1$ and $y = 1$ in (i), we get

$$1 = e^{-1+C} \Rightarrow e^0 = e^{-1+C} \Rightarrow C = 1$$

Putting $C = 1$ in (i), we get

$$\left|\frac{y}{x}\right| = e^{-1/x + 1}$$

$$\Rightarrow \quad \frac{y}{x} = \pm e^{1 - 1/x}$$

$$\Rightarrow \quad y = xe^{1-1/x} \text{ or } y = -xe^{1-1/x}$$

But, $\quad y = -x\,e^{1-1/x}$ is not satisfied by $y(1) = 1$

$\therefore \quad y = xe^{1-1/x}$, $x \neq 0$ is the required solution.

EXAMPLE 13 *Solve* $\frac{dy}{dx} = y \sin 2x$, *it being given that* $y(0) = 1$.

[CBSE 2004]

SOLUTION We have,

$$\frac{dy}{dx} = y \sin 2x$$

$$\Rightarrow \quad \frac{1}{y}\,dy = \sin 2x\,dx$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = \int \sin 2x\,dx$$

$$\Rightarrow \quad \log|y| = -\frac{1}{2}\cos 2x + C \qquad \text{...(i)}$$

It is given that $y(0) = 1$ i.e. $y = 1$ when $x = 0$.

Putting $x = 0$ and $y = 1$ in (i), we get

$$0 = -\frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

Putting $C = \frac{1}{2}$ in (i), we get

$$\log|y| = -\frac{1}{2}\cos 2x + \frac{1}{2}$$

$$\Rightarrow \quad \log|y| = \frac{1}{2}(1 - \cos 2x)$$

$$\Rightarrow \quad \log|y| = \sin^2 x$$

$\Rightarrow \quad |y| = e^{\sin^2 x}$

$\Rightarrow \quad y = \pm e^{\sin^2 x}$

$\Rightarrow \quad y = e^{\sin^2 x}$ or, $y = -e^{\sin^2 x}$

But, $\quad y = -e^{\sin^2 x}$ is not satisfied by $y(0) = 1$

$\therefore \quad y = e^{\sin^2 x}$ is the required solution.

EXAMPLE 14 *Solve the initial value problem* $x\,dy + y\,dx = xy\,dx,\ y(1) = 1$

SOLUTION We have,

$$x\,dy + y\,dx = xy\,dx$$

$$\Rightarrow \quad x\,dy = (x-1)\,y\,dx$$

$$\Rightarrow \quad \frac{1}{y}\,dy = \left(1 - \frac{1}{x}\right)dx\text{, if } x \neq 0, y \neq 0$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = \int \left(1 - \frac{1}{x}\right)dx$$

$$\Rightarrow \quad \log|y| = x - \log|x| + C$$

$$\Rightarrow \quad \log|y| + \log|x| = x + C$$

$$\Rightarrow \quad \log|xy| = x + C$$

$$\Rightarrow \quad |xy| = e^{x+C} \qquad \text{...(i)}$$

It is given that $y(1) = 1$ i.e. $y = 1$ when $x = 1$.

Putting $x = 1$ and $y = 1$ in (i), we get

$$1 = e^{1+C} \Rightarrow e^0 = e^{1+C} \Rightarrow C = -1$$

Putting $C = -1$ in (i), we get

$$\therefore \quad |xy| = e^{x-1}$$

$$\Rightarrow \quad xy = \pm e^{x-1}$$

$$\Rightarrow \quad y = \pm \frac{1}{x} e^{x-1}$$

$$\Rightarrow \quad y = \frac{1}{x} e^{x-1} \text{ or, } y = -\frac{1}{x} e^{x-1}$$

But, $y = -\frac{1}{x} e^{x-1}$ is not satisfied by $y(1) = 1$. Also, $y = \frac{1}{x} e^{x-1}$ is defined for all $x \neq 0$.

Hence, $y = \frac{1}{x} e^{x-1}, x \in R - \{0\}$ is the required solution.

EXAMPLE 15 *Find the equation of the curve passing through the point* $\left(0, \frac{\pi}{4}\right)$ *whose differential equation is* $\sin x \cos y\,dx + \cos x \sin y\,dy = 0$. **[NCERT]**

SOLUTION We have,

$$\sin x \cos y\,dx + \cos x \sin y\,dy = 0$$

$\Rightarrow \quad \dfrac{\sin x}{\cos x} dx + \dfrac{\sin y}{\cos y} dy = 0$ [On separating the variables]

$\Rightarrow \quad \tan x \, dx + \tan y \, dy = 0$

$\Rightarrow \quad \int \tan x \, dx + \int \tan y \, dy = 0$

$\Rightarrow \quad -\log | \cos x | - \log | \cos y | = \log C$

$\Rightarrow \quad -\log (| \cos x | \, | \cos y |) = \log C$

$\Rightarrow \quad \log | \cos x \cos y | = \log \left(\dfrac{1}{C}\right)$

$\Rightarrow \quad |\cos x \cos y| = \dfrac{1}{C}$

$\Rightarrow \quad \cos x \cos y = C_1 \text{ where } C_1 = \pm \dfrac{1}{C}.$ ...(i)

It is given that the curve passes through $\left(0, \dfrac{\pi}{4}\right)$.

$\therefore \quad y = \dfrac{\pi}{4} \text{ when } x = 0.$

Putting $x = 0$ and $y = \pi/4$ in (i), we get

$$\cos 0 \cos \frac{\pi}{4} = C_1 \Rightarrow C_1 = \frac{1}{\sqrt{2}}$$

Putting $C_1 = \dfrac{1}{\sqrt{2}}$ in (i), we get

$$\cos x \cos y = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad \cos y = \dfrac{1}{\sqrt{2}} \sec x$

$\Rightarrow \quad y = \cos^{-1}\left(\dfrac{1}{\sqrt{2}} \sec x\right)$

Hence, $y = \cos^{-1}\left(\dfrac{1}{\sqrt{2}} \sec x\right)$ is the required curve.

EXAMPLE 16 *Solve the initial value problem:* $dy = e^{2x+y} dx$, $y(0) = 0$.

SOLUTION We have,

$$dy = e^{2x+y} dx$$

$\Rightarrow \quad dy = e^{2x} . e^{y} dx$

$\Rightarrow \quad e^{-y} dy = e^{2x} dx$ [On separating the variables]

$\Rightarrow \quad \int e^{-y} dy = \int e^{2x} dx$

$\Rightarrow \quad -e^{-y} = \dfrac{e^{2x}}{2} + C$ ...(i)

It is given that $y(0) = 0$ i.e. $y = 0$ when $x = 0$.

Putting $x = 0$ and $y = 0$ in (i), we get

$$-1 = \frac{1}{2} + C \Rightarrow C = -\frac{3}{2}$$

Putting $C = -\frac{3}{2}$ in (i), we get

$$-e^{-y} = \frac{e^{2x}}{2} - \frac{3}{2}$$

$$\Rightarrow \quad e^{-y} = \frac{3-e^{2x}}{2}$$

$$\Rightarrow \quad e^{y} = \frac{2}{3-e^{2x}}$$

$$\Rightarrow \quad y = \log\left(\frac{2}{3-e^{2x}}\right)$$

Also, $\log\left(\frac{2}{3-e^{2x}}\right)$ is defined, if

$$\frac{2}{3-e^{2x}} > 0 \Rightarrow 3 - e^{2x} > 0$$

$$\Rightarrow \quad e^{2x} < 3 \Rightarrow 2x < \log_e 3$$

$$\Rightarrow \quad x < \frac{1}{2}\log_e 3 \Rightarrow x \in \left(-\infty, \frac{1}{2}\log_e 3\right)$$

Hence, $y = \log\left(\frac{2}{3-e^{2x}}\right), x \in \left(-\infty, \frac{1}{2}\log_e 3\right)$ is the required solution.

EXAMPLE 17 *Solve the following initial value problems:*

(i) $(x+1)\frac{dy}{dx} = 2e^{-y} - 1, y(0) = 0$ [NCERT] (ii) $y - x\frac{dy}{dx} = 2\left(1 + x^2\frac{dy}{dx}\right), y(1) = 1$

SOLUTION We have,

(i) $(x+1)\frac{dy}{dx} = 2e^{-y} - 1$

$$\Rightarrow \quad (x+1)\,dy = (2e^{-y} - 1)\,dx$$

$\Rightarrow \quad \frac{1}{x+1}dx = \frac{1}{2e^{-y}-1}dy$, if $x \neq -1$ and $y \neq \log_e 2$ [On separating the variables]

$$\Rightarrow \quad \int \frac{1}{x+1}dx = \int \frac{1}{2e^{-y}-1}dy$$

$$\Rightarrow \quad \int \frac{1}{x+1}dx = \int \frac{e^y}{2-e^y}dy$$

$$\Rightarrow \quad \int \frac{1}{x+1}dx = -\int \frac{e^y}{e^y-2}dy$$

$$\Rightarrow \quad \log|x+1| = -\log|e^y - 2| + \log C$$

$$\Rightarrow \quad \log|x+1| + \log|e^y - 2| = \log C$$

$$\Rightarrow \quad \log|(x+1)(e^y-2)| = \log C$$

$\Rightarrow \qquad |(x+1)(e^y-2)| = C$

It is given that $y(0)=0$ i.e. $y=0$ when $x=0$.

Putting $x=0$ and $y=0$ in (i), we get

$$|(0+1)(1-2)| = C \Rightarrow C = 1$$

Putting $C = 1$ in (i), we get

$$|(x+1)(e^y-2)| = 1$$

$\Rightarrow \qquad (x+1)(e^y-2) = \pm 1$

$\Rightarrow \qquad e^y - 2 = -\dfrac{1}{x+1}$, if $x \neq -1$

$\Rightarrow \qquad e^y = \left(2-\dfrac{1}{x+1}\right)$

$\Rightarrow \qquad y = \log\left(2-\dfrac{1}{x+1}\right), x \neq -1,$ which is the required solution.

(ii) $\qquad y - x\dfrac{dy}{dx} = 2\left(1+x^2\dfrac{dy}{dx}\right)$

$\Rightarrow \qquad y - 2 = x(2x+1)\dfrac{dy}{dx}$

$\Rightarrow \qquad (y-2)\,dx = x(2x+1)\,dy$

$\Rightarrow \qquad \dfrac{1}{x(2x+1)}dx = \dfrac{1}{y-2}dy$, if $x \neq 0, -\dfrac{1}{2}$ and $y \neq 2$

$\Rightarrow \qquad \displaystyle\int \frac{1}{x(2x+1)}dx = \int \frac{1}{y-2}dy$

$\Rightarrow \qquad \displaystyle\int \left(\frac{1}{x}-\frac{2}{2x+1}\right)dx = \int \frac{1}{y-2}dy$

$\Rightarrow \qquad \log|x| - \log|2x+1| = \log|y-2| + \log C$

$\Rightarrow \qquad \log\left|\dfrac{x}{2x+1}\right| = \log|y-2| + \log C$

$\Rightarrow \qquad \log\left|\dfrac{x}{2x+1}\right| - \log|y-2| = \log C$

$\Rightarrow \qquad \log\left|\dfrac{x}{2x+1}\times\dfrac{1}{y-2}\right| = \log C$

$\Rightarrow \qquad \left|\dfrac{x}{(2x+1)(y-2)}\right| = C$

It is given that $y(1)=1$ i.e. $y=1$ when $x=1$.

Putting $x=1$ and $y=1$ in (i), we get

$$\left|-\frac{1}{3}\right| = C \Rightarrow C = \frac{1}{3}$$

Putting $C = \dfrac{1}{3}$ in (i), we get

$$\left|\frac{x}{(2x+1)(y-2)}\right| = \frac{1}{3}$$

$$\Rightarrow \quad \frac{x}{(2x+1)(y-2)} = \pm\frac{1}{3}$$

$$\Rightarrow \quad y-2 = \pm\frac{3x}{2x+1}$$

$$\Rightarrow \quad y = 2\pm\frac{3x}{2x+1}$$

But, $y = 2+\frac{3x}{2x+1}$ is not satisfied by $y(1) = 1$

Hence, $y = 2-\frac{3x}{2x+1}$, where $x \neq -\frac{1}{2}$ is the required solution.

EXAMPLE 18 *Show that the general solution of the differential equation* $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1} = 0$ *is given by* $x+y+1 = A(1-x-y-2xy)$, *where A is a parameter.* **[NCERT]**

SOLUTION We have,

$$\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$$

$$\Rightarrow \quad \frac{1}{y^2+y+1}\,dy = -\frac{1}{x^2+x+1}\,dx \qquad \text{[On separating the variables]}$$

$$\Rightarrow \quad \frac{1}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dy = -\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

Integrating both sides, we get

$$\int \frac{1}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dy = -\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$\Rightarrow \quad \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = -\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2}C$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{2x+1+2y+1}{3-(2x+1)(2y+1)}\right\} = \frac{\sqrt{3}}{2}C$$

$$\Rightarrow \quad \frac{2x+2y+2}{2-2x-2y-4xy} = \tan\left(\frac{\sqrt{3}}{2}C\right)$$

$$\Rightarrow \quad x+y+1 = A(1-x-y-2xy), \text{ where } A = \tan\left(\frac{\sqrt{3}}{2}C\right)$$

This is the required solution.

EXAMPLE 19 *Find the particular solution of the differential equation* $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ *given that* $y = 0$ *when* $x = 0$. [NCERT]

SOLUTION We have,

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{3x+4y}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{3x} e^{4y}$$

$$\Rightarrow \quad e^{-4y}\, dy = e^{3x}\, dx$$

On integrating, we get

$$-\frac{1}{4} e^{-4y} = \frac{1}{3} e^{3x} + C$$

$$\Rightarrow \quad 4e^{3x} + 3e^{-4y} + 12C = 0 \quad \text{...(i)}$$

It is given that $y = 0$ when $x = 0$.

Substituting $x = 0$ and $y = 0$ in (i), we get

$$4 + 3 + 12C = 0 \Rightarrow C = -\frac{7}{12}$$

Substituting the value of C in (i), we get $4\, e^{3x} + 3\, e^{-4y} - 7 = 0$ as a particular solution of the given differential equation.

EXAMPLE 20 *Find the equation of the curve passing through the point* (1, 1) *whose differential equation is* $x\, dy = (2x^2 + 1)\, dx \ (x \neq 0)$.

SOLUTION We have,

$$x\, dy = (2x^2 + 1)\, dx$$

$$\Rightarrow \quad dy = \left(\frac{2x^2 + 1}{x}\right) dx$$

$$\Rightarrow \quad dy = \left(2x + \frac{1}{x}\right) dx$$

On integrating both sides, we get

$$\int dy = \int \left(2x + \frac{1}{x}\right) dx$$

$$\Rightarrow \quad y = x^2 + \log|x| + C \quad \text{...(i)}$$

This equation represents the family of solution curves of the given differential equation. We have to find a particular member of this family which passes through the point (1, 1).

Substituting $x = 1, y = 1$ in (i), we get

$$1 = 1 + 0 + C \Rightarrow C = 0$$

Putting $C = 0$ in (i), we get

$y = x^2 + \log |x|$ as the equation of the required curve.

EXAMPLE 21 *Find the equation of the curve passing through the point* $(-2, 3)$ *given that the slope of the tangent to the curve at any point* (x, y) *is* $\frac{2x}{y^2}$.

SOLUTION We know that the slope of the tangent to a curve is given by $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{y^2}$$

$$\Rightarrow \quad y^2 \, dy = 2x \, dx$$

On integrating both sides, we get

$$\int y^2 \, dy = \int 2x \, dx$$

$$\Rightarrow \quad \frac{y^3}{3} = x^2 + C \qquad \text{...(i)}$$

This equation represents the family of solution curves of given differential equation. We have to find a particular member of this family which passes through the point $(-2, 3)$.

Substituting $x = -2$ and $y = 3$ in (i), we get

$$9 = 4 + C \Rightarrow C = 5$$

Putting $C = 5$ in (i), we get

$\frac{y^3}{3} = x^2 + 5$ as the equation of the required curve.

EXAMPLE 22 *In a bank principal increases at the rate of 5% per year. In how many years Rs 1000 double itself.* **[NCERT]**

SOLUTION Let P be the principal at any time t. Then,

$$\frac{dP}{dt} = \frac{5P}{100}$$

$$\Rightarrow \quad \frac{dP}{dt} = \frac{P}{20}$$

$$\Rightarrow \quad \frac{1}{P} \, dP = \frac{1}{20} \, dt$$

Integrating both sides, we get

$$\int \frac{1}{P} \, dP = \int \frac{1}{20} \, dt$$

$$\Rightarrow \quad \log P = \frac{1}{20} t + \log C$$

$$\Rightarrow \quad \log \frac{P}{C} = \frac{1}{20} t$$

$$\Rightarrow \quad P = C e^{t/20} \qquad \text{...(i)}$$

When $t = 0$, we have $P = 1000$

Substituting these values in (i), we get

$$1000 = C$$

Substituting $C = 1000$ in (i), we get

$$P = 1000\ e^{t/20} \quad \text{...(ii)}$$

Let t_1 years be the time required to double the principal i.e. at $t = t_1$, $P = 2000$.

Substituting these values in (ii), we get

$$2000 = 1000\, e^{t_1/20}$$

$$\Rightarrow \quad e^{t_1/20} = 2 \Rightarrow \frac{t_1}{20} = \log_e 2 \Rightarrow t_1 = 20 \log_e 2$$

Hence, the principal doubles in $20 \log_e 2$ years.

EXAMPLE 23 *Find the equation of the curve passing through the point $(0, -2)$ given that at any point (x, y) on the curve the product of the slope of its tangent and y coordinate of the point is equal to the x-coordinate of the point.* [NCERT]

SOLUTION We know that the slope of the tangent at any point (x, y) on the curve is given by $\frac{dy}{dx}$. According to the given problem, we have

$$y\ \frac{dy}{dx} = x \quad \text{...(i)}$$

$$\Rightarrow \quad y\ dy = x\ dx$$

On integrating both sides, we get

$$\int y\ dy = \int x\ dx$$

$$\Rightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + C \quad \text{...(ii)}$$

This is the equation of the family of solution curves of differential equation (i). We have to find a particular member of the family which passes through $(0, -2)$.

Substituting $x = 0$ and $y = -2$ in (ii), we get

$$\frac{4}{2} = 0 + C \Rightarrow C = 2$$

Putting $C = 2$ in (ii), we get

$$\frac{y^2}{2} = \frac{x^2}{2} + 2 \Rightarrow y^2 = x^2 + 4$$

This is the equation of the required curve.

EXAMPLE 24 *At any point (x, y) of a curve the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4, -3)$. Find the equation of the curve given that it passes through $(-2, 1)$.* [NCERT]

SOLUTION The slope of the tangent at any point $P(x, y)$ is given by $\frac{dy}{dx}$. The slope of the line segment joining $P(x, y)$ and $A(-4, -3)$ is $\frac{y+3}{x+4}$

According to the given problem, we have

$$\frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right) \quad \text{...(i)}$$

$$\frac{1}{y+3}\, dy = \frac{2}{x+4}\ dx$$

On integrating both sides, we get

$$\int \frac{1}{y+3}\, dy = 2 \int \frac{1}{x+4}\, dx$$

$\Rightarrow \quad \log\,(y+3) = 2\,\log\,(x+4) + \log C$

$\Rightarrow \quad (y+3) = C\,(x+4)^2$ ...(ii)

This represents the family of solutions of differential equation (i). We have to find a particular member of this family which passes through $(-2, 1)$.

Substituting $x = -2, y = 1$ in (ii), we get

$$4 = C\,(-2+4)^2 \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we get $y + 3 = (x+4)^2$

This is the required equation of the curve.

EXERCISE 22.7

Solve the following differential equations :

1. $(x-1)\dfrac{dy}{dx} = 2\,xy$
2. $(1+x^2)\,dy = xy\,dx$
3. $\dfrac{dy}{dx} = (e^x + 1)\,y$
4. $(x-1)\dfrac{dy}{dx} = 2x^3 y$
5. $xy\,(y+1)\,dy = (x^2+1)\,dx$
6. $5\dfrac{dy}{dx} = e^x\,y^4$
7. $x \cos y\, dy = (x\,e^x \log x + e^x)\,dx$ [CBSE 2007]
8. $\dfrac{dy}{dx} = e^{x+y} + x^2\,e^y$
9. $x\dfrac{dy}{dx} + y = y^2$
10. $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$
11. $x \cos^2 y\, dx = y \cos^2 x\, dy$
12. $xy\, dy = (y-1)\,(x+1)\,dx$
13. $x\dfrac{dy}{dx} + \cot y = 0$
14. $\dfrac{dy}{dx} = \dfrac{x\,e^x \log x + e^x}{x \cos y}$
15. $\dfrac{dy}{dx} = e^{x+y} + e^y\,x^3$
16. $y\sqrt{1+x^2} + x\sqrt{1+y^2}\,\dfrac{dy}{dx} = 0$
17. $\sqrt{1+x^2}\,dy + \sqrt{1+y^2}\,dx = 0$
18. $\sqrt{1+x^2+y^2+x^2y^2} + xy\dfrac{dy}{dx} = 0$ [CBSE 2010]
19. $\dfrac{dy}{dx} = \dfrac{e^x\,(\sin^2 x + \sin 2x)}{y\,(2\log y + 1)}$
20. $\dfrac{dy}{dx} = \dfrac{x\,(2\log x + 1)}{\sin y + y\cos y}$
21. $(1-x^2)\,dy + xy\,dx = xy^2\,dx$
22. $\tan y\, dx + \sec^2 y \tan x\, dy = 0$
23. $(1+x)\,(1+y^2)\,dx + (1+y)\,(1+x^2)\,dy = 0$
24. $\tan y\dfrac{dy}{dx} = \sin(x+y) + \sin(x-y)$
25. $\cos x \cos y\dfrac{dy}{dx} = -\sin x \sin y$
26. $\dfrac{dy}{dx} + \dfrac{\cos x \sin y}{\cos y} = 0$
27. $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$
28. $y\,(1+e^x)\,dy = (y+1)\,e^x\,dx$
29. $(y+xy)dx + (x-xy^2)dy = 0$ [CBSE 2002]

30. $\frac{dy}{dx} = 1 - x + y - xy$ [CBSE 2002C]

31. $(y^2+1)\,dx - (x^2+1)\,dy = 0$

32. $dy + (x+1)(y+1)\,dx = 0$

33. $\frac{dy}{dx} = (1+x^2)(1+y^2)$

34. $(x-1)\frac{dy}{dx} = 2x^3 y$

35. $\frac{dy}{dx} = e^{x+y} + e^{-x+y}$

36. $\frac{dy}{dx} = (\cos^2 x - \sin^2 x)\cos^2 y$

37. $(xy^2 + 2x)\,dx + (x^2 y + 2y)\,dy = 0$

38. (i) $xy\frac{dy}{dx} = 1 + x + y + xy$ (ii) $y(1-x^2)\frac{dy}{dx} = x(1+y^2)$ [CBSE 200

Solve the following initial value problems: (39-45)

39. $\frac{dy}{dx} = y\tan 2x,\ y(0) = 2$

40. $2x\frac{dy}{dx} = 3y,\ y(1) = 2$

41. $xy\frac{dy}{dx} = y + 2,\ y(2) = 0$

42. $\frac{dy}{dx} = 2e^x y^3,\ y(0) = \frac{1}{2}$

43. $\frac{dr}{dt} = -rt,\ r(0) = r_0$

44. $\frac{dy}{dx} = y\sin 2x,\ y(0) = 1$

45. (i) $\frac{dy}{dx} = y\tan x, y(0) = 1$ [CBSE 2010] (ii) $2x\frac{dy}{dx} = 5y, y(1) = 1$

(iii) $\frac{dy}{dx} = 2e^{2x}y^2, y(0) = -1$ (iv) $\cos y\frac{dy}{dx} = e^x, y(0) = \frac{\pi}{2}$

(v) $\frac{dy}{dx} = 2xy, y(0) = 1$

46. Solve the differential equation $x\frac{dy}{dx} + \cot y = 0$, given that $y = \frac{\pi}{4}$ when $x = \sqrt{2}$.

47. Solve the differential equation $(1+x^2)\frac{dy}{dx} + (1+y^2) = 0$, given that $y = 1$ when $x = 0$

48. Solve the differential equation $\frac{dy}{dx} = \frac{2x(\log x + 1)}{\sin y + y\cos y}$, given that $y = 0$, when $x = 1$

49. Find the particular solution of $e^{dy/dx} = x + 1$, given that $y = 3$ when $x = 0$.

50. Find the solution of the differential equation $\cos y\,dy + \cos x\sin y\,dx = 0$ given that $y = \frac{\pi}{2}$ when $x = \frac{\pi}{2}$.

51. Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given th $y = 1$ when $x = 0$.

52. Find the equation of a curve passing through the point (0, 0) and whose differenti equation is $\frac{dy}{dx} = e^x\sin x$. [NCER

53. For the differential equation $xy\dfrac{dy}{dx} = (x+2)(y+2)$. Find the solution curve passing through the point $(1, -1)$. [NCERT]

54. The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t seconds. [NCERT]

55. In a bank principal increases at the rate of $r\%$ per year. Find the value of r if Rs 100 double itself in 10 years ($\log_e 2 = 0.6931$). [NCERT]

56. In a bank principal increases at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$). [NCERT]

57. In a culture the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present. [NCERT]

ANSWERS

1. $2x + 2\log|x-1| = \log y + C$
2. $y = c\sqrt{1+x^2}$
3. $\log|y| = e^x + x + C$
4. $\log|y| = \frac{2}{3}x^3 + x^2 + 2x + 2\log|x-1| + C$
5. $\frac{y^3}{3} + \frac{y^2}{2} = \frac{x^2}{2} + \log|x| + C$
6. $\frac{-5}{3y^3} = e^x + C$
7. $\sin y = e^x \log x + C$
8. $-e^{-y} = e^x + \frac{x^3}{3} + C$
9. $y - 1 = Cxy$
10. $\sin x \cdot (e^y + 1) = C$
11. $x\tan x - y\tan y = \log|\sec x| - \log|\sec y| + C$
12. $y - x = \log|x| - \log|y-1| + C$
13. $x = C\cos y$
14. $\sin y = e^x \log x + C$
15. $e^x + e^{-y} + \frac{x^4}{4} = C$
16. $\sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + \frac{1}{2}\log\left|\frac{\sqrt{1+y^2}-1}{\sqrt{1+y^2}+1}\right| = C$
17. $(y + \sqrt{1+y^2})(x + \sqrt{1+x^2}) = C$
18. $\sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| = C$
19. $y^2 \log y = e^x \sin 2x + C$
20. $y\sin y = x^2 \log x + C$
21. $\log|y-1| - \log|y| = -\frac{1}{2}\log|1-x^2| + C$

22. $\sin x \tan y = C$

23. $\tan^{-1} x + \tan^{-1} y + \frac{1}{2}\log\{(1+x^2)(1+y^2$

24. $2\cos x + \sec y = C$

25. $\sin y = C\cos x.$

26. $\log|\sin y| = -\sin x + C$

27. $\sqrt{1-x^2} + \sqrt{1-y^2} = C$

28. $y - \log|y+1| = \log|1+e^x| + C$

29. $\log x + x + \log y - \frac{1}{2}y^2 = C$

30. $\log(1+y) = x - \frac{x^2}{2} + C$

31. $\tan^{-1} x - \tan^{-1} y = C$

32. $\log|y+1| + \frac{x^2}{2} + x = C$

33. $\tan^{-1} y = x + \frac{x^3}{3} + C$

34. $y = C|x-1|^2 e^{(2/3)x^3 + x^2 + 2x}, x \in R - \{1\}$

35. $e^{-x} - e^{-y} = e^x + C$

36. $\tan y = \frac{1}{2}\sin 2x + C$

37. $y^2 + 2 = \frac{A}{x^2+2}$

38. (i) $y = x + \log x(1+y) + C, x(1+y) \neq 0$ (ii) $(1+y^2)(1-x^2) = C$

39. $y = \frac{2}{\sqrt{\cos 2x}}$

40. $y^2 = 4x^3$

41. $y - 2\log(y+2) = \log\left(\frac{x}{8}\right)$

42. $y^2(8-4e^x) = 1$

43. $r = r_0 e^{-t^2/2}$

44. $y = e^{\sin^2 x}$

45. (i) $y = \sec x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(ii) $y = |x|^{5/2}, x \in R - \{0\}$

(iii) $y = -e^{-2x}$

(iv) $y = \sin^{-1}(e^x), x \leq 0$

(v) $y = e^{x^2}, x \in R.$

46. $x = 2\cos y$

47. $x + y = 1 - xy$

48. $2y\sin y = 2x^2\log x + x^2 - 1$

49. $y = (x+1)\log|x+1| - x + 3$

50. $\log\sin y + \sin x = 1$

51. $y = \frac{1}{2x^2+1}$

52. $2y = e^x(\sin x - \cos x) + 1$

53. $y - x + 2 = \log\{x^2(y+2)^2\}$

54. $r = (63t+27)^{1/3}$

55. 6.931%

56. Rs 1648

57. $\frac{2\log 2}{\log\frac{11}{10}}$

HINTS TO SELECTED PROBLEMS

52. We have,

$$\frac{dy}{dx} = e^x \sin x$$

On integrating both sides, we get

$$y = \frac{e^x}{2}(\sin x - \cos x) + C \quad \text{...(i)}$$

It passes through (0, 0)

$$\therefore \quad 0 = -\frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

Putting $C = \frac{1}{2}$ in (i), we get

$y = \frac{1}{2}\left\{e^x(\sin x - \cos x) + \frac{1}{2}\right\}$ as the equation of the curve.

53. We have,

$$xy\frac{dy}{dx} = (x+2)(y+2)$$

$$\Rightarrow \quad \frac{y}{y+2}dy = \frac{x+2}{x}dx$$

$$\Rightarrow \quad \left(1 - \frac{2}{y+2}\right)dy = \left(1 + \frac{2}{x}\right)dx$$

$$\Rightarrow \quad \int\left(1 - \frac{2}{y+2}\right)dy = \int\left(1 + \frac{2}{x}\right)dx$$

$$\Rightarrow \quad y - 2\log(y+2) = x + 2\log x + C \quad \text{...(i)}$$

It passes through (1, − 1). Putting $x = 1$ and $y = -1$ in (i), we get

$$-1 - 2\log 1 = 1 + 2\log 1 + C \Rightarrow C = -2$$

Putting $C = -2$ in (i), we get

$$y - x + 2 = 2\log\{x(y+2)\}$$

54. Let r be the radius and V be the volume of the balloon. Then, $V = \frac{4}{3}\pi r^3$

It is given that

$$\frac{dV}{dt} = -\lambda, \text{ where } \lambda > 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -\lambda$$

$$\Rightarrow \quad 4\pi r^2\frac{dr}{dt} = -\lambda$$

$$\Rightarrow \quad 4\pi r^2\,dr = -\lambda\,dt$$

On integrating, we get

$$\frac{4}{3}\pi r^3 = -\lambda t + C$$

At $t = 0$, $r = 3$

$\therefore \quad C = 36\pi$

Putting $C = 36\pi$ in (i), we get

$$\frac{4}{3}\pi r^3 = -\lambda t + 36\pi$$

At $t = 3, r = 6$ (given)

$$\therefore \quad 288\pi = -3\lambda + 36\pi \Rightarrow \lambda = -84\pi$$

Putting $\lambda = -84\pi$ in (ii), we get

$$\frac{4}{3}\pi r^3 = 84\pi t + 36\pi$$

$$\Rightarrow \quad r^3 = 63t + 27 \Rightarrow r = (63t+27)^{1/3}$$

55. Let P be the principal. It is given that

$$\frac{dP}{dt} = \frac{r}{100}P \Rightarrow \frac{dP}{P} = \frac{r}{100}dt \Rightarrow \log P = \frac{rt}{100} + C$$

Initially i.e. at $t = 0$, let $P = P_0$. Then,

$$\log P_0 = C$$

$$\therefore \quad \log P = \frac{rt}{100} + \log P_0 \Rightarrow \log\frac{P}{P_0} = \frac{rt}{100}$$

Substituting $P_0 = 100$, $P = 2P_0 = 200$ and $t = 10$, we get

$$\log 2 = \frac{r}{10} \Rightarrow r = 10\log 2 = 10 \times 0.6931 = 6.931$$

56. We have,

$$\frac{dP}{dt} = \frac{5P}{100} \Rightarrow \frac{dP}{P} = \frac{1}{20}dt \Rightarrow \log P = \frac{t}{20} + \log C$$

Initially i.e. at $t = 0$, $P = 1000$

$$\therefore \quad \log 1000 = \log C$$

Substituting the value of log C in (i), we get

$$\log P = \frac{t}{20} + \log 1000$$

Putting $t = 10$, we get

$$\log\frac{P}{1000} = 0.5 \Rightarrow \frac{P}{1000} = e^{0.5} \Rightarrow P = 1000 \times 1.648 = 1648$$

57. Let at any time the bacteria count be N. Then,

$$\frac{dN}{dt} \propto N$$

$\Rightarrow \qquad \frac{dN}{dt} = \lambda N \Rightarrow \frac{1}{N} dN = \lambda\, dt \Rightarrow \log N = \lambda\, t + \log C$

At $t = 0, N = 100000$

$\therefore \qquad \log C = \log 100000$

So, $\log N = \lambda t + \log 100000$

At $t = 2, N = 110000$

$\therefore \qquad \log 110000 = 2\lambda + \log 100000 \Rightarrow \frac{1}{2} \log \frac{11}{10} = \lambda$

$\therefore \qquad \log N = \frac{1}{2} \log \left(\frac{11}{10}\right) t + \log 100000$

When $N = 200000$, let $t = T$. Then,

$$\log 200000 = \frac{T}{2} \log \left(\frac{11}{10}\right) + \log 100000$$

$$\Rightarrow \qquad \log 2 = \frac{T}{2} \log \frac{11}{10} \Rightarrow T = 2 \frac{\log 2}{\log \frac{11}{10}}$$

22.8.4 EQUATIONS REDUCIBLE TO VARIABLE SEPARABLE FORM

Differential equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by the substitution $ax + by + c = v$ as discussed in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve :*

(i) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$ (ii) $\frac{dy}{dx} = \cos(x + y)$

(iii) $\frac{dy}{dx} = (4x + y + 1)^2$

SOLUTION (i) We are given that

$$\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$$

$$\Rightarrow \qquad \frac{dy}{dx} = \sin(x + y)$$

Let $\quad x + y = v$. Then,

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x + y = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in the given differential equation, we get

$$\therefore \qquad \frac{dv}{dx} - 1 = \sin v$$

$$\Rightarrow \qquad \frac{dv}{dx} = 1 + \sin v$$

$$\Rightarrow \qquad \frac{dv}{1 + \sin v} = dx$$

$\Rightarrow \quad \int \frac{1}{1+\sin v}\, dv = \int dx$ [Integrating both sides]

$\Rightarrow \quad \int dx = \int \frac{1-\sin v}{1-\sin^2 v}\, dv$

$\Rightarrow \quad \int dx = \int \frac{1-\sin v}{\cos^2 v}\, dv$

$\Rightarrow \quad \int dx = \int (\sec^2 v - \tan v \sec v)\, dv$

$\Rightarrow \quad x = \tan v - \sec v + C$

$\Rightarrow \quad x = \tan(x+y) - \sec(x+y) + C$, which is the required solution.

(ii) We are given that

$$\frac{dy}{dx} = \cos(x+y)$$

Let $x + y = v$. Then,

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x + y = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in the given differential equation, we get

$$\frac{dv}{dx} - 1 = \cos v$$

$\Rightarrow \quad \frac{dv}{dx} = 1 + \cos v$

$\Rightarrow \quad \frac{1}{1+\cos v}\, dv = dx$

$\Rightarrow \quad \frac{1}{2}\sec^2 \frac{v}{2}\, dv = dx.$

$\Rightarrow \quad \int \frac{1}{2}\sec^2 \frac{v}{2}\, dv = \int 1 \cdot dx$ [Integrating both sides]

$\Rightarrow \quad \tan \frac{v}{2} = x + C$

$\Rightarrow \quad \tan\left(\frac{x+y}{2}\right) = x + C$, which is the required solution.

(iii) We are given that

$$\frac{dy}{dx} = (4x + y + 1)^2$$

Let $4x + y + 1 = v$, Then,

$$4 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4$$

Putting $4x + y + 1 = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 4$ in the given differential equation, we get

$$\frac{dv}{dx} - 4 = v^2$$

$\Rightarrow \quad \frac{dv}{dx} = v^2 + 4$

$\Rightarrow \quad dv = (v^2 + 4)\, dx$

$\Rightarrow \quad \dfrac{dv}{v^2+4} = dx$

$\Rightarrow \quad \displaystyle\int \frac{1}{v^2+4}\, dv = \int 1 \cdot dx$ [Integrating both sides]

$\Rightarrow \quad \dfrac{1}{2}\tan^{-1}\left(\dfrac{v}{2}\right) = x + C$

$\Rightarrow \quad \dfrac{1}{2}\tan^{-1}\left(\dfrac{4x+y+1}{2}\right) = x + C$, which is the required solution.

EXAMPLE 2 *Solve* : $(x+y)^2 \dfrac{dy}{dx} = a^2$.

SOLUTION Let $x + y = v$. Then,

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x + y = v$ and $\dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$ the given differential equation, we get

$v^2\left(\dfrac{dv}{dx} - 1\right) = a^2$

$\Rightarrow \quad v^2 \dfrac{dv}{dx} = a^2 + v^2$

$\Rightarrow \quad v^2\, dv = (a^2 + v^2)\, dx$

$\Rightarrow \quad \dfrac{v^2}{v^2+a^2}\, dv = dx$ [By separating the variables]

$\Rightarrow \quad \left(1 - \dfrac{a^2}{v^2+a^2}\right) dv = dx$

$\Rightarrow \quad \displaystyle\int 1 \cdot dv - a^2 \int \frac{1}{v^2+a^2}\, dv = \int dx + C$ [On integration]

$\Rightarrow \quad v - a\tan^{-1}\left(\dfrac{v}{a}\right) = x + C$

$\Rightarrow \quad (x+y) - a\tan^{-1}\left(\dfrac{x+y}{a}\right) = x + C$

EXAMPLE 3 *Solve the initial value problem :* $\cos(x+y)\, dy = dx$, $y(0) = 0$.

SOLUTION The given differential equation can be written as

$$\frac{dy}{dx} = \frac{1}{\cos(x+y)}$$

Let $x + y = v$. Then,

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x + y = v$ and $\dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$ the given differential equaтion, we get

$\Rightarrow \quad \dfrac{dv}{dx} - 1 = \dfrac{1}{\cos v}$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1+\cos v}{\cos v}$$

$$\Rightarrow \quad \frac{\cos v}{1+\cos v}\, dv = dx$$

$$\Rightarrow \quad \frac{\cos v\,(1-\cos v)}{1-\cos^2 v}\, dv = dx$$

$$\Rightarrow \quad (\cot v \operatorname{cosec} v - \cot^2 v)\, dv = dx$$

$$\Rightarrow \quad (\cot v \operatorname{cosec} v - \operatorname{cosec}^2 v + 1)\, dv = dx$$

$$\Rightarrow \quad -\operatorname{cosec} v + \cot v + v = x + C \qquad \text{[On integration]}$$

$$\Rightarrow \quad -\operatorname{cosec}(x+y) + \cot(x+y) + x + y = x + C$$

$$\Rightarrow \quad -\operatorname{cosec}(x+y) + \cot(x+y) + y = C$$

$$\Rightarrow \quad -\frac{1-\cos(x+y)}{\sin(x+y)} + y = C$$

$$\Rightarrow \quad -\tan\left(\frac{x+y}{2}\right) + y = C \qquad \text{...(i)}$$

We have,

$$y(0) = 0 \text{ i.e. } y = 0 \text{ when } x = 0$$

Putting $x = 0$ and $y = 0$ in (i), we get $C = 0$.

Putting $C = 0$ in (i), we get

$$-\tan\left(\frac{x+y}{2}\right) + y = 0 \Rightarrow y = \tan\left(\frac{x+y}{2}\right),$$ which is the required solution.

EXAMPLE 4 *Solve :* $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$.

SOLUTION Let $x + y = v$. Then,

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x + y = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in the given differential equation, we get

$$\frac{dv}{dx} - 1 = \cos v + \sin v$$

$$\Rightarrow \quad \frac{dv}{dx} = 1 + \cos v + \sin v$$

$$\Rightarrow \quad \frac{1}{1+\cos v+\sin v}\, dv = dx \qquad \text{[By separating the variables]}$$

$$\Rightarrow \quad \int \frac{1}{1+\cos v+\sin v}\, dv = \int 1 \cdot dx + C \qquad \text{[On integration]}$$

$$\Rightarrow \quad \int \frac{1}{1+\frac{1-\tan^2(v/2)}{1+\tan^2(v/2)}+\frac{2\tan(v/2)}{1+\tan^2(v/2)}}\, dv = x + C$$

$$\Rightarrow \quad \int \frac{\sec^2(v/2)}{2\{(1+\tan(v/2))\}}\,dv = x + C$$

$$\Rightarrow \quad \log|1+\tan(v/2)| = x + C$$

$$\Rightarrow \quad \log\left|1+\tan\left(\frac{x+y}{2}\right)\right| = x + C$$

EXAMPLE 5 *Solve the following initial value problems:*

(i) $(x+y+1)^2\,dy = dx$, $y(-1) = 0$

(ii) $(x-y)(dx+dy) = dx - dy$, $y(0) = -1$ **[NCERT]**

SOLUTION We have,

(i) $(x+y+1)^2\,dy = dx$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{(x+y+1)^2} \qquad ...(i)$$

Let $x+y+1 = v$. Then,

$$1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x+y+1 = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in (i), we get

$$\therefore \quad \frac{dv}{dx} - 1 = \frac{1}{v^2}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1+v^2}{v^2}$$

$$\Rightarrow \quad \frac{v^2}{v^2+1}\,dv = dx$$

$$\Rightarrow \quad \int \frac{v^2}{v^2+1}\,dv = \int dx$$

$$\Rightarrow \quad \int \frac{v^2+1-1}{v^2+1}\,dv = \int dx$$

$$\Rightarrow \quad \int \left(1 - \frac{1}{v^2+1}\right) dv = \int dx$$

$$\Rightarrow \quad v - \tan^{-1} v = x + C$$

$$\Rightarrow \quad (x+y+1) - \tan^{-1}(x+y+1) = x + C$$

$$\Rightarrow \quad y + 1 - \tan^{-1}(x+y+1) = C \qquad ...(ii)$$

It is given that $y(-1) = 0$ i.e. $y = 0$ when $x = -1$

Putting $x = 1$ and $y = 0$ in (ii), we get

$$1 - \tan^{-1} 0 = C \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we get

$y = \tan^{-1}(x+y+1) \Rightarrow x+y+1 = \tan y$ as the required solution.

(ii) $(x-y)(dx+dy) = dx - dy$

$\Rightarrow \quad (x-y-1)\,dx = -(x-y+1)\,dy$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x-y-1}{x-y+1} \quad \text{...(i)}$$

Let $x - y = v$. Then,

$$1 - \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$$

Putting $x - y = v$ and $\frac{dy}{dx} = 1 - \frac{dv}{dx}$ in (i), we get

$$1 - \frac{dv}{dx} = -\frac{v-1}{v+1}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{v-1}{v+1} + 1$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{2v}{v+1}$$

$$\Rightarrow \quad \frac{v+1}{v}\,dv = 2\,dx$$

$$\Rightarrow \quad \left(1+\frac{1}{v}\right)dv = 2\,dx$$

$$\Rightarrow \quad \int\left(1+\frac{1}{v}\right)dv = 2\int dx$$

$$\Rightarrow \quad v + \log|v| = 2x + C$$

$$\Rightarrow \quad x - y + \log|x-y| = 2x + C$$

$$\Rightarrow \quad \log|x-y| = x + y + C \quad \text{...(ii)}$$

It is given that $y(0) = -1$ i.e. when $x = 0$, $y = -1$.

Putting $x = 0$ and $y = -1$ in (ii), we get $\log 1 = -1 + C \Rightarrow C = 1$

Putting $C = 1$ in (ii), we get

$$\log|x-y| = x + y + 1$$

$$\Rightarrow \quad |x-y| = e^{x+y+1}$$

$$\Rightarrow \quad x - y = \pm e^{x+y+1}$$

But, $y(0) = -1$ does not satisfy $x - y = -e^{x+y+1}$

Hence, $x - y = e^{x+y+1}$ gives the required solution.

EXERCISE 22.8

Solve the following differential equations:

1. $\frac{dy}{dx} = (x+y+1)^2$
2. $\frac{dy}{dx} \cdot \cos(x-y) = 1$
3. $\frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5}$
4. $\frac{dy}{dx} = (x+y)^2$
5. $(x+y)^2 \frac{dy}{dx} = 1$
6. $\cos^2(x-2y) = 1 - 2\frac{dy}{dx}$
7. $\frac{dy}{dx} = \sec(x+y)$
8. $\frac{dy}{dx} = \tan(x+y)$
9. $(x+y)(dx-dy) = dx+dy$
10. $(x+y+1)\frac{dy}{dx} = 1$

ANSWERS

1. $\tan^{-1}(x+y+1)=x+C$
2. $\cot\left(\dfrac{x-y}{2}\right)=y+C$
3. $2(x-y)+\log(x-y+2)=x+C$
4. $x+y=\tan(x+C)$
5. $y-\tan^{-1}(x+y)=C$
6. $x=\tan(x-2y)+C$
7. $y=\tan\left(\dfrac{x+y}{2}\right)+C$
8. $y-x+\log|\sin(x+y)+\cos(x+y)|=C$
9. $\dfrac{1}{2}(y-x)+\dfrac{1}{2}\log|x+y|=C$
10. $x=ce^y-y-2$

22.8.5 HOMOGENEOUS DIFFERENTIAL EQUATIONS

DEFINITION *A function $f(x, y)$ is called a homogeneous function of degree n, if*

$$f(\lambda x, \lambda y)=\lambda^n f(x, y)$$

For example, $f(x, y)=x^2-y^2+3xy$ is a homogeneous function degree 2. Because,

$$f(\lambda x, \lambda y)=\lambda^2x^2-\lambda^2y^2+3\,\lambda x.\,\lambda y=\lambda^2 f(x, y)$$

Consider the following functions:

$$F(x, y)=2x-3y,\ G(x, y)=\sin\left(\frac{y}{x}\right) \text{ and, } H(x, y)=\sin x+\cos y$$

We have,

$$F(\lambda x, \lambda y)=2\lambda x-3\lambda y=\lambda^1(2x-3y)=\lambda^1 F(x, y)$$

$\therefore$ $F(x, y)$ is a homogeneous function of degree 1.

$$G(\lambda x, \lambda y)=\sin\left(\frac{\lambda y}{\lambda x}\right)=\lambda^0\ \sin\left(\frac{y}{x}\right)=\lambda^0 G(x, y)$$

$\therefore$ $G(x, y)$ is a homogeneous function of zero degree.

$H(\lambda x, \lambda y)=\sin\lambda x+\cos\lambda y\neq\lambda^n H(x, y)$ for any n

$\therefore$ $H(x, y)$ is not a homogeneous function.

Let $F_1(x, y)=x^2+2xy$ be a function of x, y. Then,

$$F_1(\lambda x, \lambda y)=\lambda^2x^2+2\lambda x\,\lambda y=\lambda^2(x^2+2xy)=\lambda^2 F_1(x, y)$$

$\therefore$ $F_1(x, y)$ is a homogeneous function of degree 2.

Also, $F_1(x, y)=x^2+2xy=x^2\left\{1+\left(\dfrac{2y}{x}\right)\right\}=x^2\ \phi_1\left(\dfrac{y}{x}\right)$

Again, $F_1(x, y)=x^2+2xy=y^2\left\{\left(\dfrac{x}{y}\right)^2+2\left(\dfrac{x}{y}\right)\right\}=y^2\Psi_1\left(\dfrac{x}{y}\right)$

Thus a homogeneous function $f(x, y)$ of degree n can always be written as

$$f(x, y)=x^n f\left(\frac{y}{x}\right) \text{or, } f(x, y)=y^n f\left(\frac{x}{y}\right)$$

DEFINITION *If a first-order first degree differential equation is expressible in the form*

$$\frac{dy}{dx}=\frac{f(x, y)}{g(x, y)}$$

where f(x, y) and g(x, y) are homogeneous functions of the same degree, then it is called a homogeneous differential equation.

Such type of equations can be reduced to variable separable form by the substitution $y = vx$ as explained below :

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x^n f(y/x)}{x^n g(y/x)} = \frac{f(y/x)}{g(y/x)} = F(y/x) \text{ (say)}$$

Let $y = vx$. Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Substituting these values in $\frac{dy}{dx} = F(y/x)$, we get

$$v + x\frac{dv}{dx} = F(v)$$

$$\Rightarrow \qquad \frac{dv}{F(v) - v} = \frac{dx}{x}$$

On integration, we get

$$\int \frac{1}{F(v) - v}\, dv = \int \frac{dx}{x} + C$$, where C is an arbitrary constant of integration.

After integration, v will be replaced by y/x to get the complete solution.

Following algorithm may be used to solve a homogeneous differential equation.

ALGORITHM

STEP I *Put the differential equation in the form* $\frac{dy}{dx} = \frac{\phi(x, y)}{\psi(x, y)}$.

STEP II *Put* $y = vx$ *and* $\frac{dy}{dx} = v + x\frac{dv}{dx}$ *in the equation in Step I and cancel out x from the right hand side. The equation reduces to the form* $v + x\frac{dv}{dx} = F(v)$.

STEP III *Shift v on RHS and separate the variables v and x.*

STEP IV *Integrate both sides to obtain the solution in terms of v and x.*

STEP V *Replace v by* $\frac{y}{x}$ *in the solution obtained in Step IV to obtain the solution in terms of x and y.*

Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the differential equation* $x^2\, dy + y(x + y)\, dx = 0$, *given that* $y = 1$ *when* $x = 1$.

SOLUTION The given differential equation is

$$x^2\, dy + y(x + y)\, dx = 0$$

$$\Rightarrow \qquad x^2\, dy = -y(x + y)\, dx$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y(x+y)}{x^2}, \text{ if } x \neq 0, y \neq 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{xy+y^2}{x^2}\right) \qquad \text{...(i)}$$

Since each of the functions $xy + y^2$ and x^2 is a homogeneous function of degree 2. Therefore, equation (i) is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = -\left(\frac{vx^2 + v^2x^2}{x^2}\right)$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = -(v + v^2)$$

$$\Rightarrow \quad x\frac{dv}{dx} = -2v - v^2$$

$$\Rightarrow \quad x\,dv = -(v^2 + 2v)\,dx$$

$$\Rightarrow \quad \frac{dv}{v^2+2v} = -\frac{dx}{x} \qquad \text{[By separating the variables]}$$

$$\Rightarrow \quad \int \frac{1}{v^2+2v}\,dv = -\int \frac{1}{x}\,dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int \frac{1}{v^2+2v+1-1}\,dv = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \int \frac{1}{(v+1)^2-1^2}\,dv = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \frac{1}{2\times 1}\log\left|\frac{v+1-1}{v+1+1}\right| = -\log x + \log C$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{v}{v+2}\right| = -\log x + \log C$$

$$\Rightarrow \quad \log\left|\frac{v}{v+2}\right| + 2\log x = 2\log C$$

$$\Rightarrow \quad \log\left|\frac{v}{v+2}\right| + \log x^2 = \log k, \text{ where } k = C^2$$

$$\Rightarrow \quad \log\left|\frac{vx^2}{v+2}\right| = \log k$$

$$\Rightarrow \quad \left|\frac{vx^2}{v+2}\right| = k$$

$$\Rightarrow \quad \left|\frac{\frac{y}{x}\times x^2}{\frac{y}{x}+2}\right| = k \Rightarrow \left|\frac{x^2y}{y+2x}\right| = k \qquad \left[\because \frac{y}{x} = v\right]$$

It is given that $y = 1$, when $x = 1$. Putting $x = 1, y = 1$ in (ii), we get

$$k = \frac{1}{3}$$

Putting $k = 1/3$ in (ii), we get

$$\left|\frac{x^2 y}{y + 2x}\right| = \frac{1}{3} \Rightarrow 3x^2 y = \pm (y + 2x)$$

But, $3x^2 y = -(y + 2x)$ is not satisfied by $y(1) = 1$.

$\therefore \quad 3x^2 y = y + 2x$

$\Rightarrow \quad y = \dfrac{2x}{3x^2 - 1}$

Now, $\quad y = \dfrac{2x}{3x^2 - 1}$ is defined for

$$3x^2 - 1 \neq 0 \text{ i.e. } x \neq \pm \frac{1}{\sqrt{3}}$$

Hence, $\quad y = \dfrac{2x}{3x^2 - 1}, x \neq \pm \dfrac{1}{\sqrt{3}}$ is the required solution.

EXAMPLE 2 *Solve the differential equation* $(x + y)\, dy + (x - y)\, dx = 0$, *given that* $y = 1$ *when* $x = 1$.

SOLUTION The given differential equation is

$$(x + y)\, dy + (x - y)\, dx = 0$$

$\Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{x - y}{x + y}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{y - x}{x + y} \quad$ ...(i)

Since each of the functions $y - x$ and $x + y$ is a homogeneous function of degree 1. Therefore, equation (i) is a homogeneous differential equation.

Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx - x}{x + vx}$$

$\Rightarrow \quad v + x\dfrac{dv}{dx} = \dfrac{v - 1}{v + 1}$

$\Rightarrow \quad x\dfrac{dv}{dx} = \dfrac{v - 1}{v + 1} - v$

$\Rightarrow \quad x\dfrac{dv}{dx} = \dfrac{v - 1 - v^2 - v}{v + 1}$

$\Rightarrow \quad x\dfrac{dv}{dx} = -\left(\dfrac{v^2 + 1}{v + 1}\right)$

$\Rightarrow \quad \dfrac{v + 1}{v^2 + 1}\, dv = -\dfrac{dx}{x}, x \neq 0 \quad$ [By separating the variables]

$\Rightarrow \quad \displaystyle\int \frac{v + 1}{v^2 + 1}\, dv = -\int \frac{dx}{x} \quad$ [Integrating both sides]

$\Rightarrow \quad \displaystyle\int \frac{v}{v^2 + 1}\, dv + \int \frac{1}{v^2 + 1}\, dv = -\int \frac{dx}{x}$

$$\Rightarrow \quad \frac{1}{2}\int \frac{2v}{v^2+1}\,dv + \int \frac{1}{v^2+1^2}\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log(v^2+1) + \tan^{-1} v = -\log|x| + C$$

$$\Rightarrow \quad \log(v^2+1) + 2\log|x| + 2\tan^{-1} v = 2C$$

$$\Rightarrow \quad \log(v^2+1) + \log x^2 + 2\tan^{-1} v = k, \text{ where } k = 2C$$

$$\Rightarrow \quad \log\{(v^2+1)\,x^2\} + 2\tan^{-1} v = k$$

$$\Rightarrow \quad \log\{((y^2/x^2)+1)\,x^2\} + 2\tan^{-1}(y/x) = k \qquad [\because v = y/x]$$

$$\Rightarrow \quad \log(x^2+y^2) + 2\tan^{-1}(y/x) = k \qquad \text{...(ii)}$$

It is given that $y = 1$, when $x = 1$. Putting $x = 1, y = 1$ in (ii), we get

$$\log 2 + 2\tan^{-1}(1) = k \Rightarrow k = \log 2 + 2(\pi/4) = (\pi/2) + \log 2$$

Substituting the value of k in (ii), we get

$$\log(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \log 2$$

Hence, $\log(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \log 2, x \neq 0$

is the required solution of the given differential equation.

EXAMPLE 3 *Solve the differential equation* $(x^2 - y^2)\,dx + 2xy\,dy = 0$; *given that* $y = 1$ *when* $x = 1$. **[NCERT, CBSE 2008]**

SOLUTION We are given that

$$(x^2 - y^2)\,dx + 2xy\,dy = 0$$

$$\Rightarrow \quad (x^2 - y^2)\,dx = -2xy\,dy$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \qquad \text{...(i)}$$

Since each of the functions $y^2 - x^2$ and $2xy$ is a homogeneous function of degree 2, the given differential equation is therefore homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x \cdot vx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left(\frac{v^2+1}{2v}\right)$$

$$\Rightarrow \quad \frac{2v}{v^2+1}\,dv = -\frac{dx}{x}, x \neq 0 \qquad \text{[By separating the variables]}$$

$$\Rightarrow \quad \int \frac{2v}{v^2+1}\,dv = -\int \frac{dx}{x} \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log(v^2+1) = -\log|x| + C$$

$$\Rightarrow \quad \log(v^2+1) + \log|x| = \log C$$

$$\Rightarrow \quad (v^2+1)\,|x| = C$$

$$\Rightarrow \quad [(y^2/x^2)+1]\,|x| = C \qquad [\because v = y/x]$$

$$\Rightarrow \quad (x^2+y^2) = C\,|x| \qquad \text{...(ii)}$$

It is given that $y = 1$ when $x = 1$.

So, putting $x = 1, y = 1$ in (ii), we get $C = 2$.

Substituting $C = 2$ in (ii), we obtain

$$x^2+y^2 = 2\,|x|$$

$$\Rightarrow \quad x^2+y^2 = \pm 2x$$

But, $x = 1\,, y = 1$ do not satisfy $x^2+y^2 = -2x$

Hence, $x^2+y^2 = 2x$ is the required solution.

EXAMPLE 4 *Solve :* $x^2 y\,dx - (x^3+y^3)\,dy = 0$ **[CBSE 2002]**

SOLUTION The given differential equation is

$$x^2 y\,dx - (x^3+y^3)\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 y}{x^3+y^3} \qquad \text{...(i)}$$

Since each of the functions $x^2 y$ and x^3+y^3 is a homogeneous function of degree 3, so the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx^3}{x^3+v^3x^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v-v-v^4}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{v^4}{1+v^3}$$

$$\Rightarrow \quad x(1+v^3)\,dv = -v^4\,dx$$

$$\Rightarrow \quad \frac{1+v^3}{v^4}\,dv = -\frac{dx}{x} \qquad \text{[By separating the variables]}$$

$$\Rightarrow \quad \left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\frac{dx}{x}, \text{ provide that } x \neq 0\,, v \neq 0$$

$$\Rightarrow \quad \frac{v^{-3}}{-3} + \log|v| = -\log|x| + C \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad -\frac{1}{3v^3} + \log|v| + \log|x| = C$$

$$\Rightarrow \quad -\frac{1}{3}\frac{x^3}{y^3} + \log\left|\frac{y}{x}\cdot x\right| = C \qquad [\because v = y/x]$$

$$\Rightarrow \quad -\frac{x^3}{3y^3} + \log|y| = C, \text{ which is the required solution.}$$

EXAMPLE 5 *Solve :* $(x^2 + xy)\,dy = (x^2 + y^2)\,dx.$ **[NCERT, CBSE 2005]**

SOLUTION The given differential equation is

$$(x^2 + xy)\,dy = (x^2 + y^2)\,dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} \qquad \text{...(i)}$$

Since each of the functions $x^2 + y^2$ and $x^2 + xy$ is a homogeneous function of degree 2, so the given differential equation is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{x^2 + vx^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1 + v^2}{1 + v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 + v^2 - v - v^2}{1 + v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - v}{1 + v}$$

$$\Rightarrow \quad x(1 + v)\,dv = (1 - v)\,dx$$

$$\Rightarrow \quad \frac{1 + v}{1 - v}\,dv = \frac{dx}{x}, x \neq 0, v \neq 1 \qquad \text{[By separating variables]}$$

$$\Rightarrow \quad \frac{2 - (1 - v)}{1 - v}\,dv = \frac{dx}{x}$$

$$\Rightarrow \quad \left(\frac{2}{1 - v} - 1\right)dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(\frac{2}{1 - v} - 1\right)dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{2}{1 - v}\,dv - \int 1.dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad -2\log|1 - v| - v = \log|x| + \log C$$

$$\Rightarrow \quad \log|x| + \log C + 2\log|1 - v| = -v$$

$\Rightarrow \quad \log\{C\,|x|\,(1-v)^2\} = -v$

$\Rightarrow \quad C\,|x|\,(1-v)^2 = e^{-v}$

$\Rightarrow \quad C\,|x|\,(1-y/x)^2 = e^{-y/x}$

$\Rightarrow \quad C\,(x-y)^2 = |x|\,e^{-y/x}$

Hence, $C\,(x-y)^2 = |x|\,e^{-y/x}$, $x \neq 0$ gives the required solution.

EXAMPLE 6 *Solve :* $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0.$ **[CBSE 2008]**

SOLUTION We are given that

$$(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{3xy + y^2}{x^2 + xy}\right) \qquad \text{...(i)}$$

Since each of the functions $(3xy + y^2)$ and $(x^2 + xy)$ is a homogeneous function of degree 2, the given equation is, therefore, a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = -\left\{\frac{3v\,x^2 + x^2\,v^2}{x^2 + vx^2}\right\}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = -\left\{\frac{3v + v^2}{1 + v}\right\}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left\{\frac{3v + v^2}{1 + v} + v\right\}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left\{\frac{2v^2 + 4v}{v + 1}\right\}$$

$$\Rightarrow \quad (v + 1)\,x\,dv = -(2v^2 + 4v)\,dx$$

$$\Rightarrow \quad \frac{v + 1}{2v^2 + 4v}\,dv = -\frac{dx}{x},\ x \neq 0,\ v \neq 0, -2 \qquad \text{[By separating variables]}$$

$$\Rightarrow \quad \frac{(2v + 2)\,dv}{v^2 + 2v} = -4\frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{2v + 2}{v^2 + 2v}\,dv = -4\int \frac{dx}{x} \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log|v^2 + 2v| = -4\log|x| + \log C$$

$$\Rightarrow \quad \log|v^2 + 2v| = \log\left(\frac{C}{x^4}\right)$$

$$\Rightarrow \quad |v^2 + 2v| = \frac{C}{x^4}$$

$$\Rightarrow \quad |y^2/x^2 + 2\,y/x| = C/x^4 \qquad [\because v = y/x]$$

$$\Rightarrow \quad |y^2 + 2xy| = \frac{C}{x^2}$$

Hence, $|y^2 + 2xy| = \frac{C}{x^2}$, $x \neq 0$, gives the required solution.

EXAMPLE 7 *Solve :* $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2 y)\,dy$.

SOLUTION We are given that

$$(x^3 - 3xy^2)\,dx = (y^3 - 3x^2 y)\,dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2 y}$$

Clearly, the given equation is a homogeneous equation. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in it, we get

$$v + x\frac{dv}{dx} = \frac{x^3 - 3v^2 x^3}{v^3 x^3 - 3vx^3}$$

$$\Rightarrow \quad v + x\frac{dy}{dx} = \frac{1 - 3v^2}{v^3 - 3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v = \frac{1 - v^4}{v^3 - 3v}$$

$$\Rightarrow \quad x(v^3 - 3v)\,dv = (1 - v^4)\,dx$$

$$\Rightarrow \quad \frac{v^3 - 3v}{1 - v^4}\,dv = \frac{dx}{x}, x \neq 0, v \neq \pm 1$$

$$\Rightarrow \quad \int \frac{v^3 - 3v}{1 - v^4}\,dv = \int \frac{dx}{x} \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int \frac{v^3}{1 - v^4}\,dv - 3\int \frac{v}{1 - v^4}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{4}\int \frac{-4v^3}{1 - v^4}\,dv - \frac{3}{2}\int \frac{2v}{1 - (v^2)^2}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{4}\int \frac{-4v^3}{1 - v^4}\,dv - \frac{3}{2}\int \frac{dt}{1 - t^2} = \int \frac{dx}{x}, \text{ where } t = v^2.$$

$$\Rightarrow \quad -\frac{1}{4}\log|1 - v^4| - \frac{3}{2} \times \frac{1}{2 \times 1}\log\left|\frac{1+t}{1-t}\right| = \log|x| + \log C$$

$$\Rightarrow \quad -\frac{1}{4}\log|1 - v^4| - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log|Cx|$$

$$\Rightarrow \quad -\log|(1 - v^4)| - 3\log\left|\frac{1+v^2}{1-v^2}\right| = 4\log|Cx|$$

$$\Rightarrow \quad \log\left|(1 - v^4)^{-1}\left(\frac{1+v^2}{1-v^2}\right)^{-3}\right| = \log|(Cx)^4|$$

$$\Rightarrow \quad \frac{1}{1 - v^4} \times \left(\frac{1-v^2}{1+v^2}\right)^3 = (Cx)^4$$

$$\Rightarrow \quad (1 - v^2)^2 = (1 + v^2)^4 (Cx)^4$$

$$\Rightarrow \quad 1 - v^2 = (1 + v^2)^2 (Cx)^2$$

$\Rightarrow \quad 1 - y^2/x^2 = (1 + y^2/x^2)^2 C^2 x^2$ $\quad [\because v = y/x]$

$\Rightarrow \quad x^2 - y^2 = (x^2 + y^2)^2 C^2$, which is the required solution

EXAMPLE 8 *Solve :* $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$ **[NCERT, CBSE 2005]**

SOLUTION The given differential equation can be written as

$$\frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} + y}{x}, x \neq 0$$

Clearly, it is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in it, we get

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2 + v^2x^2} + vx}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \sqrt{1 + v^2} + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x} \quad \text{[By separating the variables]}$$

Integrating both sides, we get

$$\int \frac{1}{\sqrt{1 + v^2}}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \log\left| v + \sqrt{1 + v^2} \right| = \log|x| + \log C$$

$$\Rightarrow \quad \left| v + \sqrt{1 + v^2} \right| = |Cx|$$

$$\Rightarrow \quad \left| \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \right| = |Cx| \quad [\because v = y/x]$$

$$\Rightarrow \quad \left\{ y + \sqrt{x^2 + y^2} \right\}^2 = C^2 x^4$$

Hence, $\left\{ y + \sqrt{x^2 + y^2} \right\}^2 = C^2 x^4$ gives the required solution

EXAMPLE 9 *Solve :*

$$y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}dx - x\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}dy = 0$$ **[NCERT, CBSE 2010]**

SOLUTION The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}}{x\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}} \quad \text{...(i)}$$

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx\{x\cos v + v\,x\sin v\}}{x\{vx\sin v - x\cos v\}} = \frac{v\{\cos v + v\sin v\}}{\{v\sin v - \cos v\}}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v} = \frac{2v\cos v}{v\sin v - \cos v}$$

$\Rightarrow \quad \dfrac{v \sin v - \cos v}{v \cos v}\, dv = 2\dfrac{dx}{x}$, if $x \neq 0, v \cos v \neq 0$ [By separating the variables]

$\Rightarrow \quad -\int \dfrac{\cos v - v \sin v}{v \cos v}\, dv = 2\int \dfrac{dx}{x}$ [Integrating both sides]

$\Rightarrow \quad -\log |v \cos v| = 2\log |x| + \log C$

$\Rightarrow \quad \log \dfrac{1}{|v \cos v|} = \log |x^2| + \log C$

$\Rightarrow \quad \left|\dfrac{1}{v \cos v}\right| = |C|\, x^2$

$\Rightarrow \quad \left|\dfrac{x}{y} \sec\left(\dfrac{y}{x}\right)\right| = |C|\, x^2$

$\Rightarrow \quad |xy \cos (y/x)| = \dfrac{1}{|C|}$

$\Rightarrow \quad |xy \cos (y/x)| = k$, where $k = 1/|C|$

Hence, $\left| xy \cos\left(\dfrac{y}{x}\right)\right| = k, x \neq 0, k > 0$ gives the required solution.

EXAMPLE 10 *Solve :* $x\dfrac{dy}{dx} = y - x \tan\left(\dfrac{y}{x}\right)$. [CBSE 2002]

SOLUTION We are given that

$$x\frac{dy}{dx} = y - x \tan \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) \qquad \text{...(i)}$$

Clearly, the given differential equation is homogeneous. Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \tan v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\tan v$$

$\Rightarrow \quad \cot v\, dv = \dfrac{-dx}{x}$, if $x \neq 0$ [By separating the variables]

$\int \cot v\, dv = -\int \dfrac{dx}{x}$ [Integrating both sides]

$$\Rightarrow \quad \log |\sin v| = -\log |x| + \log C$$

$$\Rightarrow \quad |\sin v| = \left|\frac{C}{x}\right|$$

$$\Rightarrow \quad |\sin (y/x)| = |C/x|$$

Hence, $\left|\sin \dfrac{y}{x}\right| = \left|\dfrac{C}{x}\right|$ gives the required solution.

REMARK *Sometimes a homogeneous differential equation is expressible in the form*

$$\frac{dx}{dy} = \frac{f(x, y)}{g(x, y)}$$

where $f(x, y)$ and $g(x, y)$ are homogeneous functions of same degree.

In such a situation, we substitute $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{xy}$ to solve the differential equation.

EXAMPLE 11 *Solve:* $2y\, e^{x/y}\, dx + (y - 2x\, e^{x/y})\, dy = 0.$ **[NCERT]**

SOLUTION We have,

$$2y\, e^{x/y}\, dx + \left(y - 2x\, e^{x/y}\right) dy = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{2x\, e^{x/y} - y}{2y\, e^{x/y}} \qquad \text{...(i)}$$

Clearly, the given differential equation is a homogeneous differential equation. As the right hand side of (i) is expressible as a function of $\frac{x}{y}$. So, we put $x = vy$ and $\frac{dx}{dy} = v + y\frac{dv}{dy}$ to get

$$v + y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} - v$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{1}{2e^v}$$

$$\Rightarrow \quad 2ye^v\, dv = -dy$$

$$\Rightarrow \quad 2e^v\, dv = -\frac{1}{y}\, dy \;, y \neq 0$$

$$\Rightarrow \quad 2\int e^v\, dv = -\int \frac{1}{y}\, dy$$

$$\Rightarrow \quad 2e^v = -\log|y| + \log C$$

$$\Rightarrow \quad 2\, e^v = \log\left|\frac{C}{y}\right|$$

$$\Rightarrow \quad 2e^{x/y} = \log\left|\frac{C}{y}\right|$$

Hence, $2e^{x/y} = \log\left|\frac{C}{y}\right|$, $y \neq 0$ gives the general solution of the given differential equation.

EXAMPLE 12 *Solve the following initial value problems:*

(i) $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0, y(1) = \frac{\pi}{2}$

(ii) $xe^{y/x} - y\sin\left(\frac{y}{x}\right) + x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = 0, y(1) = 0.$

SOLUTION We have,

(i) $$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x - y\sin\left(\frac{y}{x}\right)}{x\sin\left(\frac{y}{x}\right)}$$

This is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = -\frac{1 - v\sin v}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1 - v\sin v}{\sin v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\Rightarrow \quad \sin v\, dv = -\frac{1}{x}\,dx \text{, if } x \neq 0$$

$$\Rightarrow \quad \int \sin v\, dv = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad -\cos v = -\log|x| + C$$

$$\Rightarrow \quad -\cos\left(\frac{y}{x}\right) + \log|x| = C \qquad \text{...(i)}$$

It is given that $y(1) = \frac{\pi}{2}$ i.e. when $x = 1$, $y = \frac{\pi}{2}$.

Putting $x = 1$ and $y = \frac{\pi}{2}$ in (i), we get

$\frac{x}{y} = v$

$$-\cos\frac{\pi}{2} + \log 1 = C \Rightarrow C = 0.$$

Putting $C = 0$ in (i), we get

$$-\cos\left(\frac{y}{x}\right) + \log|x| = 0$$

$$\Rightarrow \quad \log|x| = \cos\left(\frac{y}{x}\right)$$

Hence, $\log|x| = \cos\left(\frac{y}{x}\right)$, $x \neq 0$ is the required solution.

(ii) $$xe^{y/x} - y\sin\left(\frac{y}{x}\right) + x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - xe^{y/x}}{x\sin\left(\frac{y}{x}\right)}$$

This is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{v\sin v - e^{v}}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin v - e^{v}}{\sin v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{e^{v}}{\sin v}$$

$$\Rightarrow \quad e^{-v}\sin v\, dv = -\frac{dx}{x}, \text{ if } x \neq 0$$

$$\Rightarrow \quad \int e^{-v}\sin v\, dv = -\int \frac{1}{x}\, dx$$

$$\Rightarrow \quad \frac{e^{-v}}{2}(-\sin v - \cos v) = -\log|x| + \log C$$

$$\left[\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}\right]$$

$$\Rightarrow \quad -\frac{1}{2}e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = -\log|x| + \log C$$

$$\Rightarrow \quad e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = 2\log|x| - 2\log C$$

$$\Rightarrow \quad e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = \log|x|^2 - 2\log C \qquad \text{...(ii)}$$

It is given that $y(1) = 0$ i.e. $y = 0$ when $x = 1$. Putting these values in (ii), we get

$$1 = 0 - 2\log C \Rightarrow \log C = -\frac{1}{2}$$

Putting $\log C = -\frac{1}{2}$ in (ii), we get

$$e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = \log|x|^2 + 1$$

Hence, $e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = 1 + \log x^2$, $x \neq 0$ gives the required solution.

EXAMPLE 13 *Solve each of the following initial value problems:*

(i) $2x^2\frac{dy}{dx} - 2xy + y^2 = 0$, $y(e) = e$

(ii) $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$, $y(1) = 2$. **[NCERT]**

SOLUTION (i) We have,

$$2x^2\frac{dy}{dx} - 2xy + y^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{2xy - y^2}{2x^2}$$

This is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{2v - v^2}{2}$$

$$\Rightarrow \quad 2x\frac{dv}{dx} = -v^2$$

$$\Rightarrow \quad -\frac{2}{v^2}dv = \frac{dx}{x}, \text{ if } x \neq 0, v \neq 0$$

$$\Rightarrow \quad \int \frac{-2}{v^2}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \quad \frac{2}{v} = \log|x| + C$$

$$\Rightarrow \quad \frac{2x}{y} = \log|x| + C \quad \text{...(i)}$$

It is given that $y(e) = e$ i.e. $y = e$ when $x = e$.

Putting $x = e$ and $y = e$ in (i), we get

$$2 = 1 + C \Rightarrow C = 1$$

Putting $C = 1$ in (i), we get

$$\frac{2x}{y} = \log|x| + 1$$

$$\Rightarrow \quad y = \frac{2x}{1 + \log|x|}$$

Clearly, y exists, if $1 + \log|x| \neq 0$ i.e. $\log|x| \neq -1$

$$\Rightarrow \quad |x| \neq e^{-1} \Rightarrow x \neq \pm\frac{1}{e}$$

Hence, $y = \frac{2x}{1 + \log|x|}$, where $x \neq 0, \frac{1}{e}, -\frac{1}{e}$ gives the required solution.

(ii) $$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$$

This is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{2v + v^2}{2}$$

$$\Rightarrow \quad 2v + 2x\frac{dv}{dx} = 2v + v^2$$

$$\Rightarrow \quad 2x\frac{dv}{dx} = v^2$$

$$\Rightarrow \quad \frac{2}{v^2}\,dv = \frac{1}{x}\,dx \text{, if } x \neq 0\text{, } v \neq 0$$

$$\Rightarrow \quad 2\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \quad -\frac{2}{v} = \log|x| + C$$

$$\Rightarrow \quad -\frac{2x}{y} = \log|x| + C \qquad \text{...(i)}$$

It is given that $y(1) = 2$ i.e. $y = 2$ when $x = 1$.

Putting $x = 1$ and $y = 2$ in (i), we get

$$-1 = 0 + C \Rightarrow C = -1$$

Putting $C = -1$ in (i), we get

$$-\frac{2x}{y} = \log|x| - 1$$

$$\Rightarrow \quad y = \frac{2x}{1 - \log|x|}$$

Clearly, y is defined if $x \neq 0$ and $1 - \log|x| = 0$.

Now, $1 - \log|x| = 0 \Rightarrow \log|x| = 1 \Rightarrow |x| = e \Rightarrow x = \pm e$.

Hence, $y = \dfrac{2x}{1 - \log|x|}$, where $x \neq 0, \pm e$ gives the solution of the given differential equation.

EXAMPLE 14 *Solve each of the following initial value problem:*

(i) $(x^2 + y^2)\,dx + xy\,dy = 0,\ y(1) = 1$ (ii) $(xe^{y/x} + y)\,dx = x\,dy,\ y(1) = 1$

(iii) $(x^2 - 2y^2)\,dx + 2xy\,dy = 0,\ y(1) = 1$

SOLUTION (i) We have,

$$(x^2 + y^2)\,dx + xy\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x^2 + y^2}{xy}$$

This is a homogeneous differential equation.

Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = -\frac{1 + v^2}{v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left(\frac{1 + 2v^2}{v}\right)$$

$$\Rightarrow \quad \frac{v\,dv}{2v^2 + 1} = -\frac{1}{x}\,dx \text{, if } x \neq 0.$$

$$\Rightarrow \quad \frac{4v}{2v^2 + 1}\,dv = -\frac{4}{x}\,dx$$

$\Rightarrow \quad \int \frac{4v}{2v^2+1}\, dv = -\int \frac{4}{x}\, dx$

$\Rightarrow \quad \log(2v^2+1) = -4\log|x| + \log C$

$\Rightarrow \quad 2v^2+1 = \frac{|C|}{x^4}$

$\Rightarrow \quad (2y^2+x^2)\, x^2 = |C|$ ...(i)

It is given that $y(1)=1$ i.e. $y=1$ when $x=1$.

Putting $x=1$, $y=1$ in (i), we get $|C| = 3$

Putting $|C|=3$ in (i), we get

$(2y^2+x^2)\, x^2 = 3.$

$\Rightarrow \quad 2x^2y^2 = 3-x^4$

$\Rightarrow \quad y = \pm\sqrt{\frac{3-x^4}{2x^2}}$

But, $\quad y = -\sqrt{\frac{3-x^4}{2x^2}}$ is not satisfied by $y(1)=1$

$\therefore \quad y = \sqrt{\frac{3-x^4}{2x^2}}$

Now, y exists, if

$3-x^4 \geq 0$ and $x \neq 0$

$\Rightarrow \quad x^4-3 \leq 0$ and $x \neq 0$

$\Rightarrow \quad x^2-3 \leq 0$ and $x \neq 0$

$\Rightarrow \quad -\sqrt{3} \leq x \leq \sqrt{3}$ and $x \neq 0 \Rightarrow x \in [-\sqrt{3}, 0) \cup (0, \sqrt{3}]$

Hence, $\quad y = \sqrt{\frac{3-x^4}{2x^2}}$, where $x \in [-\sqrt{3}, 0) \cup (0, \sqrt{3}]$

(ii) $\quad (xe^{y/x}+y)\, dx = x\, dy$

$\Rightarrow \quad \frac{dy}{dx} = e^{y/x} + \frac{y}{x}$

This is a homogeneous differential equation. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ it reduces to

$v + x\frac{dv}{dx} = e^v + v$

$\Rightarrow \quad x\frac{dv}{dx} = e^v$

$\Rightarrow \quad e^{-v}\, dv = \frac{dx}{x}$, if $x \neq 0$.

$\Rightarrow \quad \int e^{-v}\, dv = \int \frac{1}{x}\, dx$

$\Rightarrow \quad -e^{-v} = \log|x| + C$

$$\Rightarrow \quad -e^{-y/x} = \log|x| + C \qquad \text{...(i)}$$

It is given that $y(1) = 1$ i.e. when $x = 1, y = 1$.

Putting $x = 1, y = 1$ in (i), we get

$$-e^{-1} = C$$

Putting $C = -\frac{1}{e}$ in (i), we get

$$-e^{-y/x} = \log|x| - \frac{1}{e}$$

$$\Rightarrow \quad e^{-y/x} = \frac{1}{e} - \log|x|$$

$$\Rightarrow \quad -\frac{y}{x} = \log(1 - e\log|x|) - 1$$

$$\Rightarrow \quad y = x - x\log(1 - e\log|x|)$$

Hence, $y = x - x\log(1 - e\log|x|)$, is the solution of the given equation.

(iii) $\quad (x^2 - 2y^2)\,dx + 2xy\,dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2y^2 - x^2}{2xy}$$

This is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{2v^2 - 1}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v^2 - 1}{2v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1}{2v}$$

$$\Rightarrow \quad 2v\,dv = -\frac{dx}{x}, \text{ if } x \neq 0$$

$$\Rightarrow \quad \int 2v\,dv = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad v^2 = -\log|x| + C$$

$$\Rightarrow \quad y^2 = -x^2\log|x| + Cx^2 \qquad \text{...(i)}$$

It is given that $y(1) = 1$ i.e. when $x = 1$, $y = 1$.

Putting $x = 1, y = 1$ in (i), we get

$$1 = 0 + C \Rightarrow C = 1$$

Putting $C = 1$ in (i), we get

$y^2 = -x^2\log|x| + x^2$ as the required solution.

EXERCISE 22.9

Solve the following differential equations:

1. $x^2\,dy + y(x+y)\,dx = 0$ [CBSE 2010]

2. $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$ [CBSE 2004]

3. $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$

4. $x\dfrac{dy}{dx} = x + y,\ x \neq 0$

5. $(x^2 - y^2)\,dx - 2\,xy\,dy = 0$

6. $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ [NCERT]

7. $2xy\dfrac{dy}{dx} = x^2 + y^2$

8. $x^2\dfrac{dy}{dx} = x^2 - 2y^2 + xy$ [NCERT]

9. $xy\dfrac{dy}{dx} = x^2 - y^2$

10. $y\,e^{x/y}\,dx = \left(xe^{x/y} + y\right)dy$

11. $x^2\dfrac{dy}{dx} = x^2 + xy + y^2$

12. $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$

13. $2xy\,dx + (x^2 + 2y^2)\,dy = 0$

14. $3x^2\,dy = (3xy + y^2)\,dx$

15. $\dfrac{dy}{dx} = \dfrac{x}{2y + x}$

16. $(x + 2y)\,dx - (2x - y)\,dy = 0$

17. $\dfrac{dy}{dx} = \dfrac{y}{x} - \sqrt{\dfrac{y^2}{x^2} - 1}$

18. $\dfrac{dy}{dx} = \dfrac{y}{x}\left\{\log\left(\dfrac{y}{x}\right) + 1\right\}$

19. $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\left(\dfrac{y}{x}\right)$ [NCERT]

20. $y^2\,dx + (x^2 - xy + y^2)\,dy = 0$

21. $[x\sqrt{x^2 + y^2} - y^2]\,dx + xy\,dy = 0$

22. $x\dfrac{dy}{dx} = y - x\cos^2\left(\dfrac{y}{x}\right)$

23. $\dfrac{y}{x}\cos\left(\dfrac{y}{x}\right)dx - \left\{\dfrac{x}{y}\sin\left(\dfrac{y}{x}\right) + \cos\left(\dfrac{y}{x}\right)\right\}dy = 0$

24. $xy\log\left(\dfrac{x}{y}\right)dx + \left\{y^2 - x^2\log\left(\dfrac{x}{y}\right)\right\}dy = 0$ [CBSE 2010]

25. $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \dfrac{x}{y}\right)dy = 0$ [NCERT]

26. $(x^2 + y^2)\dfrac{dy}{dx} = 8x^2 - 3xy + 2y^2$

27. $(x^2 - 2xy)\,dy + (x^2 - 3xy + 2y^2)\,dx = 0$

28. $x\dfrac{dy}{dx} = y - x\cos^3\left(\dfrac{y}{x}\right)$

29. $x\dfrac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$

30. $x\cos\left(\dfrac{y}{x}\right)\cdot(y\,dx + x\,dy) = y\sin\left(\dfrac{y}{x}\right)\cdot(x\,dy - y\,dx)$

31. $(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$

32. $(x - y)\dfrac{dy}{dx} = x + 2y$ [CBSE 2010]

33. $(2x^2 y + y^3)\, dx + (xy^2 - 3x^3)\, dy = 0$

34. $x \frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$

35. $y\, dx + \left\{x \log\left(\frac{y}{x}\right)\right\} dy - 2x\, dy = 0$

36. Solve each of the following initial value problems:

(i) $(x^2 + y^2)\, dx = 2xy\, dy\,,\ y(1) = 0$

(ii) $xe^{y/x} - y + x \frac{dy}{dx} = 0\,,\ y(e) = 0$

(iii) $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec} \frac{y}{x} = 0\,,\ y(1) = 0$ [NCERT, CBSE 2009]

(iv) $(xy - y^2)\, dx - x^2\, dy = 0,\ y(1) = 1$

(v) $\frac{dy}{dx} = \frac{y(x + 2y)}{x(2x + y)}\,,\ y(1) = 2$

(vi) $(y^4 - 2x^3 y)\, dx + (x^4 - 2xy^3)\, dy = 0\,,\ y(1) = 1$

(vii) $x(x^2 + 3y^2)\, dx + y(y^2 + 3x^2)\, dy = 0\,,\ y(1) = 1$

ANSWERS

1. $x^2 y = C(y + 2x)$
2. $\log(x^2 + y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = k$
3. $x^2 + y^2 = Cx$
4. $y = x \log|x| + Cx,\ x \neq 0$
5. $x(x^2 - 3y^2) = C$
6. $\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2 + y^2) + C.$
7. $x = C(x^2 - y^2)$
8. $\frac{x + \sqrt{2}\, y}{x - \sqrt{2}\, y} = (Cx)^{2\sqrt{2}}$
9. $x^2(x^2 - 2y^2) = C$
10. $e^{x/y} = \log y + C$
11. $\tan^{-1}\left(\frac{y}{x}\right) = C + \log|x|$
12. $x^2 y - xy^2 = C,\ x \neq 0$
13. $3x^2 y + 2y^3 = C,\ xy \neq 0$
14. $\frac{-3x}{y} = \log|x| + C$
15. $(x + y)(2y - x)^2 = C$
16. $\sqrt{x^2 + y^2} = C\, e^{2\tan^{-1} y/x},\ x \neq 0$
17. $y + \sqrt{y^2 - x^2} = C$
18. $\log\left(\frac{y}{x}\right) = Cx$
19. $\tan\left(\frac{y}{2x}\right) = Cx$
20. $y = Ce^{\tan^{-1}(y/x)}$
21. $\sqrt{x^2 + y^2} = x \log\left|\frac{C}{x}\right|$
22. $\tan\left(\frac{y}{x}\right) = \log\left|\frac{C}{x}\right|$
23. $\left|y \sin\left(\frac{y}{x}\right)\right| = C$
24. $\frac{x^2}{y^2}\left\{\log\left(\frac{x}{y}\right) - \frac{1}{2}\right\} + \log y^2 = C$
25. $x + y\, e^{x/y} = C$
26. $C\,|2x - y|^{5/8}(4x^2 + y^2)^{3/16} = e^{-3/8 \tan^{-1}(y/2x)},\ x \neq 0$

27. $\frac{y}{x} + \log x = C$

28. $\tan\left(\frac{y}{x}\right) = \log\left|\frac{C}{x}\right|$

29. $y + \sqrt{y^2 - x^2} = C x^2$

30. $\sec\left(\frac{y}{x}\right) = Cxy$

31. $\frac{x}{x+y} + \log x = C$

32. $\log\left|x^2 + xy + y^2\right| = 2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}\,x}\right) + C, x \in R - \{0\}$

33. $x^2 y^{12} = C^4 \left|2y^2 - x^2\right|^5$

34. $x\sin\left(\frac{y}{x}\right) = c\left(1 + \cos\frac{y}{x}\right); x\sin\left(\frac{y}{x}\right) \neq 0$

35. $Cy = \log\left|\frac{y}{x}\right| - 1, x \neq 0$

36. (i) $(x^2 - y^2)^2 = x^2$

(ii) $y = x\log(\log|x|); x \in (-\infty, -1) \cup (1, \infty)$.

(iii) $\log|x| = \cos\left(\frac{y}{x}\right) - 1, x \neq 0$

(iv) $y = \frac{x}{1 + \log|x|}, x \neq 0; \pm e^{-1}$.

(v) $xy = 2\,|y - x|^{3/2}, x \in R$

(vi) $(x^3 + y^3)^2 = 4x^2 y^2, x \in R$

(vii) $x^4 + 6x^2 y^2 + y^4 = 8, x \in R$.

22.8.6 LINEAR DIFFERENTIAL EQUATIONS

A differential equation is linear if the dependent variable (y) and its derivative appear only in first degree. The general form of a linear differential equation of first order is

$$\frac{dy}{dx} + Py = Q \quad \text{...(i)}$$

where P and Q are functions of x (or constants)

For example,

(i) $\frac{dy}{dx} + xy = x^3$,

(ii) $x\frac{dy}{dx} + 2y = x^3$,

(iii) $\frac{dy}{dx} + 2y = \sin x$ etc. are linear differential equations.

This type of differential equations are solved when they are multiplied by a factor, which is called integrating factor, because by multiplication of this factor the left hand side of the differential equation (i) becomes exact differential of some function.

Multiplying both sides of (i) by $e^{\int P dx}$, we get

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Q e^{\int P dx}$$

$$\Rightarrow \qquad \frac{d}{dx}\left\{y\, e^{\int P dx}\right\} = Q\, e^{\int P dx}$$

On integrating both sides w.r.t. x, we get

$$y\, e^{\int P dx} = \int Q\, e^{\int P dx} + C \qquad \text{...(ii)}$$

This is the required solution, where C is the constant of integration.

Here, $e^{\int P dx}$ is called the integrating factor.

The solution (ii) in short may also be written as

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + C$$

Following algorithm may be used to solve a linear differential equation.

ALGORITHM

STEP I *Write the differential equation in the form $\frac{dy}{dx} + Py = Q$ and obtain P and Q.*

STEP II *Find integrating factor (I.F.) given by I.F. $= e^{\int P dx}$*

STEP III *Multiply both sides of equation in Step I by I.F.*

STEP IV *Integrate both sides of the equation obtained in step III w.r.t. x to obtain*

$$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C$$

This gives the required solution

Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the differential equation*

$$\frac{dy}{dx} - \frac{y}{x} = 2x^2, x > 0 \qquad \text{[CBSE 2004, 2007]}$$

SOLUTION We are given that

$$\frac{dy}{dx} + \left(\frac{-1}{x}\right) y = 2x^2 \qquad \text{...(i)}$$

Clearly, it is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{-1}{x} \text{ and } Q = 2x^2.$$

Now, $\quad \text{I.F.} = e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}.$

Multiplying both sides of (i) by I.F. $= \frac{1}{x}$, we get

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = 2x$$

Integrating both sides w.r.t. x, we get

$$y \cdot \left(\frac{1}{x}\right) = \int 2x\, dx + C \qquad \text{[Using : } y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\,]$$

$$\Rightarrow \quad y \cdot \frac{1}{x} = x^2 + C$$

$$\Rightarrow \quad y = x^3 + Cx$$

Hence, the required solution is given by $y = x^3 + cx$, $x > 0$.

EXAMPLE 2 *Solve the differential equation*

$$\frac{dy}{dx} + \frac{y}{2x} = 3x^2, x > 0$$

SOLUTION The given differential equation is

$$\frac{dy}{dx} + \frac{1}{2x} \cdot y = 3x^2 \qquad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{1}{2x} \text{ and } Q = 3x^2.$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int (1/2x)\, dx} = e^{(1/2) \log x} = e^{\log x^{1/2}} = x^{1/2}$$

Multiplying both sides of (i) by I.F. $= \sqrt{x}$, we get

$$\sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y = 3x^{5/2}$$

Integrating both sides w.r.t. x, we get

$$y\sqrt{x} = \int 3x^{5/2}\, dx + C \qquad \text{[Using : } y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\,]$$

$$\Rightarrow \quad y\sqrt{x} = 3\left(\frac{x^{7/2}}{7/2}\right) + C$$

$$\Rightarrow \quad y\sqrt{x} = \frac{6}{7} x^{7/2} + C$$

Hence, the required solution is given by $y\sqrt{x} = \frac{6}{7} x^{7/2} + C, x > 0$

EXAMPLE 3 *Solve the differential equation*

$$x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x, x > 0 \qquad \text{[CBSE 2010]}$$

SOLUTION The given differential equation is

$$x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x^2} \qquad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{1}{x \log x} \text{ and } Q = \frac{2}{x^2}$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x \log x} dx} = e^{\int \frac{1}{t} dt}, \text{ where } t = \log x$$

$$= e^{\log t} = t = \log x$$

Multiplying both sides of (i) by I.F. = $\log x$, we get

$$\log x \frac{dy}{dx} + \frac{1}{x} y = \frac{2}{x^2} \log x$$

Integrating both sides w.r.t. x, we get

$$y \cdot \log x = \int \frac{2}{x^2} \log x \, dx + C \qquad \left[\text{Using : } y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad y \cdot \log x = 2 \cdot \int \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{x^{-2}}\, dx + C$$

$$\Rightarrow \quad y \cdot \log x = 2\left[\log x \cdot \left(\frac{x^{-1}}{-1}\right) - \int \frac{1}{x} \cdot \left(\frac{x^{-1}}{-1}\right) dx\right] + C$$

$$\Rightarrow \quad y \log x = 2\left[-\frac{\log x}{x} + \int x^{-2}\, dx\right] + C$$

$$\Rightarrow \quad y \log x = 2\left[-\frac{\log x}{x} - \frac{1}{x}\right] + C$$

$$\Rightarrow \quad y \log x = -\frac{2}{x}(1 + \log x) + C$$

Hence, the required solution is given by

$$y \log x = -\frac{2}{x}(1 + \log x) + C, x > 0$$

EXAMPLE 4 *Solve the differential equation* $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$ **[CBSE 2010]**

SOLUTION The given differential equation is

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2 - 1} y = \frac{1}{(x^2 - 1)^2} \qquad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{2x}{x^2 - 1} \text{ and } Q = \frac{1}{(x^2 - 1)^2}$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int 2x/(x^2 - 1)\, dx} = e^{\log (x^2 - 1)} = (x^2 - 1)$$

Multiplying both sides of (i) by I.F. = $(x^2 - 1)$, we get

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$$

Integrating both sides, we get

$$y\,(x^2 - 1) = \int \frac{1}{x^2 - 1} dx + C \qquad \left[\text{Using : } y\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad y\,(x^2 - 1) = \frac{1}{2} \log \left|\frac{x - 1}{x + 1}\right| + C.$$ This is the required solution.

EXAMPLE 5 *Solve* $\frac{dy}{dx} + y \sec x = \tan x \quad \left(0 \le x < \frac{\pi}{2}\right)$ **[CBSE 2008]**

SOLUTION The given differential equation is

$$\frac{dy}{dx} + (\sec x)\, y = \tan x \qquad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \sec x \text{ and } Q = \tan x$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \sec x\, dx} = e^{\log(\sec x + \tan x)} = (\sec x + \tan x)$$

Multiplying both sides of (i) by I.F. = ($\sec x + \tan x$), we get

$$(\sec x + \tan x)\frac{dy}{dx} + y \sec x (\sec x + \tan x) = \tan x (\sec x + \tan x)$$

Integrating both sides w.r.t. x, we get

$$y(\sec x + \tan x) = \int \tan x (\sec x + \tan x)\, dx + C$$

[Using : $y\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + C$]

$$\Rightarrow \quad y(\sec x + \tan x) = \int (\tan x \sec x + \tan^2 x)\, dx + C$$

$$y(\sec x + \tan x) = \int (\tan x \sec x + \sec^2 x - 1)\, dx + C$$

$$\Rightarrow \quad y(\sec x + \tan x) = \sec x + \tan x - x + C, \text{ which is the required solution.}$$

EXAMPLE 6 *Solve:* $\cos^2 x \frac{dy}{dx} + y = \tan x \quad \left(0 \le x < \frac{\pi}{2}\right)$ **[NCERT, CBSE 2008]**

SOLUTION We are given that

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \quad \frac{dy}{dx} + (\sec^2 x)\, y = \tan x \sec^2 x \qquad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \sec^2 x \text{ and } Q = \tan x \sec^2 x$$

$$\therefore \quad I = \text{I.F.} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

Multiplying both sides of (i) by I.F. = $e^{\tan x}$, we get

$$e^{\tan x}\frac{dy}{dx} + \sec^2 x\, e^{\tan x} \cdot y = e^{\tan x} \cdot \tan x \sec^2 x$$

Integrating both sides w.r.t. x, we get

$$y\, e^{\tan x} = \int e^{\tan x} \cdot \tan x \sec^2 x\, dx + C \qquad [\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C]$$

$$\Rightarrow \quad y\, e^{\tan x} = \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^t}\, dt + C, \text{ where } t = \tan x$$

$$\Rightarrow \quad y\, e^{\tan x} = t e^t - \int e^t\, dt + C \qquad \text{[Integrating by parts]}$$

$$\Rightarrow \quad y e^{\tan x} = t e^t - e^t + C$$

$$\Rightarrow \quad y e^{\tan x} = e^{\tan x}(\tan x - 1) + C, \text{ which is the required solution.}$$

EXAMPLE 7 *Solve :* $(x^2+1)\dfrac{dy}{dx}+2xy = \sqrt{x^2+4}.$ **[CBSE 2010]**

SOLUTION We are given that

$$(x^2+1)\frac{dy}{dx}+2xy = \sqrt{x^2+4}$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{2x}{x^2+1}y = \frac{\sqrt{x^2+4}}{x^2+1} \quad \text{...(i)}$$

This is a linear differential equation of the form $\dfrac{dy}{dx}+Py=Q$, where

$$P = \frac{2x}{x^2+1} \text{ and } Q=\frac{\sqrt{x^2+4}}{x^2+1}.$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int 2x/(x^2+1)\,dx} = e^{\log(x^2+1)} = (x^2+1)$$

Multiplying both sides of (i) by I.F. $=(x^2+1)$, we get

$$(x^2+1)\frac{dy}{dx}+2xy = \sqrt{x^2+4}$$

Integrating both sides w.r.t. x, we obtain

$$y(x^2+1) = \int \sqrt{x^2+4}\,dx + C \qquad [\text{Using} : y\,(I.\,F.) = \int Q\,(I.\,F.)\,dx + C]$$

$$\Rightarrow \quad y(x^2+1) = \frac{1}{2}x\sqrt{x^2+4}+\frac{1}{2}(2)^2\log\left|\,x+\sqrt{x^2+4}\,\right| + C$$

$$\Rightarrow \quad y(x^2+1) = \frac{1}{2}x\sqrt{x^2+4}+2\log\left|\,x+\sqrt{x^2+4}\,\right| + C,$$

which is the required solution.

EXAMPLE 8 *Solve :* $\dfrac{dy}{dx}-2y = \cos 3x$

SOLUTION We are given that

$$\frac{dy}{dx}+(-2)\,y = \cos 3x \quad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py = Q, \text{ where } P = -2 \text{ and } Q = \cos 3x.$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int -2\,dx} = e^{-2x}$$

Multiplying both sides of (*i*) by I.F. $= e^{-2x}$, we get

$$e^{-2x}\frac{dy}{dx}-2ye^{-2x} = \cos 3x \cdot e^{-2x}$$

Integrating both sides w.r.t. x, we get

$$ye^{-2x} = \int e^{-2x}\cos 3x\,dx + C \qquad [\text{Using} : \; y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C]$$

$$\Rightarrow \quad y\,e^{-2x} = I + C, \text{ where } I = e^{-2x}\cos 3x$$

Now, $\quad I = \int \underset{\text{I}}{e^{-2x}}\,\underset{\text{II}}{\cos 3x}\,dx$

$$\Rightarrow \quad I = \frac{1}{3}e^{-2x}\sin 3x - \int \frac{(-2)}{3}e^{-2x}\sin 3x\,dx$$

$$\Rightarrow \quad I = \frac{1}{3}e^{-2x}\sin 3x + \frac{2}{3}\int e^{-2x}\sin 3x\,dx$$

$$\Rightarrow \quad I = \frac{1}{3}e^{-2x}\sin 3x + \frac{2}{3}\left[\frac{-1}{3}e^{-2x}\cos 3x - \int (-2)\,e^{-2x}\left(\frac{-\cos 3x}{3}\right)dx\right]$$

$$\Rightarrow \quad I = \frac{1}{3}e^{-2x}\sin 3x + \frac{2}{3}\left[-\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{3}\int e^{-2x}\cos 3x\,dx\right]$$

$$\Rightarrow \quad I = \frac{1}{3}e^{-2x}\sin 3x - \frac{2}{9}e^{-2x}\cos 3x - \frac{4}{9}I$$

$$\Rightarrow \quad \left(I + \frac{4}{9}I\right) = \frac{e^{-2x}}{9}(3\sin 3x - 2\cos 3x)$$

$$\Rightarrow \quad I = \frac{e^{-2x}}{13}(3\sin 3x - 2\cos 3x)$$

Substituting the value of I in (ii), we get

$$ye^{-2x} = \frac{e^{-2x}}{13}(3\sin 3x - 2\cos 3x) + C, \text{ which is the required solution.}$$

EXAMPLE 9 *Solve :* $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ *subject to the initial conditon* $y(0) = 0$.

SOLUTION The given differential equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2} \qquad \text{...(i)}$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where

$$P = \frac{2x}{1+x^2} \text{ and } Q = \frac{4x^2}{1+x^2}$$

We have, I.F. $= e^{\int P\,dx} = e^{\int 2x/(1+x^2)\,dx} = e^{\log(1+x^2)} = 1 + x^2$

Multiplying both sides of (i) by I.F. $(1+x^2)$, we get

$$(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$$

Integrating both sides w.r.t. x, we get

$$y(1+x^2) = \int 4x^2\,dx + C \qquad \left[\text{Using: } y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad y(1+x^2) = \frac{4x^3}{3} + C \qquad \text{...(ii)}$$

It is given that $y = 0$, when $x = 0$. Putting $x = 0$ and $y = 0$ in (i), we get

$$0 = 0 + C \Rightarrow C = 0$$

Substituting $C = 0$ in (ii), we get

$$y = \frac{4x^3}{3(1+x^2)}, \text{ which is the required solution.}$$

EXAMPLE 10 *Solve :* $\frac{dy}{dx} + y = \cos x - \sin x$

SOLUTION The given differential equation is

$$\frac{dy}{dx} + y = \cos x - \sin x \quad \text{...(i)}$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = 1$ and $Q = \cos x - \sin x$.

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int 1\, dx} = e^x$$

Multiplying both sides of (i) by I.F. $= e^x$, we get

$$e^x \frac{dy}{dx} + y = e^x (\cos x - \sin x)$$

Integrating both sides w.r.t. x, we get

$$ye^x = \int e^x (\cos x - \sin x)\, dx + C \qquad [\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C]$$

$$\Rightarrow \quad ye^x = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\cos x}\, dx - \int e^x \sin x\, dx + C$$

$$\Rightarrow \quad ye^x = e^x \cos x - \int -\sin x\, e^x\, dx - \int e^x \sin x + C \qquad [\text{Integ. 1st integral by parts}]$$

$$\Rightarrow \quad ye^x = e^x \cos x + \int e^x \sin x\, dx - \int e^x \sin x\, dx + C$$

$\Rightarrow \quad ye^x = e^x \cos x + C$, which is the required solution.

EXAMPLE 11 *Solve :* $\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x.$

SOLUTION The given differential equation is

$$\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x \quad \text{...(i)}$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \tan x$ and $Q = 2x + x^2 \tan x$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

Multiplying both sides of (i) by I.F. $= \sec x$, we get

$$\sec x \frac{dy}{dx} + y \sec x \tan x = 2x \sec x + x^2 \sec x \tan x$$

Integrating both sides w.r.t. x, we get

$$y \sec x = \int (2x \sec x + x^2 \sec x \tan x)\, dx + C \quad [\text{Using: } y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C]$$

$$\Rightarrow \quad y \sec x = \int 2x \sec x\, dx + \int \underset{\text{I}}{x^2} \underset{\text{II}}{\sec x \tan x}\, dx + C$$

$$\Rightarrow \quad y \sec x = \int 2x \sec x\, dx + x^2 \sec x - \int 2x \sec x\, dx + C$$

$\Rightarrow \quad y \sec x = x^2 \sec x + C$, which is the required solution

EXAMPLE 12 *Solve :* $\frac{dy}{dx}+\frac{y}{x} = \cos x+\frac{\sin x}{x}, x>0.$

SOLUTION The given differential equation is

$$\frac{dy}{dx}+\left(\frac{1}{x}\right)y = \cos x+\frac{\sin x}{x} \quad ...(i)$$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q, \text{ where } P=\frac{1}{x} \text{ and } Q=\cos x+\frac{\sin x}{x}$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Multiplying both sides of (i) by I.F. $= x$, we get

$$x\frac{dy}{dx}+y = x\cos x+\sin x$$

Integrating both sides w.r.t. x, we get

$$yx = \int (x\cos x+\sin x)\,dx+C \quad \text{[Using : } y\text{ (I.F.)} = \int Q\text{ (I.F.)}\,dx+C]$$

$$\Rightarrow \quad xy = \int \underset{\text{I}}{x}\underset{\text{II}}{\cos x}\,dx+\int \sin x\,dx+C$$

$$\Rightarrow \quad xy = x\sin x-\int \sin x\,dx+\int \sin x\,dx+C \quad \text{[Integrating 1st integral by parts]}$$

$$\Rightarrow \quad xy = x\sin x+C$$

Hence, $y=\sin x+\frac{C}{x}, x>0$ gives the required solution.

EXAMPLE 13 *Solve :* $\frac{dy}{dx}+x\sin 2y = x^3\cos^2 y$

SOLUTION The given differential equation can be written as

$$\sec^2 y\frac{dy}{dx}+x\frac{\sin 2y}{\cos^2 y} = x^3$$

$$\Rightarrow \quad \sec^2 y\frac{dy}{dx}+2x\tan y = x^3$$

Let $\tan y=v$. Then,

$$\sec^2 y\frac{dy}{dx}=\frac{dv}{dx} \quad ...(i)$$

Putting $\tan y=v$ and $\sec^2 y\frac{dy}{dx}=\frac{dv}{dx}$ in (i), we get

$$\therefore \quad \frac{dv}{dx}+(2x)v = x^3 \quad ...(ii)$$

This is a linear differential equation of the form $\frac{dv}{dx}+Pv=Q$, where $P=2x$ and $Q=x^3$.

$$\therefore \quad \text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

Multiplying both sides of (i) by e^{x^2}, we obtain

$$e^{x^2}\frac{dy}{dx}+e^{x^2}(2x)\,v = x^3 e^{x^2}$$

Integrating both sides w.r.t. x, we get

$$ve^{x^2} = \int x^3 e^{x^2}\,dx + C \qquad \left[\text{Using} : v\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad ve^{x^2} = \frac{1}{2}\int t\,e^t\,dt + C, \text{ where } t = x^2$$

$$\Rightarrow \quad ve^{x^2} = \frac{1}{2}(t-1)\,e^t + C$$

$$\Rightarrow \quad e^{x^2}\tan y = \frac{1}{2}(x^2-1)\,e^{x^2} + C, \text{ which gives the required solution.}$$

EXAMPLE 14 *Solve :* $\dfrac{dy}{dx} = -\dfrac{x + y\cos x}{1+\sin x}$

SOLUTION We have,

$$\frac{dy}{dx} = -\frac{x+y\cos x}{1+\sin x}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{\cos x}{1+\sin x}\,y = -\frac{x}{1+\sin x} \qquad \text{...(i)}$$

This is a linear differential equation with $P = \dfrac{\cos x}{1+\sin x}$ and $Q = \dfrac{-x}{1+\sin x}$

$$\therefore \quad \text{I. F.} = e^{\int \frac{\cos x}{1+\sin x}dx} = e^{\log(1+\sin x)} = (1+\sin x)$$

Multiplying both sides of (i) by I.F. ($= 1+\sin x$), we get

$$(1+\sin x)\frac{dy}{dx} + y\cos x = -x$$

Integrating with respect to x, we get

$$y\,(1+\sin x) = \int -x\,dx + C \qquad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad y\,(1+\sin x) = -\frac{x^2}{2} + C$$

$$\Rightarrow \quad y = \frac{2C - x^2}{2\,(1+\sin x)}$$

Clearly, it is defined for $1+\sin x \neq 0$ i.e. for $x \neq n\pi + (-1)^n \pi/2,\ n \in Z$

Hence, $y = \dfrac{2C-x^2}{2\,(1+\sin x)},\ x \neq n\pi + (-1)^n \pi/2,\ n \in Z$ gives the solution of the given differential equation.

EXAMPLE 15 *Solve each of the following initial value problems:*

(i) $\frac{dy}{dx} - y = e^x, y(0) = 1$

(ii) $x\frac{dy}{dx} + y = x \log x, y(1) = \frac{1}{4}$

SOLUTION (i) We have,

$$\frac{dy}{dx} - y = e^x \quad \text{...(i)}$$

This is a linear differential equation with $P = -1$ and $Q = e^x$

$\therefore$ I.F. $= e^{\int -1\, dx} = e^{-x}$

Multiplying both sides of (i) by e^{-x}, we get

$$\frac{dy}{dx} e^{-x} - y e^{-x} = e^x \cdot e^{-x}$$

Integrating both sides with respect to x, we get

$$y e^{-x} = \int e^x . e^{-x}\, dx + C \quad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad y e^{-x} = x + C \quad \text{...(ii)}$$

It is given that $y(0) = 1$ i.e. $y = 1$ when $x = 0$

Putting $x = 0, y = 1$ in (ii), we get

$$1 = 0 + C \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we get

$$y e^{-x} = x + 1$$

$$\Rightarrow \quad y = (x+1)e^x$$

Clearly, y is defined for all $x \in R$.

Hence, $y = (x+1)e^x, x \in R$ is the required solution.

(ii) $$x\frac{dy}{dx} + y = x \log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x} y = \log x, x > 0 \quad \text{...(i)}$$

This is linear differential equation of the form $\frac{dy}{dx} + Py = Q$ with

$$P = \frac{1}{x} \text{ and } Q = \log x$$

$\therefore$ I.F. $= e^{\int \frac{1}{x} dx} = e^{\log x} = x$ $\quad [\because x > 0]$

Multiplying both sides of (i) by I.F. $= x$, we get

$$x\frac{dy}{dx} + y = x \log x$$

Integrating with respect to x, we get

$$y x = \int \underset{II}{x} \underset{I}{\log x}\, dx \quad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad yx = \frac{x^2}{2}(\log x) - \frac{1}{2}\int x\,dx$$

$$\Rightarrow \quad xy = \frac{x^2}{2}(\log x) - \frac{x^2}{4} + C \quad \text{...(ii)}$$

It is given that $y(1) = \frac{1}{4}$ *i.e.* $y = \frac{1}{4}$ when $x = 1$

$$\frac{1}{4} = 0 - \frac{1}{4} + C \Rightarrow C = \frac{1}{2}$$

Putting $C = \frac{1}{2}$ in (ii), we get

$$xy = \frac{x^2}{2}(\log x) - \frac{x^2}{4} + \frac{1}{2}$$

$$\Rightarrow \quad y = \frac{1}{2}x\log x - \frac{x}{4} + \frac{1}{2x}$$

Hence, $y = \frac{1}{2}x\log x - \frac{x}{4} + \frac{1}{2x}$, $x > 0$ is the solution of the given differential equation.

EXAMPLE 16 *Solve each of the following initial value problems:*

(i) $\frac{dy}{dx} + \frac{2x}{x^2+1}y = \frac{1}{(x^2+1)^2}, y(0) = 0$

(ii) $(x^2+1)y' - 2xy = (x^4 + 2x^2 + 1)\cos x, y(0) = 0$

SOLUTION (i) We have,

$$\frac{dy}{dx} + \frac{2x}{x^2+1}y = \frac{1}{(x^2+1)^2} \quad \text{..(i)}$$

This is a linear differential equation with $P = \frac{2x}{x^2+1}$ and $Q = \frac{1}{(x^2+1)^2}$

$$\therefore \quad \text{I.F.} = e^{\int \frac{2x}{x^2+1}dx} = e^{\log(x^2+1)} = x^2+1$$

Multiplying both sides of (i) by I.F. $(= x^2+1)$, we get

$$(x^2+1)\frac{dy}{dx} + 2xy = \frac{1}{x^2+1}$$

Integrating both sides with respect to x, we get

$$y(x^2+1) = \int \frac{1}{x^2+1}dx + C \quad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad y(x^2+1) = \tan^{-1}x + C \quad \text{...(ii)}$$

It is given that $y(0) = 0$ i.e. $y = 0$ when $x = 0$

Putting $x = 0, y = 0$ in (ii), we get

$$0 = 0 + C \Rightarrow C = 0$$

Putting C = 0 in (ii), we get

$$y(x^2+1) = \tan^{-1} x$$

$$\Rightarrow \quad y = \frac{\tan^{-1} x}{x^2+1}$$

Clearly, it is defined for all $x \in R$.

Hence, $y = \dfrac{\tan^{-1} x}{x^2+1}$, $x \in R$ gives the required solution.

(ii) $$(x^2+1)y' - 2xy = (x^4+2x^2+1)\cos x$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{2x}{x^2+1} y = (x^2+1)\cos x \qquad \text{...(i)}$$

This is a linear differential equation with $P = \dfrac{-2x}{x^2+1}$ and $Q = (x^2+1)\cos x$

$$\therefore \quad \text{I.F.} = e^{\int \frac{-2x}{x^2+1} dx} = e^{-\log(x^2+1)} = (x^2+1)^{-1}$$

Multiplying (i) by $\dfrac{1}{x^2+1}$, we get

$$\frac{1}{x^2+1}\frac{dy}{dx} - \frac{2x}{(x^2+1)^2} y = \cos x$$

Integrating both sides, we get

$$y \times \frac{1}{x^2+1} = \int \cos x \, dx + C$$

$$\Rightarrow \quad \frac{y}{x^2+1} = \sin x + C \qquad \text{...(ii)}$$

It is given that $y(0) = 0$ i.e. $y = 0$ when $x = 0$

Putting $x = 0$, $y = 0$ in (ii), we get $C = 0$

Putting $C = 0$ in (ii), we get

$$\frac{y}{x^2+1} = \sin x$$

$$\Rightarrow \quad y = (x^2+1)\sin x$$

Clearly, it is defined for all $x \in R$.

Hence, $y = (x^2+1)\sin x$, $x \in R$ gives the solution.

EXAMPLE 17 *Solve:* (i) $x\dfrac{dy}{dx} + y - x + xy\cot x = 0, x \neq 0$ **[NCERT]**

(ii) $(1+x^2)\,dy + 2xy\,dx = \cot x\,dx, x \neq 0$

SOLUTION (i) We have,

$$x\frac{dy}{dx} + y - x + xy\cot x = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} + (1 + x\cot x)y = x$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right) y = 1 \qquad \text{...(i)}$$

This is a linear differential equation with $P = \frac{1}{x} + \cot x$ and $Q = 1$.

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \left(\frac{1}{x} + \cot x\right) dx} = e^{\log x + \log \sin x} = e^{\log (x \sin x)} = x \sin x$$

Multiplying both sides of (i) by I.F. $= x \sin x$, we get

$$x \sin x \frac{dy}{dx} + (\sin x + x \cos x)\, y = x \sin x$$

Integrating with respect to x, we get

$$y\,(x \sin x) = \int \underset{\text{I}}{x} \underset{\text{II}}{\sin x}\, dx + C$$

$$\Rightarrow \quad xy \sin x = -x \cos x + \sin x + C$$

(ii) We have,

$$(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{1 + x^2} y = \frac{\cot x}{1 + x^2} \qquad \text{...(i)}$$

This is a linear differential equation with $P = \frac{2x}{1 + x^2}$ and $Q = \frac{\cot x}{1 + x^2}$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log (1 + x^2)} = 1 + x^2$$

Multiplying both sides of (i) by I.F. $= 1 + x^2$, we get

$$(1 + x^2) \frac{dy}{dx} + 2\,xy = \cot x$$

Integrating both sides with respect to x, we get

$$y\,(1 + x^2) = \int \cot x\, dx + C$$

$$\Rightarrow \quad y\,(1 + x^2) = \log |\sin x| + C$$

22.8.7 LINEAR DIFFERENTIAL EQUATIONS OF THE FORM

Sometimes a linear differential equation can be put in the form

$$\frac{dx}{dy} + Rx = S$$

where R and S are functions of y or constants.

Note that y is independent variable and x is a dependent variable.

The following algorithm is used to solve these types of equations.

ALGORITHM

<u>STEP I</u> *Write the differential equation in the form* $\frac{dx}{dy} + Rx = S$ *and obtain* R *and* S.

<u>STEP II</u> *Find I.F. by using* I.F. $= e^{\int R\,dy}$

<u>STEP III</u> *Multiply both sides of the differential equation in Step I by I.F.*

<u>STEP IV</u> *Integrate both sides of the equation obtained is Step III w.r.t.* y *to obtain the solution given by*

$$x\,(\text{I.F.}) = \int S\,(\text{I.F.})\, dy + C, \text{ where } C \text{ is the constant of integration.}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve:* $y\,dx - (x + 2y^2)\,dy = 0$ **[NCERT]**

SOLUTION The given differential equation is

$$y\,dx - (x + 2y^2)\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x + 2y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x + 2y^2}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 2y \qquad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S, \text{ where } R = -\frac{1}{y} \text{ and } S = 2y$$

$$\therefore \quad \text{I.F.} = e^{\int R\,dy} = e^{\int \frac{-1}{y}\,dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1}$$

Multiplying both sides of (i) by I.F. y^{-1}, we obtain

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = 2$$

Integrating both sides w.r.t. y, we get

$$x \cdot \frac{1}{y} = \int 2\,dy + C \qquad [\text{Using}: x\,(\text{I.F.}) = \int S\,(\text{I.F.})\,dy + C\,]$$

$$\Rightarrow \quad \frac{x}{y} = 2y + C\text{, which is the required solution.}$$

EXAMPLE 2 *Solve:* $y\,dx + (x - y^3)\,dy = 0$

SOLUTION The given differential equation is

$$y\,dx + (x - y^3)\,dy = 0$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{y} = y^2 \qquad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S, \text{ where } R = \frac{1}{y} \text{ and } S = y^2$$

$$\therefore \quad \text{I.F.} = e^{\int R\,dy} = e^{\int \frac{1}{y}\,dy} = e^{\log y} = y$$

Multiplying both sides of (i) by I.F. $= y$, we obtain

$$y\frac{dx}{dy} + x = y^3$$

Integrating both sides w.r.t. y, we get

$$xy = \int y^3\,dy + C \qquad [\text{Using}: x\,(\text{I.F.}) = \int S\,(\text{I.F.})\,dy + C\,]$$

$\Rightarrow \quad xy = \frac{y^4}{4} + C$, which is the required solution.

EXAMPLE 3 *Solve :* $(x + 2y^3)\,dy = y\,dx$.

SOLUTION The given differential equation can be written as

$$\frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 2y^2, y \neq 0 \quad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S, \text{ where } R = \frac{-1}{y} \text{ and } S = 2y^2$$

We have, $\text{I.F.} = e^{\int R\,dy} = e^{\int \frac{-1}{y}\,dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$

Multiplying both sides of (i) by I.F. $= \frac{1}{y}$, we get

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = 2y$$

Integrating both sides w.r.t. y, we get

$$x\left(\frac{1}{y}\right) = \int 2y\,dy + C \qquad \left[\text{Using}: x\,(\text{I.F.}) = \int S\,(\text{I.F.})\,dy + C\right]$$

$$\Rightarrow \quad \frac{x}{y} = y^2 + C$$

$$\Rightarrow \quad x = y^3 + cy$$

Hence, $x = y^3 + cy$, where $y \neq 0$ gives the required solution.

EXAMPLE 4 *Solve:* $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1, x \neq 0.$ **[NCERT]**

SOLUTION We have,

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \quad \text{...(i)}$$

This is a linear differential equation with $P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{\sqrt{x}}\,dx} = e^{2\sqrt{x}}$$

Multiplying both sides of (i) by I.F. $= e^{2\sqrt{x}}$, we get

$$\frac{dy}{dx}e^{2\sqrt{x}} + \frac{y\,e^{2\sqrt{x}}}{\sqrt{x}} = \frac{1}{\sqrt{x}}$$

Integrating both sides with respect to x, we get

$$ye^{2\sqrt{x}} = \int e^{2\sqrt{x}} \cdot \frac{e^{-2\sqrt{x}}}{\sqrt{x}}\, dx + C \qquad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad y\,e^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\, dx + C$$

$$\Rightarrow \quad y\,e^{2\sqrt{x}} = 2\sqrt{x} + C$$

$\Rightarrow \quad y = (2\sqrt{x} + C)e^{-2\sqrt{x}}$, which gives the required solution.

EXAMPLE 5 *Solve each of the following initial value problems:*

(i) $(x - \sin y)\, dy + (\tan y)\, dx = 0,\ y(0) = 0$

(ii) $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy,\ y(0) = 0$ **[NCERT]**

SOLUTION We have,

(i) $(x - \sin y)\, dy + (\tan y)\, dx = 0$

$$\Rightarrow \quad \frac{dx}{dy} = -\left(\frac{x - \sin y}{\tan y}\right)$$

$$\Rightarrow \quad \frac{dx}{dy} + (\cot y)\, x = \cos y \qquad \text{...(i)}$$

This is a linear differential equation with $R = \cot y$ and $S = \cos y$.

$$\therefore \quad \text{I.F.} = e^{\int \cot y} = e^{\log \sin y} = \sin y$$

Multiplying both sides of (i) by $\sin y$, we get

$$\frac{dx}{dy} \sin y + x \cos y = \cos y \sin y$$

Integrating both sides with respect to y, we get

$$x \sin y = \int \cos y \sin y\, dy + C \qquad [\text{Using} : x\,(\text{I.F.}) = \int S\,(\text{I.F.})\, dy + C]$$

$$\Rightarrow \quad x \sin y = \frac{1}{2}\int (\sin 2y)\, dy + C$$

$$\Rightarrow \quad x \sin y = -\frac{1}{4}\cos 2y + C \qquad \text{...(ii)}$$

It is given that $y(0) = 0$ i.e. $y = 0$ when $x = 0$.

Putting $x = 0, y = 0$ in (ii), we get

$$0 = -\frac{1}{4} + C \Rightarrow C = \frac{1}{4}$$

Putting $C = \frac{1}{4}$ in (ii), we get

$$x \sin y = -\frac{1}{4}\cos 2y + \frac{1}{4}$$

$$\Rightarrow \quad x \sin y = \frac{1}{2}\sin^2 y$$

$$\Rightarrow \quad 2x = \sin y$$

$\Rightarrow \quad y = \sin^{-1} 2x$

Clearly, it is defined for

$$-1 \le 2x \le 1 \Rightarrow -\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Hence, $y = \sin^{-1} 2x,\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ gives the required solution.

(ii) $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2} \qquad ...(i)$$

This is a linear differential equation with $R = \frac{1}{1 + y^2}$ and $S = \frac{\tan^{-1} y}{1 + y^2}$.

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{1 + y^2} dy} = e^{\tan^{-1} y}$$

Multiplying both sides of (i) by I.F. $= e^{\tan^{-1} y}$, we get

$$\frac{dx}{dy} e^{\tan^{-1} y} + \frac{x}{1 + y^2} e^{\tan^{-1} y} = \frac{\tan^{-1} y}{1 + y^2} e^{\tan^{-1} y}$$

Integrating both sides with respect to y, we get

$$x e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1 + y^2} e^{\tan^{-1} y} dy + C \qquad \left[\text{Using} : x\,(\text{I.F.}) = \int S\,(\text{I.F.})\, dy + C\right]$$

$$\Rightarrow \quad x e^{\tan^{-1} y} = \int t\, e^t\, dt + C, \text{ where } t = \tan^{-1} y$$

$$\Rightarrow \quad x e^{\tan^{-1} y} = e^t (t - 1) + C$$

$$\Rightarrow \quad x e^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y - 1) + C \qquad ...(ii)$$

It is given that $y(0) = 0$ i.e. $y = 0$ when $x = 0$

Putting $x = 0, y = 0$ in (ii), we get

$$0 = e^0 (0 - 1) + C \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we get

$$x e^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y - 1) + 1$$

$$\Rightarrow \quad (x - \tan^{-1} y + 1)\, e^{\tan^{-1} y} = 1$$

This gives the required solution.

EXAMPLE 6 *Solve each of the following initial value problems:*

(i) $y e^y\, dx = (y^3 + 2x e^y)\, dy,\ y(0) = 1$

(ii) $\sqrt{1 - y^2}\, dx = (\sin^{-1} y - x) dy,\ y(0) = 0$

SOLUTION We have,

(i) $$ye^y\,dx = (y^3 + 2x\,e^y)\,dy$$

$$\Rightarrow \quad \frac{dx}{dy} = y^2 e^{-y} + \frac{2x}{y}, \text{ if } y \neq 0$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y} \qquad \text{...(i)}$$

This is a linear differential equation with $R = -2/y$ and $S = y^2 e^{-y}$

$$\therefore \quad \text{I.F.} = e^{\int -\frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$$

Multiplying both sides of (i) by $\frac{1}{y^2}$, we get

$$\frac{1}{y^2}\frac{dx}{dy} - \frac{2}{y^3}x = e^{-y} \qquad \text{...(ii)}$$

Integrating both sides of (ii) with respect to y, we get

$$x\left(\frac{1}{y^2}\right) = \int e^{-y}\,dy + C \qquad \left[\text{Using} : x\,(\text{I.F.}) = \int S\,(\text{I.F.})\,dy + C\right]$$

$$\Rightarrow \quad \frac{x}{y^2} = -e^{-y} + C \qquad \text{...(iii)}$$

It is given that $y(0) = 1$ i.e. $y = 1$ when $x = 0$.

Putting $x = 0$, $y = 1$ in (iii), we get

$$0 = -e^{-1} + C \Rightarrow C = \frac{1}{e}$$

Putting $C = \frac{1}{e}$ in (ii), we get

$$\frac{x}{y^2} = -e^{-y} + \frac{1}{e} \Rightarrow x = y^2(e^{-1} - e^{-y})$$

Hence, $x = y^2(e^{-1} - e^{-y})$, $y \neq 0$ gives the required solution.

(ii) $$\sqrt{1-y^2}\,dx = (\sin^{-1} y - x)\,dy$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin^{-1} y}{\sqrt{1-y^2}} - \frac{x}{\sqrt{1-y^2}}, \text{ if } 1 - y^2 \neq 0 \text{ i.e. } y \neq \pm 1$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{\sqrt{1-y^2}}x = \frac{\sin^{-1} y}{\sqrt{1-y^2}} \qquad \text{...(i)}$$

This is a linear differential equation with $R = \frac{1}{\sqrt{1-y^2}}$ and $S = \frac{\sin^{-1} y}{\sqrt{1-y^2}}$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{\sqrt{1-y^2}}dy} = e^{\sin^{-1} y}$$

Multiplying both sides of (i) by I.F. $= (e^{\sin^{-1} y})$, we get

$$e^{\sin^{-1} y}\frac{dx}{dy}+\frac{x}{\sqrt{1-y^2}}e^{\sin^{-1} y} = e^{\sin^{-1} y}\cdot\frac{\sin^{-1} y}{\sqrt{1-y^2}}$$

Integrating both sides with respect to y, we get

$$x\, e^{\sin^{-1} y} = \int e^{\sin^{-1} y}\frac{\sin^{-1} y}{\sqrt{1-y^2}}\,dy + C \qquad \left[\text{Using : } x\,(\text{I.F.}) = \int S\,(\text{I.F.})\,dy + C\right]$$

$\Rightarrow \quad x e^{\sin^{-1} y} = \int t e^t\,dt + C$, where $t = \sin^{-1} y$

$$\Rightarrow \quad x e^{\sin^{-1} y} = e^t(t-1)+C$$

$$\Rightarrow \quad x\, e^{\sin^{-1} y} = e^{\sin^{-1} y}(\sin^{-1} y - 1)+C \qquad \text{...(ii)}$$

It is given that $y(0) = 0$ *i.e.* $y = 0$ when $x = 0$. Putting $x = 0$, $y = 0$ in (ii), we get

$$0 = e^0(0-1)+C \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we get

$$x e^{\sin^{-1} y} = e^{\sin^{-1} y}(\sin^{-1} y - 1)+1$$

$$\Rightarrow \quad e^{\sin^{-1} y}(x - \sin^{-1} y + 1) = 1.$$

$$\Rightarrow \quad x - \sin^{-1} y + 1 = e^{\sin^{-1} y},\ y \in (-1, 1)$$

This gives the required solution.

EXERCISE 22.10

Solve the following differential equations :

1. $\frac{dy}{dx}+2y = e^{3x}$

2. $4\frac{dy}{dx}+8y = 5\,e^{-3x}$ [CBSE 2007]

3. $\frac{dy}{dx}+2y = 6\,e^x$ [CBSE 2007C]

4. $\frac{dy}{dx}+y = e^{-2x}$

5. $x\frac{dy}{dx} = x+y$

6. $\frac{dy}{dx}+2y = 4x$

7. $x\frac{dy}{dx}+y = x\,e^x$, $x>0$

8. $\frac{dy}{dx}+\frac{4x}{x^2+1}y+\frac{1}{(x^2+1)^2} = 0$ [CBSE 2005]

9. $x\frac{dy}{dx}+y = x\log x$

10. $x\frac{dy}{dx}-y = (x-1)\,e^x$

11. $\frac{dy}{dx}+\frac{y}{x} = x^3$

12. $\frac{dy}{dx}+y = \sin x$ [NCERT]

13. $\frac{dy}{dx}+y = \cos x$

14. $\frac{dy}{dx}+2y = \sin x$ [NCERT]

15. $\frac{dy}{dx} = y\tan x - 2\sin x$

16. $(1+x^2)\frac{dy}{dx}+y = \tan^{-1} x$ [CBSE 2009]

17. $\frac{dy}{dx}+y\tan x = \cos x$

18. $\frac{dy}{dx}+y\cot x = x^2\cot x + 2x$ [NCERT, CBSE 2005]

19. $\frac{dy}{dx} + y \tan x = x^2 \cos^2 x$

20. $(1 + x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$ [CBSE 2002]

21. $x\, dy = (2y + 2x^4 + x^2)\, dx$

22. $(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$

23. $y^2 \frac{dx}{dy} + x - \frac{1}{y} = 0$

24. $(2x - 10y^3) \frac{dy}{dx} + y = 0$

25. $(x + \tan y)\, dy = \sin 2y\, dx$

26. $dx + x dy = e^{-y} \sec^2 y\, dy$

27. $\frac{dy}{dx} = y \tan x - 2 \sin x$

28. $\frac{dy}{dx} + y \cos x = \sin x \cos x$

29. $(1 + x^2) \frac{dy}{dx} - 2xy = (x^2 + 2)(x^2 + 1)$ [CBSE 2005]

30. $(\sin x) \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$

31. $(x^2 - 1) \frac{dy}{dx} + 2(x + 2)y = 2(x + 1)$

32. $x \frac{dy}{dx} + 2y = x \cos x$

33. $\frac{dy}{dx} - y = xe^x$ [CBSE 2002]

34. $\frac{dy}{dx} + 2y = xe^{4x}$ [CBSE 2002C]

35. Solve the differential equation $(x + 2y^2) \frac{dy}{dx} = y$, give that when $x = 2, y = 1$ [CBSE 2000]

36. Find one-parameter families of solution curves of the following differential equations: (or Solve the following differential equations)

(i) $\frac{dy}{dx} + 3y = e^{mx}$, m is a given real number

$\frac{dy}{dx} = \frac{y}{x+2y^2}$

(ii) $\frac{dy}{dx} - y = \cos 2x$

(iii) $x \frac{dy}{dx} - y = (x + 1) e^{-x}$

(iv) $x \frac{dy}{dx} + y = x^4$

(v) $(x \log x) \frac{dy}{dx} + y = \log x$

(vi) $\frac{dy}{dx} - \frac{2xy}{1 + x^2} = x^2 + 2$

(vii) $\frac{dy}{dx} + y \cos x = e^{\sin x} \cos x$

(viii) $(x + y) \frac{dy}{dx} = 1$

(ix) $\frac{dy}{dx} \cos^2 x = \tan x - y$

(x) $e^{-y} \sec^2 y\, dy = dx + x\, dy$

(xi) $x \log x \frac{dy}{dx} + y = 2 \log x$ [CBSE 2009]

(xii) $x \frac{dy}{dx} + 2y = x^2 \log x$

$\frac{y}{x} = V$

$y = Vx$

37. Solve each of the following initial value problems:

(i) $y' + y = e^x, y(0) = \frac{1}{2}$

(ii) $x \frac{dy}{dx} - y = \log x, y(1) = 0$

(iii) $\frac{dy}{dx} + 2y = e^{-2x} \sin x, y(0) = 0$

(iv) $x \frac{dy}{dx} - y = (x + 1) e^{-x}, y(1) = 0$

(v) $(1+y^2)\,dx+(x-e^{-\tan^{-1}y})\,dy = 0$, $y(0) = 0$

(vi) $\frac{dy}{dx}+y\tan x = 2x+x^2\tan x$, $y(0) = 1$

(vii) $x\frac{dy}{dx}+y = x\cos x+\sin x$, $y\left(\frac{\pi}{2}\right) = 1$

(viii) $\frac{dy}{dx}+y\cot x = 4x\operatorname{cosec} x$, $y\left(\frac{\pi}{2}\right) = 0$ [NCERT]

(ix) $\frac{dy}{dx}+2y\tan x = \sin x$; $y = 0$ when $x=\frac{\pi}{3}$ [NCERT]

(x) $\frac{dy}{dx}-3y\cot x = \sin 2x$; $y = 2$ when $x = \frac{\pi}{2}$ [NCERT]

38. Find the genreal solution of the differential equation $x\frac{dy}{dx}+2y = x^2\ (x\neq 0)$ [NCERT]

39. Find the general solution of the differential equation $\frac{dy}{dx}-y=\cos x$ [NCERT]

ANSWERS

1. $y=\frac{1}{5}e^{3x}+Ce^{-2x}$
2. $y = -\frac{5}{4}e^{-3x}+Ce^{-2x}$
3. $ye^{2x}=2e^{3x}+C$
4. $y = -e^{-2x}+Ce^{-x}$
5. $\frac{y}{x}=\log|x|+C, x\neq 0$
6. $y = (2x-1)+Ce^{-2x}$
7. $y = \left(\frac{x-1}{x}\right)e^x+\frac{C}{x}, x>0$
8. $y(x^2+1)^2=-x+C$
9. $4xy=2x^2\log|x|-x^2+C, x\neq 0$
10. $y=e^x+Cx$
11. $5xy=x^5+C$
12. $y=Ce^{-x}+\frac{1}{2}(\sin x-\cos x)$
13. $y=Ce^{-x}+\frac{1}{2}(\cos x+\sin x)$
14. $y=Ce^{-2x}+\frac{1}{5}(2\sin x-\cos x)$
15. $2y\cos x=\cos 2x+C$
16. $y=\tan^{-1}x-1+Ce^{-\tan^{-1}x}$
17. $y\sec x=x+C$
18. $y\sin x=x^2\sin x+C$
19. $y\sec x=x^2\sin x+2x\cos x-2\sin x+C$
20. $2ye^{\tan^{-1}x}=e^{2\tan^{-1}x}+C$
21. $y=x^4+x^2\log x+Cx^2$
22. $2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+C$
23. $y=\left(\frac{y+1}{y}\right)+Ce^{1/y}$
24. $x=2y^3+Cy^{-2}$
25. $x=\tan y+c\sqrt{\tan y}$
26. $xe^y=\tan y+C$
27. $y=\sec x(-\sin^2 x+C)$
28. $y=\sin x-1+Ce^{-\sin x}$
29. $y=(x+\tan^{-1}x+C)(x^2+1)$
30. $y\sin x=\frac{2}{3}\sin^3 x+C$
31. $y=\frac{2(x+1)}{(x-1)^3}\left\{x^2-6x+8\log(x+1)\right\}+C$

32. $x^2 y = x^2 \sin x + 2x \cos x - 2 \sin x + C$

33. $y = \left(\frac{x^2}{2} + c\right) e^x$

34. $y = \frac{x}{6} e^{4x} - \frac{1}{36} e^{4x} + ce^{-2x}$

35. $x = 2y^2$

36. (i) $y = \frac{e^{mx}}{m+3} + Ce^{-3x}$, if $m + 3 \neq 0$; $y = (x + C) e^{-3x}$, if $m + 3 = 0$

(ii) $y = \frac{1}{5}(-\cos 2x + 2 \sin 2x) + C e^x$ (iii) $y = -e^{-x} + Cx, x \in R$

(iv) $y = \frac{1}{5} x^4 + \frac{C}{x}, x \neq 0$ (v) $y = \frac{1}{2} \log x + \frac{C}{\log x}, x > 0, x \neq 1$

(vi) $y = (x^2 + 1)\{(x + \tan^{-1} x) + C\}, x \in R$

(vii) $y = \frac{1}{2} e^{\sin x} + C e^{-\sin x}, x \in R$ (viii) $x + y - 1 = C e^y$

(ix) $y = \tan x - 1 + Ce^{-\tan x}, \cos x \neq 0$ (x) $x = e^{-y}(\tan y + C), x \in R$

(xi) $y = \log x + \frac{C}{\log x}$ (xii) $y = \frac{x^2}{16}(4 \log |x| - 1) + \frac{C}{x^2}$

37. (i) $y = \frac{1}{2} e^x, x \in R$ (ii) $y = x - 1 - \log x, x > 0$

(iii) $y e^{2x} = 1 - \cos x, x \in R$ (iv) $y = xe^{-1} - e^{-x}, x \in R$

(v) $xe^{\tan^{-1} y} = \tan^{-1} y$ (vi) $y = x^2 + \cos x, x \in R$

(vii) $y = \sin x, x \in R$ (viii) $y \sin x = 2x^2 - \frac{\pi^2}{2}, x \neq n\pi, n \in Z$

(ix) $y = \cos x - 2 \cos^2 x$ (x) $y = 4 \sin^3 x - 2 \sin^2 x$

38. $y = \frac{x^2}{4} + Cx^{-2}$

39. $y = \left(\frac{\sin x - \cos x}{2}\right) + Ce^{-x}$

22.9 APPLICATIONS OF DIFFERENTIAL EQUATIONS

In this section, we shall discuss some problems on the applications of differential equations in science and engineering. We shall also discuss problems on applications to other disciplines.

Type I APPLICATIONS ON GROWTH AND DECAY

EXMAPLE 1 *The surface area of a balloon being inflated changes at a constant rate. If initially, its radius is 3 units and after 2 seconds, it is 5 units, find the radius after t seconds.*

SOLUTION Let r be the radius and S be the surface area of the balloon at any time t. Then,

$$S = 4\pi r^2$$

$$\Rightarrow \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt} \qquad \text{...(i)}$$

It is given that $\frac{dS}{dt} = \text{const} = k$ (say). Putting $\frac{dS}{dt} = k$ in (i), we get

$$k = 8\pi r \frac{dr}{dt}$$

$\Rightarrow \quad 8\pi r\, dr = k\, dt$ [By separating the variables]

Integrating both sides, we get

$$4\pi r^2 = kt + C \quad ...(ii)$$

We are given that at $t = 0, r = 3$ and at $t = 2, r = 5$

$\therefore \quad 36\pi = k(0) + C$ and $100\pi = 2k + C \Rightarrow C = 36\pi$ and $k = 32\pi$

Substituting the values of C and k in (ii), we obtain

$$4\pi r^2 = 32\pi t + 36\pi \Rightarrow r^2 = 8t + 9 \Rightarrow r = \sqrt{8t+9}$$

EXAMPLE 2 *A population grows at the rate of 8% per year. How long does it take for the population to double ? Use differential equation for it.*

SOLUTION Let P_0 be the initial population and let the population after t years be P. Then,

$$\frac{dP}{dt} = \frac{8P}{100} \quad \text{(Given)}$$

$$\Rightarrow \quad \frac{dP}{dt} = \frac{2P}{25}$$

$$\Rightarrow \quad \frac{dP}{P} = \frac{2}{25}dt \quad \text{[By separating the variables]}$$

$$\Rightarrow \quad \int \frac{1}{P}dP = \frac{2}{25}\int 1.dt \quad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log P = \frac{2}{25}t + C \quad ...(i)$$

At $t = 0$, we have $P = P_0$

$$\therefore \quad \log P_0 = \frac{2 \times 0}{25} + C \quad [\text{Putting } t = 0 \text{ and } P = P_0 \text{ in (i)}]$$

$$\Rightarrow \quad C = \log P_0$$

Substituting $C = \log P_0$ in (i), we get

$$\log P = \frac{2}{25}t + \log P_0$$

$$\Rightarrow \quad \log \frac{P}{P_0} = \frac{2}{25}t$$

$$\Rightarrow \quad t = \frac{25}{2}\log\left(\frac{P}{P_0}\right)$$

When $P = 2P_0$, we have

$$t = \frac{25}{2}\log\left(\frac{2P_0}{P_0}\right) = \frac{25}{2}\log 2$$

Thus, the population is doubled in $\frac{25}{2}\log 2$ years.

EXAMPLE 3 *Suppose the growth of a population is proportional to the number present. If the population of a colony doubles in 25 days, in how many days will the population become triple?*

SOLUTION Let P_0 be the initial population and P be the population at any time t. Then,

$$\frac{dP}{dt} \propto P \qquad \text{[Given]}$$

$$\Rightarrow \quad \frac{dP}{dt} = \lambda P, \ \lambda \text{ is a constant}$$

$$\Rightarrow \quad \frac{dP}{P} = \lambda\, dt$$

$$\Rightarrow \quad \int \frac{1}{P}\, dP = \lambda \int dt$$

$$\Rightarrow \quad \log P = \lambda t + C \qquad \text{...(i)}$$

At $t = 0$, we have $P = P_0$

$$\therefore \quad \log P_0 = 0 + C \Rightarrow C = \log P_0$$

Putting $C = \log P_0$ in (i), we get

$$\log P = \lambda t + \log P_0$$

$$\Rightarrow \quad \log\left(\frac{P}{P_0}\right) = \lambda t \qquad \text{...(ii)}$$

It is given that $P = 2P_0$ when $t = 25$ days.

Putting $t = 25$ and $P = 2P_0$ in (ii), we get

$$\log 2 = 25\lambda \Rightarrow \lambda = \frac{1}{25}\log 2$$

Putting $\lambda = \frac{1}{25}\log 2$ in (ii), we get

$$\log\left(\frac{P}{P_0}\right) = \left(\frac{1}{25}\log 2\right)t \qquad \text{...(iii)}$$

Suppose the population is tripled in t_1 days. i.e. $P = 3\,P_0$ when $t = t_1$.

Putting $P = 3P_0$ and $t = t_1$ in (iii), we get

$$\log 3 = \left(\frac{1}{25}\log 2\right)t_1 \Rightarrow t_1 = 25\left(\frac{\log 3}{\log 2}\right)\text{days}$$

Hence, the population is tripled in $25\left(\frac{\log 3}{\log 2}\right)$ days.

EXAMPLE 4 *It is known that, if the interest is compounded continuously, the principal changes at the rate equal to the product of the rate of bank interest per annum and the principal.*

(i) *If the interest is compounded continuously at 5% per annum, in how many years will Rs.100 double itself?*

(ii) *At what interest rate will Rs. 100 double itself in 10 years?* ($\log_e 2 = 0.6931$)

(iii) *How much will Rs. 1000 be worth at 5% interest after 10 years?* ($e^{0.5} = 1.648$).

SOLUTION If P denotes the principal at any time t and the rate of interest be r % per annum compounded continuously, then according to the law given in the problem, we have

$$\frac{dP}{dt} = \frac{Pr}{100}$$

$$\Rightarrow \quad \frac{dP}{P} = \frac{r}{100}\,dt$$

$$\Rightarrow \quad \int \frac{1}{P}\,dP = \frac{r}{100}\int dt$$

$$\Rightarrow \quad \log P = \frac{rt}{100} + C \qquad \text{...(i)}$$

Let P_0 be the initial principal i.e. at $t = 0$, $P = P_0$

Putting $P = P_0$ in (i), we get

$$\log P_0 = C$$

Putting $C = \log P_0$ in (i), we get

$$\log P = \frac{rt}{100} + \log P_0$$

$$\Rightarrow \quad \log\left(\frac{P}{P_0}\right) = \frac{rt}{100} \qquad \text{...(ii)}$$

(i) In this case, we have

$$r = 5,\ P_0 = \text{Rs.}100 \text{ and } P = \text{Rs.}200 = 2P_0$$

Substituting these values in (ii), we have

$$\log 2 = \frac{5}{100}t \Rightarrow t = 20\log_e 2 = 20 \times 0.6931 \text{ years} = 13.862 \text{ years.}$$

(ii) In this case, we have

$$P_0 = \text{Rs.}100,\ P = \text{Rs.}200 = 2P_0 \text{ and } t = 10 \text{ years.}$$

Substituting these values in (ii), we get

$$\log 2 = \frac{10r}{100} \Rightarrow r = 10\log 2 = 10 \times 0.6931 = 6.931$$

Hence, $r = 6.931\%$ per annum.

(iii) In this case, we have

$$P_0 = \text{Rs.}1000,\ r = 5 \text{ and } t = 10$$

Substituting these values in (ii), we get

$$\log\left(\frac{P}{1000}\right) = \frac{5 \times 10}{100} = \frac{1}{2} = 0.5$$

$$\Rightarrow \quad \frac{P}{1000} = e^{0.5}$$

$$\Rightarrow \quad P = 1000 \times 1.648 = 1648$$

Hence, $P = \text{Rs.}1648$.

EXAMPLE 5 *It is given that the rate at which some bacteria multiply is proportional to the instantaneous number present. If the original number of bacteria doubles in two hours, in how many hours will it be five times?*

SOLUTION Let the original count of bacteria be N_0 and at any time t the count of bacteria be N. Then,

$$\frac{dN}{dt} \propto N \qquad \text{[Given]}$$

$\Rightarrow \quad \frac{dN}{dt} = \lambda N$, λ is a constant

$$\Rightarrow \quad \frac{dN}{N} = \lambda\, dt$$

$$\Rightarrow \quad \int \frac{1}{N} dN = \lambda \int dt$$

$$\Rightarrow \quad \log N = \lambda t + C \qquad \text{...(i)}$$

We have, $N = N_0$ at $t = 0$.

$$\therefore \quad \log N_0 = 0 + C \Rightarrow C = \log N_0$$

Putting $C = \log N_0$ in (i), we get

$$\log N = \lambda t + \log N_0$$

$$\Rightarrow \quad \log\left(\frac{N}{N_0}\right) = \lambda t \qquad \text{...(ii)}$$

When $t = 2$ hours, $N = 2N_0$ [Given]

$$\therefore \quad \log\left(\frac{2N_0}{N_0}\right) = 2\lambda \qquad \text{[Putting } N = 2N_0 \text{ and } t = 2 \text{ in (ii)]}$$

$$\Rightarrow \quad \lambda = \frac{1}{2}\log 2$$

Putting $\lambda = \frac{1}{2}\log 2$ in (ii), we get

$$\log\left(\frac{N}{N_0}\right) = \left(\frac{1}{2}\log 2\right)t$$

$$\Rightarrow \quad t = \frac{2}{\log 2}\log\left(\frac{N}{N_0}\right)$$

Suppose the count of bacteria becomes 5 times i.e. $5N_0$ in t_1 hours. Then,

$$t_1 = \frac{2}{\log 2}\log\left(\frac{5N_0}{N_0}\right)$$

$$\Rightarrow \quad t_1 = \frac{2}{\log 2}(\log 5) = \frac{2\log 5}{\log 2} \text{ hours.}$$

EXAMPLE 6 *It is given that radium decomposes at a rate proportional to the amount present. If p % of the original amount of radium disappears in l years, what percentage of it will remain after $2l$ years?*

SOLUTION Let A_0 be the original amount of radium and A be the amount of radium at any time t. Then,

$$\frac{dA}{dt} \propto A \qquad \text{[Given]}$$

$$\Rightarrow \quad \frac{dA}{dt} = -\lambda A, \lambda \text{ is a positive constant}$$

$$\Rightarrow \quad \frac{dA}{A} = -\lambda\, dt$$

$$\Rightarrow \quad \log A = -\lambda t + C \qquad ...(i)$$

At $t = 0$, we have $A = A_0$

$$\therefore \quad \log A_0 = 0 + C \Rightarrow C = \log A_0$$

Putting $C = \log A_0$ in (i), we get

$$\log A = -\lambda t + \log A_0$$

$$\Rightarrow \quad \log\left(\frac{A}{A_0}\right) = -\lambda t \qquad ...(ii)$$

It is given that p % of the original amount of radium disintegrates in l years. This means that

$$A = \left(A_0 - \frac{A_0 p}{100}\right) \text{ when } t = l \text{ i.e. } \frac{A}{A_0} = 1 - \frac{p}{100} \text{ when } t = l$$

Substituting these values in (ii), we get

$$\log\left(1 - \frac{p}{100}\right) = -\lambda l$$

$$\Rightarrow \quad \lambda = -\frac{1}{l}\log\left(1 - \frac{p}{100}\right)$$

Substituting the value of λ in (ii), we get

$$\log\left(\frac{A}{A_0}\right) = \frac{t}{l}\log\left(1 - \frac{p}{100}\right)$$

When $t = 2l$, we have

$$\log\left(\frac{A}{A_0}\right) = 2\log\left(1 - \frac{p}{100}\right)$$

$$\Rightarrow \quad \frac{A}{A_0} = \left(1 - \frac{p}{100}\right)^2$$

$$\Rightarrow \quad \frac{A}{A_0} \times 100 = \left(1 - \frac{p}{100}\right)^2 \times 100$$

$$\Rightarrow \quad \frac{A}{A_0} \times 100 = \left(10 - \frac{p}{10}\right)^2$$

Hence, required percent $= \left(10 - \frac{p}{10}\right)^2$.

EXAMPLE 7 *A radioactive substance disintegrates at a rate proportional to the amount of substance present. If 50% of the given amount disintegrates in 1600 years. What percentage of the substance disintegrates in 10 years?* $\left(\textit{Take } e^{\frac{-\log 2}{160}} = 0.9957\right)$

SOLUTION Let A denote the amount of the radioactive substance present at any instant t and let A_0 be the initial amount of the substance.

It is given that

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = -\lambda A \qquad \text{...(i)}$$

where λ is the constant of proportionality such that $\lambda > 0$. Here negative sign indicates that A decreases with the increase in t.

Now,

$$\frac{dA}{dt} = -\lambda A$$

$$\Rightarrow \quad \frac{1}{A} dA = -\lambda\, dt$$

$$\Rightarrow \quad \int \frac{1}{A} dA = -\lambda \int 1 \, . \, dt$$

$$\Rightarrow \quad \log A = -\lambda t + C \qquad \text{...(ii)}$$

Initially i.e. at $t = 0$, we have $A = A_0$

$$\therefore \quad \log A_0 = 0 + C \Rightarrow C = \log A_0$$

Putting $C = \log A_0$ in (ii), we get

$$\log A = -\lambda t + \log A_0$$

$$\Rightarrow \quad \log\left(\frac{A}{A_0}\right) = -\lambda t \qquad \text{...(iii)}$$

It is given that $A = \dfrac{A_0}{2}$ at $t = 1600$ years

Putting $A = \dfrac{A_0}{2}$ and $t = 1600$ in (iii), we get

$$\log\left(\frac{1}{2}\right) = -1600\lambda \Rightarrow \lambda = \frac{1}{1600} \log 2$$

Substituting the value of λ in (iii), we get

$$\log\left(\frac{A}{A_0}\right) = -\left(\frac{1}{1600} \log 2\right) t$$

$$\Rightarrow \quad \frac{A}{A_0} = e^{-\frac{\log 2}{1600} t}$$

$$\Rightarrow \quad A = A_0 e^{-\frac{\log 2}{1600} t}$$

At $t = 10$, we have

$$A = A_0(0.9957) \qquad \left[\because e^{-\frac{\log 2}{160}} = 0.9957\right]$$

Amount that disintegrates in 10 years .

$$\Rightarrow \quad A_0 - A = A_0 - 0.9957\,A_0$$

$$\Rightarrow \quad A_0 - A = 0.0043\,A_0$$

$$\Rightarrow \quad \frac{A_0 - A}{A_0} = 0.0043$$

$$\Rightarrow \quad \frac{A_0 - A}{A_0} \times 100 = 0.43$$

Hence, 0.43% of the original amount disintegrates in 10 years.

EXAMPLE 8 *The rate at which radioactive substances decay is known to be proportional to the number of such nuclei that are present at the time in a given sample.*

(i) *In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. Find what percentage of the original radioactive nuclei will remain after 1000 years.*

(ii) *If 100 grams of a radioactive substance is present 1 year after the substance was produced and 75 grams is present 2 years after the substance was produced, how much radioactive substance was produced?*

SOLUTION (i) Let there be N radioactive nuclei in a sample at any time t and let N_0 be the initial number of radioactive nuclei. Then,

$$\frac{dN}{dt} \propto N \qquad \text{[Given]}$$

$$\Rightarrow \quad \frac{dN}{dt} = -\lambda N, \text{ where } \lambda > 0 \text{ is a constant}$$

$$\Rightarrow \quad \frac{dN}{N} = -\lambda\, dt$$

$$\Rightarrow \quad \int \frac{1}{N}\, dN = -\lambda \int dt$$

$$\Rightarrow \quad \log N = -\lambda t + C \qquad \text{...(i)}$$

At $t = 0$, we have $N = N_0$

$$\therefore \quad \log N_0 = 0 + C \Rightarrow C = \log N_0$$

Putting $C = \log N_0$ in (i), we get

$$\log N = -\lambda t + \log N_0$$

$$\Rightarrow \quad \log \frac{N}{N_0} = -\lambda t \qquad \text{...(ii)}$$

It is given that 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years.

$\therefore$ At $t = 100$, $N = N_0 - \dfrac{10\,N_0}{100} = \dfrac{9\,N_0}{10}$

Substituting these values in (ii), we get

$$\log \frac{9}{10} = -100\,\lambda \Rightarrow \lambda = -\frac{1}{100} \log \frac{9}{10}$$

Putting the value of λ in (ii), we get

$$\log \frac{N}{N_0} = \left(\frac{1}{100} \log \frac{9}{10}\right) t \qquad \text{...(iii)}$$

We have to find the value of N at $t = 1000$ years.
Putting $t = 1000$ years in (iii), we get

$$\log \frac{N}{N_0} = 10 \log \left(\frac{9}{10}\right)$$

$$\Rightarrow \quad \log \frac{N}{N_0} = \log \left(\frac{9}{10}\right)^{10}$$

$$\Rightarrow \quad \frac{N}{N_0} = \left(\frac{9}{10}\right)^{10}$$

$$\Rightarrow \quad \frac{N}{N_0} \times 100 = \left(\frac{9}{10}\right)^{10} \times 100 = \frac{9^{10}}{10^8}$$

Hence, $\dfrac{9^{10}}{10^8}$ % of radioactive nuclei will remain after 1000 years.

(ii) Suppose N_0 grams of radioactive substance was produced and at any time t, N grams of substance is present. Then,

$$\frac{dN}{dt} \propto N$$

$$\Rightarrow \quad \frac{dN}{dt} = -\lambda N, \text{ where } \lambda > 0 \text{ is a constant}$$

$$\Rightarrow \quad \frac{dN}{N} = -\lambda\, dt$$

$$\Rightarrow \quad \int \frac{1}{N}\, dN = -\lambda \int dt$$

$$\Rightarrow \quad \log N = -\lambda t + C \qquad \text{...(i)}$$

At $t = 0$, we have $N = N_0$.

$$\therefore \quad \log N_0 = 0 + C \Rightarrow C = \log N_0$$

Putting $C = \log N_0$ in (i), we get

$$\log N = -\lambda t + \log N_0$$

$$\Rightarrow \quad \log \frac{N}{N_0} = -\lambda t \qquad \text{...(ii)}$$

It is given that

$N = 100$ grams at $t = 1$ and $N = 75$ at $t = 2$.

$\therefore \quad \log\left(\frac{100}{N_0}\right) = -\lambda$ and $\log\left(\frac{75}{N_0}\right) = -2\lambda$

$\Rightarrow \quad \log\left(\frac{75}{N_0}\right) = 2\log\left(\frac{100}{N_0}\right) \Rightarrow \log\left(\frac{75}{N_0}\right) = \log\left(\frac{100}{N_0}\right)^2$

$\Rightarrow \quad \frac{75}{N_0} = \left(\frac{100}{N_0}\right)^2$

$\Rightarrow \quad N_0 = \frac{100^2}{75} \Rightarrow N_0 = \frac{400}{3}$ grams.

EXAMPLE 9 *In a college hostel accommodating 1000 students, one of them came in carrying a flue virus, then the hostel was isolated. If the rate at which the virus spreads is assumed to be proportional to the product of the number N of infected students and the number of non-infected students, and if the number of infected students is 50 after 4 years, then show that more than 95% of the students will be infected after 10 days.*

SOLUTION We have,

$$\frac{dN}{dt} \propto N(1000-N)$$

$\Rightarrow \quad \frac{dN}{dt} = \lambda N(1000-N)$, where λ is a constant

$$\Rightarrow \quad \frac{1}{N(1000-N)}\,dN = \lambda\,dt$$

$$\Rightarrow \quad \int \frac{1}{N(1000-N)}\,dN = \lambda \int dt$$

$$\Rightarrow \quad \frac{1}{1000}\int\left(\frac{1}{1000-N}+\frac{1}{N}\right)dN = \int dt$$

$$\Rightarrow \quad \frac{1}{1000}\{\log N - \log(1000-N)\} = \lambda t + C$$

$$\Rightarrow \quad \frac{1}{1000}\log\left(\frac{N}{1000-N}\right) = \lambda t + C \qquad \text{...(i)}$$

At $t=0$, we have $N=1$.

$$\Rightarrow \quad \frac{1}{1000}\log\frac{1}{999} = 0 + C$$

$$\Rightarrow \quad C = \frac{1}{1000}\log\left(\frac{1}{999}\right) = -\frac{\log 999}{1000}$$

Subst ituting the value of C in (i), we get

$$\frac{1}{1000}\log\left(\frac{N}{1000-N}\right) = \lambda t - \frac{\log 999}{1000}$$

$$\Rightarrow \quad \frac{1}{1000}\log\left(\frac{N}{1000-N}\right) + \frac{1}{1000}\log 999 = \lambda t$$

$$\Rightarrow \quad \frac{1}{1000}\log\left(\frac{999\,N}{1000-N}\right) = \lambda t \qquad \text{...(ii)}$$

If $t=4$, then $N=50$ (Given)

Substituting these values in (ii), we get

$$\frac{1}{1000}\log\left(\frac{49950}{950}\right) = 4\lambda$$

$$\Rightarrow \quad \lambda = \frac{1}{4000}\log\left(\frac{4995}{95}\right) = \frac{1}{4000}\log\left(\frac{999}{19}\right)$$

Putting the value of λ in (ii), we get

$$\frac{1}{1000}\log\left(\frac{999N}{1000-N}\right) = \left\{\frac{1}{4000}\log\left(\frac{999}{19}\right)\right\} t$$

$$\Rightarrow \quad 4\log\left(\frac{999N}{1000-N}\right) = \left(\log\frac{999}{19}\right) t$$

When $t = 10$, the value of N is given by

$$4\log\left(\frac{999N}{1000-N}\right) = 10\times\log\left(\frac{999}{19}\right)$$

$$\Rightarrow \quad \log\left(\frac{999N}{1000-N}\right) = \frac{5}{2}\log\left(\frac{999}{19}\right)$$

$$\Rightarrow \quad \log\left(\frac{1000-N}{999N}\right) = -\frac{5}{2}\log\left(\frac{999}{19}\right) = \log\left(\frac{999}{19}\right)^{-5/2}$$

$$\Rightarrow \quad \frac{1000-N}{999N} = \left(\frac{999}{19}\right)^{-5/2}$$

$$\Rightarrow \quad \frac{1000}{999N} - \frac{1}{999} = \left(\frac{999}{19}\right)^{-5/2}$$

$$\Rightarrow \quad \frac{1000}{999N} = \frac{1}{999} + \left(\frac{999}{19}\right)^{-5/2}$$

$$\Rightarrow \quad \frac{1000}{N} = 1 + (999)^{-3/2}\times 19^{5/2}$$

$$\Rightarrow \quad N = \frac{1000}{1+(999)^{-3/2}\times 19^{5/2}} = 952 \text{ approximately}$$

$$\therefore \quad \frac{N}{1000}\times 100 = \frac{N}{10} = 95.2$$

Hence, more than 95% students will be infected after 10 days.

EXAMPLE 10 *Assume that a spherical rain drop evaporates at a rate proportional to its surface area. If its radius originally is 3 mm and 1 hour later has been reduced to 2 mm, find an expression for the radius of the rain drop at any time.*

SOLUTION Let r be the radius, V be the volume and S be the surface area of the rain drop at any time t. Then,

$$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2$$

We are given that

$$\frac{dV}{dt} \propto S$$

$\Rightarrow \quad \frac{dV}{dt} = kS$, k is the constant of proportionality

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k(4\pi r^2)$$

$$\Rightarrow \quad 4\pi r^2 \frac{dr}{dt} = k(4\pi r^2)$$

$$\Rightarrow \quad \frac{dr}{dt} = k$$

$$\Rightarrow \quad dr = k\,dt$$

Integrating both sides, we get

$$\int dr = k\int dt$$

$$\Rightarrow \quad r = kt + C \quad \text{...(i)}$$

We are given that $r = 3$ at $t = 0$ and $r = 2$ at $t = 1$

$$\therefore \quad 3 = k(0) + C \text{ and } 2 = k + C$$

$$\Rightarrow \quad C = 3 \text{ and } k = -1$$

Putting the values of C and k in (*i*), we get $r = 3 - t$, and $0 \le t \le 3$.

Type II ON NEWTON'S LAW OF COOLING

The Newton's law of cooling states that the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.

EXAMPLE 11 *The temperature T of a cooling object drops at a rate proportional to the difference T – S, where S is constant temperature of surrounding medium. If initially T = 150°C, find the temperature of the cooling object at any time t.*

SOLUTION Let T be the temperature of the cooling object at any time t. Then

$$\frac{dT}{dt} = -k(T - S), \text{ where } k > 0 \text{ is a constant.}$$

$$\Rightarrow \quad \frac{dT}{T - S} = -k\,dt$$

Integrating both sides, we get

$$\int \frac{1}{T - S}\,dT = -k\int dt$$

$$\Rightarrow \quad \log|T - S| = -kt + \log C \quad \text{...(i)}$$

It is given that at $t = 0$, $T = 150$°C. Putting $t = 0$ and $T = 150$ in (i), we get

$$\therefore \quad \log|150 - S| = 0 + \log C$$

Putting the value of log C in (*i*), we get

$$\log|T - S| = -kt + \log|150 - S|$$

$$\Rightarrow \quad \log\left|\frac{T - S}{150 - S}\right| = -kt$$

$$\Rightarrow \quad \frac{T - S}{150 - S} = e^{-kt}$$

$\Rightarrow \quad T - S = (150 - S)\,e^{-kt}$, this gives the temperature T at any time t.

EXAMPLE 12 *Water at temperature 100° C cools in 10 minutes to 80° C in a room of temperature 25° C. Find*

(i) *the temperature of water after 20 minutes*

(ii) *the time when the temperature is 40° C.* $\left[\text{Given : } \log_e \frac{11}{15} = -0.3101, \log 5 = 1.6094\right]$

SOLUTION Let T be the temperature of water at any time t. Then, by Newton's law of cooling

$$\frac{dT}{dt} \propto (T-25)$$

$$\Rightarrow \quad \frac{dT}{dt} = -\lambda (T-25), \text{ where } \lambda > 0 \text{ is a constant.}$$

$$\Rightarrow \quad \frac{dT}{T-25} = -\lambda\, dt$$

$$\Rightarrow \quad \int \frac{1}{T-25}\, dT = -\lambda \int dt$$

$$\Rightarrow \quad \log|T-25| = -\lambda t + C \qquad \text{...(i)}$$

At $t = 0$, we have $T = 100°$ C

Substituting these values in (i), we get

$$\log 75 = 0 + C \Rightarrow C = \log 75$$

Putting $C = \log 75$ in (i), we get

$$\log|T-25| = -\lambda t + \log 75$$

$$\Rightarrow \quad \log\left|\frac{T-25}{75}\right| = -\lambda t \qquad \text{...(ii)}$$

It is also given that $T = 80°$ C at $t = 10$.

$$\therefore \quad \log\left|\frac{80-25}{75}\right| = -10\lambda$$

$$\Rightarrow \quad \log\frac{11}{15} = -10\lambda$$

$$\Rightarrow \quad \lambda = -\frac{1}{10}\log\frac{11}{15}$$

Putting the value of λ in (ii), we get

$$\log\left|\frac{T-25}{75}\right| = \left(\frac{1}{10}\log\frac{11}{15}\right)t \qquad \text{...(iii)}$$

(i) When $t = 20$ minutes, we have

$$\log\left|\frac{T-25}{75}\right| = \left(\frac{1}{10}\log\frac{11}{15}\right)\times 20 \qquad [\text{Putting } t = 20 \text{ in (iii)}]$$

$$\Rightarrow \quad \log\left|\frac{T-25}{75}\right| = \log\left(\frac{11}{15}\right)^2$$

$$\Rightarrow \quad \frac{T-25}{75} = \left(\frac{11}{15}\right)^2$$

$$\Rightarrow \quad T-25 = \frac{121}{225} \times 75$$

$$\Rightarrow \quad T = 65.33° \text{ C}$$

So, the temperature of water after 20 minutes is 65.33° C

(ii) Putting $T = 40°$ C in (iii), we get

$$\log\left|\frac{40-25}{75}\right| = \left(\frac{1}{10}\log\frac{11}{15}\right)t$$

$$\Rightarrow \quad \log\frac{1}{5} = \frac{1}{10}\left(\log\frac{11}{15}\right)t$$

$$\Rightarrow \quad t = \frac{10\log\left(\frac{1}{5}\right)}{\log\left(\frac{11}{15}\right)} = \frac{-10\log 5}{\log\frac{11}{15}} = \frac{-10 \times 1.6094}{-0.3101} = 53.46$$

Hence, $t = 53.46$ minutes.

EXAMPLE 13 *A thermometer reading 80° F is taken outside. Five minutes later the thermometer reads 60° F. After another 5 minutes the thermometer reads 50° F. What is the temperature outside?*

SOLUTION Let at any time t the thermometer reading be $T°$ F and the outside temperature be $S°$ F. Then, by Newton's law of cooling

$$\frac{dT}{dt} \propto (T-S)$$

$$\Rightarrow \quad \frac{dT}{dt} = -\lambda(T-S)$$

$$\Rightarrow \quad \frac{dT}{T-S} = -\lambda\, dt$$

$$\Rightarrow \quad \int \frac{1}{T-S}\, dT = -\lambda \int dt$$

$$\Rightarrow \quad \log(T-S) = -\lambda t + C \qquad \text{...(i)}$$

It is given that $T = 80°$ F at $t = 0$

$$\therefore \quad \log(80-S) = 0 + C \Rightarrow C = \log(80-S)$$

Putting the value of C in (i), we get

$$\log(T-S) = -\lambda t + \log(80-S)$$

$$\Rightarrow \quad \log\left(\frac{T-S}{80-S}\right) = -\lambda t \qquad \text{...(ii)}$$

It is given that

$$T = 60° \text{ F at } t = 5$$

and,

$$T = 50° \text{ F at } t = 10$$

Substituting these values in (ii), we get

$$\log\left(\frac{60-S}{80-S}\right) = -5\lambda \text{ and } \log\left(\frac{50-S}{80-S}\right) = -10\lambda$$

$$\Rightarrow \quad 2\log\left(\frac{60-S}{80-S}\right) = \log\left(\frac{50-S}{80-S}\right)$$

$$\Rightarrow \quad \left(\frac{60-S}{80-S}\right)^2 = \left(\frac{50-S}{80-S}\right)$$

$$\Rightarrow \quad (60-S)^2 = (50-S)(80-S)$$

$$\Rightarrow \quad 3600 - 120S + S^2 = 4000 - 130S + S^2$$

$$\Rightarrow \quad 10S = 400$$

$$\Rightarrow \quad S = 40^\circ \text{ F}.$$

Hence, the out side temperature is 40° F.

EXAMPLE 14 *The doctor took the temperature of a dead body at 11.30 PM which was 94.6° F. He took the temperature of the body again after one hour, which was 93.4° F. If the temperature of the room was 70° F, estimate the time of death. Taking normal temperature of human body as 98.6° F.* $\left[\text{Given}: \log\frac{143}{123} = 0.15066, \log\frac{123}{117} = 0.05\right]$

SOLUTION Let T be the temperature of the body at time t. Then, by Newton's law of cooling, we have

$$\frac{dT}{dt} \propto (T-70)$$

$$\Rightarrow \quad \frac{dT}{dt} = -\lambda(T-70), \text{ where } \lambda > 0 \text{ is a constant}$$

$$\Rightarrow \quad \frac{dT}{T-70} = -\lambda\, dt$$

$$\Rightarrow \quad \int \frac{1}{T-70} = -\lambda \int dt$$

$$\Rightarrow \quad \log|T-70| = -\lambda t + C \qquad \text{...(i)}$$

At $t=0$, we have $T = 94.6°$ F and at $t=1$, $T = 93.4°$ F

$$\therefore \quad \log(94.6-70) = 0 + C \text{ and } \log(93.4-70) = -\lambda + C$$

$$\Rightarrow \quad \log 24.6 = C \text{ and } \log 23.4 = -\lambda + C$$

$$\Rightarrow \quad C = \log 24.6 \text{ and } \lambda = \log 24.6 - \log 23.4$$

$$\Rightarrow \quad C = \log 24.6 \text{ and } \lambda = \log\left(\frac{24.6}{23.4}\right) = \log\left(\frac{123}{117}\right)$$

Substituting these values in (i), we get

$$\log|T-70| = -\left(\log\frac{123}{117}\right)t + \log 24.6 \qquad \text{...(ii)}$$

Let t_1 be the time that has elapsed after the death. Then, at $t = t_1$ we have $T = 98.6$

Substituting these values in (ii), we have

$$\log(98.6-70) = -\left(\log\frac{123}{117}\right)t_1 + \log 24.6$$

$$\Rightarrow \quad \log 28.6 = -\left(\log\frac{123}{117}\right)t_1 + \log 24.6$$

$$\Rightarrow \quad \log\left(\frac{28.6}{24.6}\right) = -\left(\log\frac{123}{117}\right)t_1$$

$$\Rightarrow \quad \log\frac{143}{123} = -\left(\log\frac{123}{117}\right)t_1$$

$$\Rightarrow \quad t_1 = -\frac{\log\left(\frac{143}{123}\right)}{\log\frac{123}{117}}$$

$$\Rightarrow \quad t_1 = -\frac{0.15066}{0.05} = -3.0132.$$

Hence, estimated time of death is 11.30 – 3.01 = 8.30 pm. (approx.)

Type III APPLICATIONS ON CO-ORDINATE GEOMETRY

EXAMPLE 15 *The slope of the tangent to the curve at any point is twice the ordinate at that point. The curve passes through the point (4, 3). Determine its equation.*

SOLUTION Let $P(x, y)$ be any point on the curve. Then, slope of the tangent at P is $\frac{dy}{dx}$.

It is given that the slope of the tangent at $P(x, y)$ is twice the ordinate i.e. $2y$.

$$\therefore \quad \frac{dy}{dx} = 2y$$

$$\Rightarrow \quad \frac{dy}{y} = 2\,dx$$

$$\Rightarrow \quad \log y = 2x + \log C$$

$$\Rightarrow \quad y = C e^{2x} \qquad \text{...(i)}$$

Since the curve passes through (4, 3). Therefore, $y = 3$ for $x = 4$.

Putting $x = 4$ and $y = 3$ in (i), we get

$$3 = C e^{8} \Rightarrow C = 3e^{-8}$$

Putting the value of C in (i), we get

$$y = 3e^{2x-8}$$

This is the required equation of the curve.

EXAMPLE 16 *The normal lines to a given curve at each point pass through (2, 0). The curve passes through (2, 3). Formulate the differential equation and hence find out the equation of the curve.*

SOLUTION Let $P(x, y)$ be any point on the curve. The equation of the normal at $P(x, y)$ to the given curve is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) \qquad \text{...(i)}$$

It is given that the normal at each point passes through (2, 0). Therefore, (i) also passes through (2, 0). Putting $Y = 0$ and $X = 2$ in (i), we get

$$0 - y = -\frac{1}{\frac{dy}{dx}}(2 - x)$$

$$\Rightarrow \quad y\frac{dy}{dx} = 2 - x$$

$$\Rightarrow \quad y\,dy = (2 - x)\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = -\frac{(2-x)^2}{2} + C \qquad \text{[On integrating both sides]}$$

$$\Rightarrow \quad y^2 = -(2-x)^2 + 2C \qquad \text{...(ii)}$$

This passes through (2, 3). Therefore,

$$9 = 0 + 2C \Rightarrow C = \frac{9}{2}$$

Putting $C = \frac{9}{2}$ in (ii), we get

$$y^2 = -(2-x)^2 + 9$$

This is the equation of the required curve.

EXAMPLE 17 *The slope of the tangent to a curve at any point is reciprocal of twice the ordinate of that point. The curve passes through (4, 3). Formulate the differential equation and hence find the equation of the curve.*

SOLUTION Let $P(x, y)$ be any point on the curve. Then the slope of the tangent at $P(x, y)$ is $\frac{dy}{dx}$. It is given that the slope of the tangent at P is reciprocal of twice the ordinate of the point P.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2y}$$

$$\Rightarrow \quad 2y\,dy = dx$$

On integrating both sides, we obtain

$$y^2 = x + C \qquad \text{...(i)}$$

Since the curve passes through (4, 3). Therefore, putting $x = 4$ and $y = 3$ in (i), we get

$$9 = 4 + C \Rightarrow C = 5$$

Putting $C = 5$ in (i), we get

$$y^2 = x + 5$$

This is the required equation of the curve.

EXAMPLE 18 *The slope of the tangent at any point on a curve is λ times the slope of the line joining the point of contact to the origin. Formulate the differential equation and hence find the equation of the curve.*

SOLUTION Let $P(x, y)$ be any point on the given curve. Then,

Slope of the tangent at $P = \lambda$ (Slope of the line OP)

$$\Rightarrow \quad \frac{dy}{dx} = \lambda\left(\frac{y-0}{x-0}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\lambda y}{x}$$

This is the required differential equation. Now,

$$\frac{dy}{dx} = \frac{\lambda y}{x}$$

$$\Rightarrow \quad \frac{dy}{y} = \lambda \frac{dx}{x}$$

$$\Rightarrow \quad \log y = \lambda \log x + \log C$$

[On integrating both sides]

$$\Rightarrow \quad y = Cx^{\lambda}$$

This the equation of the required curve.

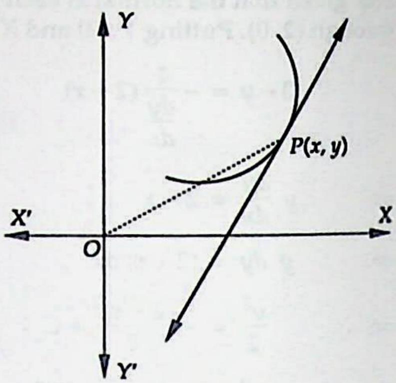


Fig. 22.5

EXAMPLE 19 *The slope of the tangent to a curve at any point (x, y) on it is given by $\frac{y}{x} - \cot\frac{y}{x} . \cos\frac{y}{x}$, $(x > 0, y > 0)$ and the curve passes through the point $(1, \pi/4)$. Find the equation of the curve.*

SOLUTION Let $y = f(x)$ be the given curve. Then, the slope of the tangent at $P(x, y)$ is $\frac{dy}{dx}$. But, the slope of the tangent at P is given as $\frac{y}{x} - \cot\frac{y}{x}\cos\frac{y}{x}$. Therefore,

$$\frac{dy}{dx} = \frac{y}{x} - \cot\frac{y}{x}\cos\frac{y}{x} \quad \text{...(i)}$$

This is a homogeneous differential equation. Let $y = vx$ Then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \cot v \cos v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\cot v \cos v$$

$$\Rightarrow \quad \sec v \tan v \, dv = -\frac{dx}{x}$$

$$\Rightarrow \quad \sec v = -\log|x| + \log C \quad \text{[On integrating]}$$

$$\Rightarrow \quad \sec\left(\frac{y}{x}\right) = -\log|x| + C \quad \text{...(ii)}$$

Since the curve passes through $(1, \pi/4)$. Therefore,

$$\sec \pi/4 = -\log 1 + C \quad \Rightarrow \sqrt{2} = C$$

Putting $C = \sqrt{2}$ in (ii), we get

$$\sec\left(\frac{y}{x}\right) = -\log|x| + \sqrt{2}$$

This is the equation of the required curve.

EXAMPLE 20 *If the tangent at any point P of a curve meets the axis of X in T. Find the curve for which OP = PT, O being the origin.*

SOLUTION Let $y = f(x)$ be the given curve and let $P(x, y)$ be a point on it. The equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{...(i)}$$

This meets X–axis at T. So, the x-coordinate of T is obtained by putting $Y = 0$ in (i).
Putting $Y = 0$ in (i), we get

$$0 - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow \quad X = x - y\frac{dx}{dy}$$

Thus, the coordinates of T are $\left(x - y\frac{dx}{dy}, 0\right)$

$$\therefore \quad PT = \sqrt{\left(x - \left(x - y\frac{dx}{dy}\right)\right)^2 + (y - 0)^2} = \sqrt{y^2\left(\frac{dx}{dy}\right)^2 + y^2} \quad \text{[Given]}$$

Now,

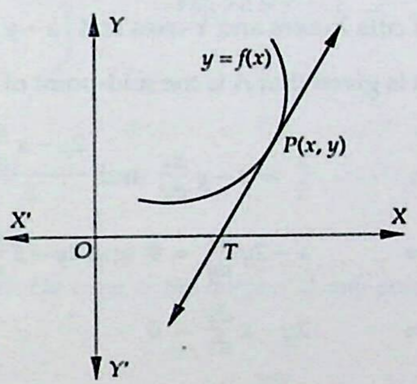


Fig. 22.6

$$PT = OP$$

$$\Rightarrow \quad \sqrt{y^2 + y^2\left(\frac{dx}{dy}\right)^2} = \sqrt{x^2 + y^2}$$

$$\Rightarrow \quad y^2 + y^2\left(\frac{dx}{dy}\right)^2 = x^2 + y^2$$

$$\Rightarrow \quad y^2\left(\frac{dx}{dy}\right)^2 = x^2$$

$$\Rightarrow \quad y\frac{dx}{dy} = \pm x$$

$$\Rightarrow \quad \frac{dx}{x} = \pm\frac{dy}{y}$$

$$\Rightarrow \quad \log x = \pm \log y + \log C \quad \text{[On integrating]}$$

$$\Rightarrow \quad \log x = \log(C\,y) \text{ or, } \log x = \log\left(\frac{C}{y}\right)$$

$$\Rightarrow \quad x = C\,y \text{ or } x = \frac{C}{y}.$$

Hence, the equations of the curve are $x = Cy$ or, $xy = C$.

EXAMPLE 21 *Show that the curve for which the normal at every point passes through a fixed point is a circle.*

SOLUTION Let $P(x, y)$ be any point on the given curve. The equation of the normal to the given curve at (x, y) is

$$Y - y = \frac{1}{\frac{dy}{dx}}(X - x) \quad \text{...(i)}$$

It is given that the normal at every point passes through a fixed point (a, b) (say). Therefore,

$$b - y = -\frac{dx}{dy}(a - x) \qquad \text{[Putting } X = a \text{ and } Y = b \text{ in (i)]}$$

$$\Rightarrow \quad -(y - b)\,dy = (x - a)\,dx$$

$$\Rightarrow \quad (x - a)\,dx + (y - b)\,dy = 0$$

$$\Rightarrow \quad \frac{(x-a)^2}{2} + \frac{(y-b)^2}{2} = C \qquad \text{[On integrating]}$$

$$\Rightarrow \quad (x - a)^2 + (y - b)^2 = 2C$$

$$\Rightarrow \quad (x - a)^2 + (y - b)^2 = r^2, \text{ where } r^2 = 2C$$

Clearly, this equation represents a circle.

EXAMPLE22 *Find the curve in the xy-plane, passing through the point (1, 1), so that the segment of any tangent drawn to the curve between its point of tangency and the y-axis is bisected at the x-axis.*

SOLUTION Let $P(x, y)$ be the point of tangency. The equation of the tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

It cuts X-axes and Y-axes at $A\left(x - y\frac{dx}{dy}, 0\right)$ and $B\left(0, y - x\frac{dy}{dx}\right)$ respectively.

It is given that A is the mid-point of BP.

$$\therefore \quad \frac{x}{2} = x - y\frac{dx}{dy} \text{ and } \frac{2y - x\frac{dy}{dx}}{2} = 0$$

$$\Rightarrow \quad x - 2y\frac{dx}{dy} = 0 \text{ and } 2y - x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y - x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} = 2y$$

$$\Rightarrow \quad \frac{1}{y}\,dy = \frac{2}{x}\,dx$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = 2\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \log y = 2\log x + \log c \qquad [\because x > 0, y > 0]$$

$$\Rightarrow \quad y = Cx^2 \qquad \text{...(i)}$$

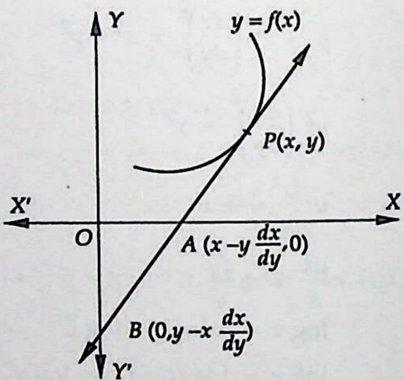


Fig. 22.7

It is given that the curve passes through (1, 1).

$$\therefore \quad 1 = C$$

Putting $C = 1$ in (i), we get

$y = x^2$ as the equation of the required curve.

EXAMPLE 23 *Show that the curve for which the portion of the tangent at any point of it included between the coordinate axes is bisected at the point of contact is a rectangular hyperbola $xy = C$.*

SOLUTION Let $P(x, y)$ be any point on the curve $y = f(x)$ such that the tangent at P cuts the coordinate axes at A and B. The coordinate of A and B are $\left(x - y\frac{dx}{dy}, 0\right)$ and $\left(0, y - x\frac{dy}{dx}\right)$ respectively.

It is given that $P(x, y)$ is the mid-point of AB.

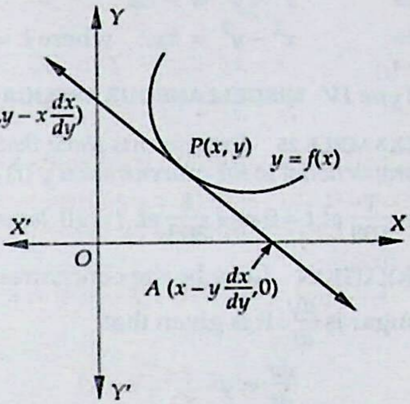


Fig. 22.8

$$\therefore \quad \frac{x - y\frac{dx}{dy} + 0}{2} = x \text{ and } \frac{0 + y - x\frac{dy}{dx}}{2} = y$$

$$\Rightarrow \quad x - y\frac{dx}{dy} = 2x \text{ and } y - x\frac{dy}{dx} = 2y$$

$$\Rightarrow \quad x = -y\frac{dx}{dy} \text{ and } y = -x\frac{dy}{dx}$$

$$\Rightarrow \quad y = -x\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}dy = -\frac{1}{x}dx$$

$$\Rightarrow \quad \int \frac{1}{y}dy = \int -\frac{1}{x}dx$$

$$\Rightarrow \quad \log|y| = -\log|x| + \log|C|$$

$$\Rightarrow \quad \log|xy| = \log|C|$$

$$\Rightarrow \quad |xy| = |C|$$

$$\Rightarrow \quad xy = C, \text{ which is a rectangular hyperbola.}$$

EXAMPLE 24 *Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2 + y^2}{2xy}$ is given by $x^2 - y^2 = kx$.*

SOLUTION We know that the slope of the tangent at any point on a curve is $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

Clearly, it is a homogeneous differential equation.

Substituting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - v^2}{2v}$$

$$\Rightarrow \quad \frac{2v}{1 - v^2}dv = \frac{1}{x}dx$$

$$\Rightarrow \quad \int \frac{2v}{v^2 - 1}dv = -\int \frac{1}{x}dx$$

$$\Rightarrow \quad \log|v^2-1| = -\log|x| + \log|C|$$

$$\Rightarrow \quad \log|v^2-1| = \log\left|\frac{C}{x}\right|$$

$$\Rightarrow \quad v^2 - 1 = \frac{C}{x}$$

$$\Rightarrow \quad y^2 - x^2 = Cx$$

$$\Rightarrow \quad x^2 - y^2 = -Cx$$

$$\Rightarrow \quad x^2 - y^2 = kx, \quad \text{where } k = -C$$

Type IV MISCELLANEOUS APPLICATIONS

EXAMPLE 25 *Experiments show that the rate of inversion of cane-sugar in dilute solution is proportional to the concentration $y(t)$ of the unaltered solution. Suppose that the concentration is $\frac{1}{100}$ at $t=0$ and $\frac{1}{300}$ at $t=10$ hours. Find $y(t)$.*

SOLUTION Let y be the concentration at time t. Then, the rate of inversion of the cane sugar is $\frac{dy}{dt}$. It is given that

$$\frac{dy}{dt} \propto y$$

$$\Rightarrow \quad \frac{dy}{dt} = -ky \qquad \left[\because \frac{dy}{dt} < 0\right]$$

$$\Rightarrow \quad \frac{dy}{y} = -k\,dt$$

$$\Rightarrow \quad \log y = -kt + C \qquad \text{...(i)}$$

When $t=0$, it is given that $y=\frac{1}{300}$. So, putting $t=0$ and $y=\frac{1}{100}$ in (i), we get

$$\log\frac{1}{100} = C \Rightarrow C = \log 10^{-2} = -2$$

Putting $C=-2$ in (i), we get

$$\log y = -kt - 2 \qquad \text{...(ii)}$$

When $t = 10$, it is given that $y=\frac{1}{300}$. Putting $y=\frac{1}{300}$ and $t=10$ in (ii), we get

$$\therefore \quad \log\frac{1}{300} = -10k - 2$$

$$\Rightarrow \quad -\log 300 = -10k - 2$$

$$\Rightarrow \quad -(\log 3 + \log 100) = -10k - 2$$

$$\Rightarrow \quad -(\log 3 + 2) = -10k - 2$$

$$\Rightarrow \quad -\log 3 = -10k \Rightarrow k = \frac{1}{10}\log 3$$

Putting the value of k in (ii), we get

$$\log y = \left(-\frac{1}{10}\log 3\right)t - 2$$

$$\Rightarrow \quad y = e^{\left(-\frac{1}{10}\log 3\right)t - 2}$$

EXAMPLE 26 *The equation of electromotive forces for an electric circuit containing resistance and self inductance is* $E = Ri + L\dfrac{di}{dt}$*, where E is the electromotive force given to the circuit, R, the resistance and L, the coefficient of induction. Find the current i at time t when (i) E = 0 and (ii) E = a non-zero constant.*

SOLUTION We are given that

$$L\frac{di}{dt} + Ri = E$$

$$\Rightarrow \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad \text{...(i)}$$

This is a linear differential equation in i and t.

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{(R/L)\,t}$$

Multiplying both sides of (i) by I.F. $= e^{(R/L)\,t}$, we get

$$e^{(R/L)\,t}\frac{di}{dt} + e^{(R/L)\,t}\frac{R}{L}i = \frac{E}{L}e^{(R/L)\,t}$$

Integrating both sides w.r.t. t, we get

$$ie^{(R/L)\,t} = \int \frac{E}{L}e^{(R/L)\,t}\,dt + C$$

$$\Rightarrow \quad ie^{(R/L)\,t} = \frac{E}{L}\left(\frac{e^{(R/L)\,t}}{\frac{R}{L}}\right) + C$$

$$\Rightarrow \quad ie^{(R/L)\,t} = \frac{E}{R}e^{(R/L)\,t} + C$$

$$\Rightarrow \quad i = \frac{E}{R} + Ce^{(-R/L)\,t} \qquad \text{...(ii)}$$

(i) When $E = 0$, from (ii), we get

$$i = Ce^{-Rt/L}$$

(ii) When E = a non-zero constant, from (ii), we get

$$i = \frac{E}{R} + Ce^{(-R/L)\,t}$$

EXERCISE 22.11

1. The surface area of a balloon being inflated, changes at a rate proportional to time t. If initially its radius is 1 unit and after 3 seconds it is 2 units, find the radius after time t.
2. A population grows at the rate of 5% per year. How long does it take for the population to double ?
3. The rate of growth of a population is proportional to the number present. If the population of a city doubled in the past 25 years, and the present population is 100000, when will the city have a population of 500000?

   $\left[\text{Given } \log_e 5 = 1.609, \log_e 2 = 0.6931.\right]$
4. In a culture, the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present?

5. If the interest is compounded continuously at 6% per annum, how much worth Rs. 1000 will be after 10 years? How long will it take to double Rs. 1000?

[Given $e^{0.6} = 1.822$]

6. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given the number triples in 5 hrs, find how many bacteria will be present after 10 hours. Also find the time necessary for the number of bacteria to be 10 times the number of initial present.

[Given $\log_e 3 = 1.0986, e^{2.1972} = 9$]

7. The population of a city increases at a rate proportional to the number of inhabitants present at any time t. If the population of the city was 200000 in 1990 and 250000 in 2000, what will be the population in 2010? [NCERT]

8. If the marginal cost of manufacturing a certain item is given by $C'(x) = \frac{dC}{dx} = 2 + 0.15\,x$.

Find the total cost function $C(x)$, given that $C(0) = 100$.

9. A bank pays interest by continuous compounding, that is, by treating the interest rate as the instantaneous rate of change of principal. Suppose in an account interest accrues at 8% per year, compounded continuously. Calculate the percentage increase in such an account over one year.[Take $e^{0.08} \approx 1.0833$]

10. In a simple circuit of resistance R, self inductance L and voltage E, the current i at any time t is given by $L\frac{di}{dt} + R\,i = E$. If E is constant and initially no current passes through the circuit, prove that $i = \frac{E}{R}\left(1 - e^{-(R/L)\,t}\right)$

11. The decay rate of radium at any time t is proportional to its mass at that time. Find the time when the mass will be halved of its initial mass.

12. Experiments show that radium disintegrates at a rate proportional to the amount of radium present at the moment. Its half-life is 1590 years. What percentage will disappear in one year ?

[Use : $e^{-\frac{\log 2}{1590}} = 0.9996$]

13. The slope of the tangent at a point $P(x, y)$ on a curve is $\frac{-x}{y}$. If the curve passes through the point $(3, -4)$, find the equation of the curve.

14. Find the equation of the curve which passes through the point $(2, 2)$ and satisfies the differential equation $y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$.

15. Find the equation of the curve passing through the point $\left(1, \frac{\pi}{4}\right)$ and tangent at any point of which makes an angle $\tan^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)$ with x-axis.

16. Find the curve for which the intercept cut off by a tangent on x-axis is equal to four times the ordinate of the point of contact.

17. Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is $y + 2(x+1) = 2e^{2x}$.

18. The tangent at any point (x, y) of a curve makes an angle $\tan^{-1}(2x + 3y)$ with x-axis. Find the equation of the curve if it passes through $(1, 2)$.

19. Find the equation of the curve such that the portion of the x-axis cut off between the origin and the tangent at a point is twice the abscissa and which passes through the point (1, 2).

20. Find the equation to the curve satisfying $x(x+1)\frac{dy}{dx} - y = x(x+1)$ and passing through (1, 0).

21. Find the equation of the curve which passes through the point (3, – 4) and has the slope $\frac{2y}{x}$ at any point (x, y) on it.

22. Find the equation of the curve which passes through the origin and has the slope $x + 3y - 1$ at any point (x, y) on it.

23. At every point on a curve the slope is the sum of the abscissa and the product of the ordinate and the abscissa, and the curve passes through (0, 1). Find the equation of the curve.

24. A curve is such that the length of the perpendicular from the origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is $y^2 - 2xy\frac{dy}{dx} - x^2 = 0$, and hence find the curve.

25. Find the equation of the curve which passes through the point (1, 2) and the distance between the foot of the ordinate of the point of contact and the point of intersection of the tangent with x-axis is twice the abscissa of the point of contact.

26. The normal to a given curve at each point (x, y) on the curve passes through the point (3, 0). If the curve contains the point (3, 4), find its equation.

27. The rate of increase of bacteria in a culture is proportional to the number of bacteria present and it is found that the number doubles in 6 hours. Prove that the bacteria becomes 8 times at the end of 18 hours.

28. Radium decomposes at a rate proportional to the quantity of radium present. It is found that in 25 years, approximately 1.1% of a certain quantity of radium has decomposed. Determine approximately how long it will take for one-half of the original amount of radium to decompose?

[Given $\log_e 0.989 = 0.01106$ and $\log_e 2 = 0.6931$]

29. Show that all curves for which the slope at any point (x, y) on it is $\frac{x^2 + y^2}{2xy}$ are rectangular hyperbola.

30. The slope of the tangent at each point of a curve is equal to the sum of the coordinates of the point. Find the curve that passes through the origin.

31. Find the equation of the curve passing through the point (0, 1) if the slope of the tangent to the curve at each of its point is equal to the sum of the abscissa and the product of the abscissa and the ordinate of the point. [NCERT]

32. The slope of a curve at each of its points is equal to the square of the abscissae of the point. Find the particular curve through the point (– 1, 1).

33. Find the equation of the curve that passes through the point $(0, a)$ and is such that at any point (x, y) on it, the product of its slope and the ordinate is equal to the abscissa.

34. The x-intercept of the tangent line to a curve is equal to the ordinate of the point of contact. Find the particular curve through the point (1, 1).

ANSWERS

1. $r = \sqrt{1 + \frac{1}{3} t^2}$
2. $20 \log 2$ years
3. 58 years
4. $\frac{2 \log 2}{\log\left(\frac{11}{10}\right)}$ hours
5. Rs. 1822, 12 years
6. 9 times, $\frac{5 \log 10}{\log 3}$
7. 312500
8. $C(x) = 0.075x^2 + 2x + 100$
9. 8.33%
11. $\frac{1}{k} \log 2$, k is the constant of proportionality
12. 0.04%
13. $x^2 + y^2 = 25$
14. $2xy - 2x - y - 2 = 0$
15. $\tan\left(\frac{y}{x}\right) = \log\left(\frac{e}{x}\right)$
16. $e^{-xy} = Cy^4$
18. $y e^{-3x} = \left(-\frac{2}{3} x - \frac{1}{9}\right) e^{-3x} + \frac{25}{9} e^{-3}$
19. $xy = 2$
20. $y = \frac{x}{x+1} (x - 1 + \log x)$
21. $9y + 4x^2 = 0$
22. $3(x + 3y) = 2(1 - e^{3x})$
23. $y + 1 = 2 e^{x^2/2}$
24. $x^2 + y^2 = Cx$
25. $y^2 = 4x$
26. $x^2 + y^2 - 6x - 7 = 0$
28. 1567 years.
30. $x + y = e^x - 1$
31. $1 + y = 2 e^{x^2/2}$
32. $3y = x^3 + 4$
33. $4x^2 + 9y = 0$
34. $x + y \log y = y$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Define a differential equation.
2. Define order of a differential equation.
3. Define degree of a differential equation.
4. Write the differential equation representing the family of straight lines $y = Cx + 5$, where C is an arbitrary constant.
5. Write the differential equation obtained by eliminating the arbitrary constant C in the equation $x^2 - y^2 = C^2$.
6. Write the differential equation obtained eliminating the arbitrary constant C in the equation $xy = C^2$.
7. Write the degree of the differential equation $a^2 \frac{d^2 y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/4}$
8. Write the order of the differential equation $1 + \left(\frac{dy}{dx}\right)^2 = 7 \left(\frac{d^2 y}{dx^2}\right)^3$

9. Write the order and degree of the differential equation

$$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

10. Write the degree of the differential equation

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2 \log\left(\frac{d^2y}{dx^2}\right).$$

11. Write the order of the differential equation of the family of circles touching X-axis at the origin.
12. Write the order of the differential equation of all non-horizontal lines in a plane.
13. If $\sin x$ is an integrating factor of the differential equation $\frac{dy}{dx} + Py = Q$, then write the value of P.
14. Write the order of the differential equation of the family of circles of radius r.
15. Write the order of the differential equation whose solution is

$$y = a\cos x + b\sin x + c\,e^{-x}.$$

16. Write the order of the differential equation associated with the primitive $y = C_1 + C_2 e^x + C_3 e^{-2x + C_4}$, where C_1, C_2, C_3, C_4 are arbitrary constants.
17. What is the degree of the following differential equation?

$$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$$ [CBSE 2010]

ANSWERS

4. $x\frac{dy}{dx} - y + 5 = 0$ 5. $x\,dx - y\,dy = 0$ 6. $x\,dy + y\,dx = 0$ 7. 4
8. 2 9. 1, 2 10. Not defined 11. 1
12. 2 13. $\cot x$ 14. 2 15. 3 16. 3 17. 1

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The integrating factor of the differential equation $(x\log x)\frac{dy}{dx} + y = 2\log x$, is given by
(a) $\log(\log x)$ (b) e^x (c) $\log x$ (d) x
2. The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ is
(a) $\log y = kx$ (b) $y = kx$ (c) $xy = k$ (d) $y = k\log x$
3. Integrating factor of the differential equation $\cos x\frac{dy}{dx} + y\sin x = 1$, is
(a) $\sin x$ (b) $\sec x$ (c) $\tan x$ (d) $\cos x$
4. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right) = y^3$, is
(a) 1/2 (b) 2 (c) 3 (d) 4
5. The degree of the differential equation $\left[5 + \left(\frac{dy}{dx}\right)^2\right]^{5/3} = x^5\left(\frac{d^2y}{dx^2}\right)$, is
(a) 4 (b) 2 (c) 5 (d) 10

6. The general solution of the differential equation $\frac{dy}{dx} + y \cot x = \operatorname{cosec} x$, is

(a) $x + y \sin x = C$ (b) $x + y \cos x = C$
(c) $y + x(\sin x + \cos x) = C$ (d) $y \sin x = x + C$

7. The differential equation obtained on eliminating A and B from $y = A \cos \omega t + B \sin \omega t$, is

(a) $y'' + y' = 0$ (b) $y'' - \omega^2 y = 0$
(c) $y'' = -\omega^2 y$ (d) $y'' + y = 0$

8. The equation of the curve whose slope is given by $\frac{dy}{dx} = \frac{2y}{x}$; $x > 0, y > 0$ and which passes through the point (1, 1) is

(a) $x^2 = y$ (b) $y^2 = x$
(c) $x^2 = 2y$ (d) $y^2 = 2x$

9. The order of the differential equation whose general solution is given by $y = c_1 \cos(2x + C_2) - (c_3 + c_4) a^{x + c_5} + c_6 \sin(x - c_7)$ is

(a) 3 (b) 4 (c) 5 (d) 2

10. The solution of the differential equation $\frac{dy}{dx} = \frac{ax + g}{by + f}$ represents a circle when

(a) $a = b$ (b) $a = -b$ (c) $a = -2b$ (d) $a = 2b$

11. The solution of the differential equation $\frac{dy}{dx} + \frac{2y}{x} = 0$ with $y(1) = 1$ is given by

(a) $y = \frac{1}{x^2}$ (b) $x = \frac{1}{y^2}$ (c) $x = \frac{1}{y}$ (d) $y = \frac{1}{x}$

12. The solution of the differential equation $\frac{dy}{dx} - \frac{y(x+1)}{x} = 0$ is given by

(a) $y = xe^{x + C}$ (b) $x = ye^x$
(c) $y = x + C$ (d) $xy = e^x + C$

13. The order of the differential equation satisfying $\sqrt{1 - x^4}\, a + \sqrt{1 - y^4} = a(x^2 - y^2)$ is

(a) 1 (b) 2 (c) 3 (d) 4

14. The solution of the differential equation $y_1 y_3 = y_2^2$ is

(a) $x = C_1 e^{C_2 y} + C_3$ (b) $y = C_1 e^{C_2 x} + C_3$
(c) $2x = C_1 e^{C_2 y} + C_3$ (d) none of these

15. The general solution of the differential equation $\frac{dy}{dx} + y g'(x) = g(x) g'(x)$, where $g(x)$ is a given function of x, is

(a) $g(x) + \log\{1 + y + g(x)\} = C$ (b) $g(x) + \log\{1 + y - g(x)\} = C$
(c) $g(x) - \log\{1 + y - g(x)\} = C$ (d) none of these

16. The solution of the differential equation $\frac{dy}{dx} = 1 + x + y^2 + xy^2$, $y(0) = 0$ is

(a) $y^2 = \exp\left(x + \frac{x^2}{2}\right) - 1$ (b) $y^2 = 1 + C \exp\left(x + \frac{x^2}{2}\right)$

(c) $y = \tan(C + x + x^2)$ (d) $y = \tan\left(x + \frac{x^2}{2}\right)$

17. The differential equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=C$ is

(a) $\frac{y''}{y'}+\frac{y'}{y}-\frac{1}{x}=0$ (b) $\frac{y''}{y'}+\frac{y'}{y}+\frac{1}{x}=0$

(c) $\frac{y''}{y'}-\frac{y'}{y}-\frac{1}{x}=0$ (d) none of these

18. Solution of the differential equation $\frac{dy}{dx}+\frac{y}{x}=\sin x$ is

(a) $x(y+\cos x)=\sin x+C$ (b) $x(y-\cos x)=\sin x+C$
(c) $x(y+\cos x)=\cos x+C$ (d) none of these

19. The equation of the curve satisfying the differential equation $y(x+y^3)\,dx=x(y^3-x)\,dy$ and passing through the point (1, 1) is

(a) $y^3-2x+3x^2y=0$ (b) $y^3+2x+3x^2y=0$
(c) $y^3+2x-3x^2y=0$ (d) none of these

20. The solution of the differential equation $2x\frac{dy}{dx}-y=3$ represents

(a) circles (b) straight lines
(c) ellipses (d) parabolas

21. The solution of the differential equation $x\frac{dy}{dx}=y+x\tan\frac{y}{x}$, is

(a) $\sin\frac{x}{y}=x+C$ (b) $\sin\frac{y}{x}=Cx$

(c) $\sin\frac{x}{y}=Cy$ (d) $\sin\frac{y}{x}=Cy$

22. The differential equation satisfied by $ax^2+by^2=1$ is

(a) $xyy_2+y_1^2+yy_1=0$ (b) $xyy_2+xy_1^2-yy_1=0$
(c) $xyy_2-xy_1^2+yy_1=0$ (d) none of these

23. The differential equation which represents the family of curves $y=e^{Cx}$ is

(a) $y_1=C^2y$ (b) $xy_1-\ln y=0$
(c) $x\ln y=yy_1$ (d) $y\ln y=xy_1$

24. Which of the following transformations reduce the differential equation $\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x^2}(\log z)^2$ into the form $\frac{du}{dx}+P(x)\,u=Q(x)$

(a) $u=\log x$ (b) $u=e^z$
(c) $u=(\log z)^{-1}$ (d) $u=(\log z)^2$

25. The solution of the differential equation $\frac{dy}{dx}=\frac{y}{x}+\frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ is

(a) $\phi\left(\frac{y}{x}\right)=kx$ (b) $x\,\phi\left(\frac{y}{x}\right)=k$

(c) $\phi\left(\frac{y}{x}\right)=ky$ (d) $y\,\phi\left(\frac{y}{x}\right)=k$

26. If m and n are the order and degree of the differential equation

$$(y_2)^5 + \frac{4(y_2)^3}{y_3} + y_3 = x^2 - 1, \text{ then}$$

(a) $m = 3, n = 3$ (b) $m = 3, n = 2$
(c) $m = 3, n = 5$ (d) $m = 3, n = 1$

27. The solution of the differential equation $\frac{dy}{dx} + 1 = e^{x+y}$, is

(a) $(x + y)\, e^{x+y} = 0$ (b) $(x + C)\, e^{x+y} = 0$
(c) $(x - C)\, e^{x+y} = 1$ (d) $(x - C)\, e^{x+y} + 1 = 0$

28. The solution of $x^2 + y^2 \frac{dy}{dx} = 4$, is

(a) $x^2 + y^2 = 12x + C$ (b) $x^2 + y^2 = 3x + C$
(c) $x^3 + y^3 = 3x + C$ (d) $x^3 + y^3 = 12x + C$

29. The family of curves in which the subtangent at any point of a curve is double the abscissae, is given by

(a) $x = Cy^2$ (b) $y = Cx^2$ (c) $x^2 = Cy^2$ (d) $y = Cx$

30. The solution of the differential equation $x\,dx + y\,dy = x^2 y\,dy - y^2 x\,dx$, is

(a) $x^2 - 1 = C(1 + y^2)$ (b) $x^2 + 1 = C(1 - y^2)$
(c) $x^3 - 1 = C(1 + y^3)$ (d) $x^3 + 1 = C(1 - y^3)$

31. The solution of the differential equation $(x^2 + 1)\frac{dy}{dx} + (y^2 + 1) = 0$, is

(a) $y = 2 + x^2$ (b) $y = \frac{1 + x}{1 - x}$
(c) $y = x(x - 1)$ (d) $y = \frac{1 - x}{1 + x}$

32. The differential equation $x\frac{dy}{dx} - y = x^2$, has the general solution

(a) $y - x^3 = 2cx$ (b) $2y - x^3 = cx$
(c) $2y + x^2 = 2cx$ (d) $y + x^2 = 2cx$

33. The solution of the differential equation $\frac{dy}{dx} - ky = 0,\ y(0) = 1$ approaches to zero when $x \to \infty$, if

(a) $k = 0$ (b) $k > 0$
(c) $k < 0$ (d) none of these

34. The solution of the differential equation $(1 + x^2)\frac{dy}{dx} + 1 + y^2 = 0$, is

(a) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} C$ (b) $\tan^{-1} y - \tan^{-1} x = \tan^{-1} C$
(c) $\tan^{-1} y \pm \tan^{-1} x = \tan C$ (d) $\tan^{-1} y + \tan^{-1} x = \tan^{-1} C$

35. The solution of the differential equation $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$, is

(a) $\tan^{-1}\left(\frac{x}{y}\right) = \log y + C$ (b) $\tan^{-1}\left(\frac{y}{x}\right) = \log x + C$
(c) $\tan^{-1}\left(\frac{x}{y}\right) = \log x + C$ (d) $\tan^{-1}\left(\frac{y}{x}\right) = \log y + C$

36. The differential equation $\frac{dy}{dx} + Py = Qy^n, n > 2$ can be reduced to linear form by substituting
(a) $z = y^{n-1}$ (b) $z = y^n$ (c) $z = y^{n+1}$ (d) $z = y^{1-n}$

37. If p and q are the order and degree of the differential equation $y\frac{dy}{dx} + x^3\frac{d^2y}{dx^2} + xy = \cos x$, then
(a) $p < q$ (b) $p = q$ (c) $p > q$ (d) none of these

38. Which of the following is the integrating factor of $(x \log x)\frac{dy}{dx} + y = 2\log x$?
(a) x (b) e^x (c) $\log x$ (d) $\log(\log x)$

39. What is integrating factor of $\frac{dy}{dx} + y\sec x = \tan x$?
(a) $\sec x + \tan x$ (b) $\log(\sec x + \tan x)$
(c) $e^{\sec x}$ (d) $\sec x$

40. Integrating factor of the differential equation $\cos x\frac{dy}{dx} + y\sin x = 1$, is
(a) $\cos x$ (b) $\tan x$ (c) $\sec x$ (d) $\sin x$

41. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$, is
(a) 3 (b) 2 (c) 1 (d) not defined

42. The order of the differential equation $2x^2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$, is
(a) 2 (b) 1 (c) 0 (d) not defined

43. The number of arbitrary constants in the general solution of differential equation of fourth order is
(a) 0 (b) 2 (c) 3 (d) 4

44. The number of arbitrary constants in the particular solution of a differential equation of third order is
(a) 3 (b) 2 (c) 1 (d) 0

45. Which of the following differential equations has $y = C_1e^x + C_2e^{-x}$ as the general solution?
(a) $\frac{d^2y}{dx^2} + y = 0$ (b) $\frac{d^2y}{dx^2} - y = 0$
(c) $\frac{d^2y}{dx^2} + 1 = 0$ (d) $\frac{d^2y}{dx^2} - 1 = 0$

46. Which of the following differential equations has $y = x$ as one of its particular solution?
(a) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$ (b) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = x$
(c) $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$ (d) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$

47. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$, is

(a) $e^x + e^{-y} = C$ (b) $e^x + e^y = C$
(c) $e^{-x} + e^y = C$ (d) $e^{-x} + e^{-y} = C$

48. A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution

(a) $y = vx$ (b) $v = yx$ (c) $x = vy$ (d) $x = v$

49. Which of the following is a homogeneous differential equation?

(a) $(4x + 6y + 5)\,dy - (3y + 2x + 4)\,dx = 0$ (b) $xy\,dx - (x^3 + y^3)\,dy = 0$
(c) $(x^3 + 2y^2)\,dx + 2xy\,dy = 0$ (d) $y^2\,dx + (x^2 - xy - y^2)\,dy = 0$

50. The integrating factor of the differential equation $x\frac{dy}{dx} - y - 2x^2$

(a) e^{-x} (b) e^{-y} (c) $\frac{1}{x}$ (d) x

51. The integrating factor of the differential equation $(1 - y^2)\frac{dx}{dy} + yx = ay\ (-1 < y < 1)$ is

(a) $\frac{1}{y^2 - 1}$ (b) $\frac{1}{\sqrt{y^2 - 1}}$ (c) $\frac{1}{1 - y^2}$ (d) $\frac{1}{\sqrt{1 - y^2}}$

52. The general solution of the differential equation $\frac{y\,dx - x\,dy}{y} = 0$, is

(a) $xy = C$ (b) $x = Cy^2$ (c) $y = Cx$ (d) $y = Cx^2$

53. The general solution of a differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$ is

(a) $y e^{\int P_1 dy} = \int \left\{Q_1 e^{\int P_1 dy}\right\} dy + C$
(b) $y e^{\int P_1 dx} = \int \left\{Q_1 e^{\int P_1 dx}\right\} dx + C$
(c) $x e^{\int P_1 dy} = \int \left\{Q_1 e^{\int P_1 dy}\right\} dy + C$
(d) $x e^{\int P_1 dx} = \int \left\{Q_1 e^{\int P_1 dx}\right\} dx + C$

54. The general solution of the differential equation $e^x\,dy + (y\,e^x + 2x)\,dx = 0$ is

(a) $x e^y + x^2 = C$ (b) $x e^y + y^2 = C$
(c) $y e^x + x^2 = C$ (d) $y e^y + x^2 = C$

ANSWERS

1. (c)	2. (b)	3. (b)	4. (b)	5. (e)	6. (d)	7. (c)	8. (a)
9. (c)	10. (b)	11. (a)	12. (a)	13. (a)	14. (b)	15. (b)	16. (d)
17. (a)	18. (a)	19. (c)	20. (d)	21. (b)	22. (b)	23. (d)	24. (c)
25. (a)	26. (b)	27. (d)	28. (d)	29. (a)	30. (a)	31. (d)	32. (b)
33. (c)	34. (d)	35. (b)	36. (d)	37. (a)	38. (c)	39. (a)	40. (c)
41. (d)	42. (a)	43. (d)	44. (d)	45. (b)	46. (c)	47. (a)	48. (c)
49. (d)	50. (c)	51. (d)	52. (c)	53. (c)	54. (c)		

REVISION EXERCISE

1. Determine the order and degree (if defined) of the following differential equations:

(i) $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$ [NCERT] (ii) $y''' + 2y'' + y' = 0$ [NCERT]

(iii) $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ [NCERT] (iv) $y''' + 2y'' + y' = 0$ [NCERT]

(v) $y'' + (y')^2 + 2y = 0$ [NCERT] (vi) $y'' + 2y' + \sin y = 0$ [NCERT]

(vii) $y''' + y^2 + e^{y'} = 0$ [NCERT]

2. Verify that the function $y = e^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

3. In each of the following verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

(i) $y = e^x + 1$ $\qquad y'' - y' = 0$

(ii) $y = x^2 + 2x + C$ $\qquad y' - 2x - 2 = 0$

(iii) $y = \cos x + C$ $\qquad y' + \sin x = 0$

(iv) $y = \sqrt{1+x^2}$ $\qquad y' = \frac{xy}{1+x^2}$

(v) $y = x\sin x$ $\qquad xy' = y + x\sqrt{x^2 - y^2}$ ($x \neq 0$ and $x > y$ or $x < -y$)

(vi) $y = \sqrt{a^2 - x^2}, x \in (-a, a)$ $\qquad x + y\frac{dy}{dx} = 0,\ y \neq 0$

4. Form the differential equation respresenting the family of curves $y = mx$, where m is an arbitrary constant.

5. Form the differential equation representing the family of curves $y = a\sin(x+b)$, where a, b are arbitrary constant.

6. Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.

7. Form the differential equation of the family of circles having centre on y-axis and radius 3 unit.

8. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.

9. Form the differential equation of the family of ellipses having foci on y-axis and centre at the origin.

10. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at the origin.

11. Verify that $xy = a\,e^x + b\,e^{-x} + x^2$ is a solution of the differential equation $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0.$

12. Show that $y = Cx + 2C^2$ is a solution of the differential equation $2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0.$

13. Show that $y^2 - x^2 - xy = a$ is a solution of the differential equation $(x - 2y)\frac{dy}{dx} + 2x + y = 0.$

14. Verify that $y = A\cos x + \sin x$ satisfies the differential equation $\cos x \frac{dy}{dx} + (\sin x)\, y = 1$.

15. Find the differential equation corresponding to $y = ae^{2x} + be^{-3x} + ce^{x}$ where a, b, c are arbitrary constants.

16. Show that the differential equation of all parabolas which have their axes parallel to y-axis is $\frac{d^3 y}{dx^3} = 0$.

17. From $x^2 + y^2 + 2ax + 2by + c = 0$, derive a differential equation not containing a, b and c.

Solve the following differential equations:

18. $\frac{dy}{dx} = \sin^3 x \cos^4 x + x\sqrt{x+1}$

19. $\frac{dy}{dx} = \frac{1}{x^2 + 4x + 5}$

20. $\frac{dy}{dx} = y^2 + 2y + 2$

21. $\frac{dy}{dx} + 4x = e^x$

22. $\frac{dy}{dx} = x^2 e^x$

23. $\frac{dy}{dx} - x\sin^2 x = \frac{1}{x \log x}$

24. $(\tan^2 x + 2\tan x + 5)\frac{dy}{dx} = 2(1 + \tan x)\sec^2 x$

25. $\frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x$

26. $\tan y\, dx + \tan x\, dy = 0$

27. $(1 + x)\, y\, dx + (1 + y)\, x\, dy = 0$

28. $x\cos^2 y\, dx = y\cos^2 x\, dy$

29. $\cos y \log(\sec x + \tan x)\, dx = \cos x \log(\sec y + \tan y)\, dy$

30. $\operatorname{cosec} x\ (\log y)\, dy + x^2 y\, dx = 0$

31. $(1 - x^2)\, dy + xy\, dx = xy^2\, dx$

32. $\frac{dy}{dx} = \frac{\sin x + x\cos x}{y(2\log y + 1)}$

33. $x(e^{2y} - 1)\, dy + (x^2 - 1)\, e^y\, dx = 0$

34. $\frac{dy}{dx} + 1 = e^{x+y}$

35. $\frac{dy}{dx} = (x + y)^2$

36. $\cos(x + y)\, dy = dx$

37. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

38. $\frac{dy}{dx} = \frac{y(x - y)}{x(x + y)}$

39. $(x + y - 1)\, dy = (x + y)\, dx$

40. $\frac{dy}{dx} - y\cot x = \operatorname{cosec} x$

41. $\frac{dy}{dx} - y\tan x = -2\sin x$

42. $\frac{dy}{dx} - y\tan x = e^x \sec x$

43. $\frac{dy}{dx} - y\tan x = e^x$

44. $(1 + y + x^2 y)\, dx + (x + x^3)\, dy = 0$

45. $(x^2 + 1)\, dy + (2y - 1)\, dx = 0$

46. $y\sec^2 x + (y + 7)\tan x \frac{dy}{dx} = 0$

47. $(2ax + x^2)\frac{dy}{dx} = a^2 + 2ax$

48. $(x^3 - 2y^3)\, dx + 3x^2 y\, dy = 0$

49. $x^2\, dy + (x^2 - xy + y^2)\, dx = 0$

50. $y - x\frac{dy}{dx} = b\left(1 + x^2 \frac{dy}{dx}\right)$

51. $\frac{dy}{dx} + 2y = \sin 3x$

52. $\frac{dy}{dx}+y=4x$

53. $\frac{dy}{dx}+5y=\cos 4x$

54. $x\frac{dy}{dx}+x\cos^2\left(\frac{y}{x}\right)=y$

55. $\cos^2 x\frac{dy}{dx}+y=\tan x$

56. $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$

57. $(1+y^2)+(x-e^{-\tan^{-1}y})\frac{dy}{dx}=0$

58. $y^2+\left(x+\frac{1}{y}\right)\frac{dy}{dx}=0$

59. $2\cos x\frac{dy}{dx}+4y\sin x=\sin 2x$, given that $y=0$ when $x=\frac{\pi}{3}$

60. $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy$

61. $\frac{dy}{dx}+y\tan x=x^n\cos x, n\neq -1$

62. Find the general solution of the differential equation $\frac{dy}{dx}=\frac{x+1}{2-y}, y\neq 2.$

63. Find the particular solution of the differential equation $\frac{dy}{dx}=-4xy^2$ given that $y=1$, when $x=0$.

64. For each of the following differential equations, find the general solution:

(i) $\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$ (ii) $\frac{dy}{dx}=\sqrt{4-y^2}, -2<y<2$ **[NCERT]**

(iii) $\frac{dy}{dx}=(1+x^2)(1+y^2)$ (iv) $y\log y\,dx-x\,dy=0$

(v) $\frac{dy}{dx}=\sin^{-1}x$ (vi) $\frac{dy}{dx}+y=1, y\neq 1$

65. For each of the following differential equations, find a particular solution satisfying the given condition: **[NCERT]**

(i) $x(x^2-1)\frac{dy}{dx}=1,\ y=0$ when $x=2$

(ii) $\cos\left(\frac{dy}{dx}\right)=a,\quad y=1$ when $x=0$

(iii) $\frac{dy}{dx}=y\tan x,\quad y=1$ when $x=0$

66. Solve the each of the following differential equations: **[NCERT]**

(i) $(x-y)\frac{dy}{dx}=x+2y$

(ii) $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x$

(iii) $y\,dx+x\log\left(\frac{y}{x}\right)dy-2x\,dy=0$

(iv) $\frac{dy}{dx}-y=\cos x$

(v) $x\frac{dy}{dx}+2y=x^2, x\neq 0$

(vi) $\frac{dy}{dx}+2y=\sin x$

(vii) $\frac{dy}{dx}+3y=e^{-2x}$

(viii) $\frac{dy}{dx}+\frac{y}{x}=x^2$

(ix) $\frac{dy}{dx}+(\sec x)y=\tan x\left(0<x<\frac{\pi}{2}\right)$

(x) $x\frac{dy}{dx}+2y=x^2\log x$

(xi) $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$

(xii) $(1+x^2)\,dy+2xy\,dx=\cot x\,dx$

(xiii) $(x+y)\frac{dy}{dx}=1$

(xiv) $y\,dx+(x-y^2)\,dy=0$

(xv) $(x+3y^2)\frac{dy}{dx} = y, y>0$

67. Find a particular solution of each of the following differential equations: [NCERT]

(i) $(1+x^2)\frac{dy}{dx}+2xy = \frac{1}{1+x^2}; y = 0$, when $x = 1$

(ii) $(x+y)\,dy+(x-y)\,dx = 0; y = 1$ when $x = 1$

(iii) $x^2\,dy+(xy+y^2)\,dx = 0;\ y = 1$ when $x = 1$

68. Find the equation of the curve passing through the point (1, 1) whose differential equation is $x\,dy = (2x^2+1)\,dx, x \neq 0$. [NCERT]

69. Find the equation of a curve passing through the point $(-2, 3)$, given that the slope of the tangent to the curve at any point (x, y) is $\frac{2x}{y^2}$. [NCERT]

70. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $\frac{dy}{dx} = e^x \sin x$. [NCERT]

71. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4, -3)$. Find the equation of the curve given that it passes through $(-2, 1)$. [NCERT]

72. Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2+y^2}{2xy}$ is given by $x^2-y^2 = Cx$. [NCERT]

73. Find the equation of a curve passing throught the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x-coordinate and the product of the x-coordinate and y-coordinate of that point. [NCERT]

74. Find the equation of the curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point. [NCERT]

75. Find the equation of the curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5. [NCERT]

76. The slope of the tangent to the curve at any point is the reciprocal of twice the ordinate at that point. The curve passes through the point (4, 3). Determine its equation. [NCERT]

77. The decay rate of radium at any time t is proportional to its mass at that time. Find the time when the mass will be halved of its initial mass.

78. Experiments show that radium disintegrates at a rate proportional to the amount of radium present at the moment. Its half-life is 1590 years. What percentage will disappear in one year ?

[Use $e^{-\frac{\log 2}{1590}} = 0.9996$]

79. A wet porous substance in the open air loeses its moisture at a rate proportional to the moisture content. If a sheet hung in the wind loses half of its moisture during the first hour, when will it have lost 95% moisture, wheather conditions remaining the same.

ANSWER

1.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
Order	2	3	3	3	2	2	3
Degree	1	1	2	1	1	1	Not specified

4. $x\frac{dy}{dx} - y = 0$

5. $\frac{d^2y}{dx^2} + y = 0$

6. $y^2 - 2xy\frac{dy}{dx} = 0$

7. $(x^2-9)(y')^2 + x^2 = 0$

8. $xy' - 2y = 0$

9. $xyy'' + x(y')^2 - yy' = 0$

10. $xyy'' + x(y')^2 - yy' = 0$

15. $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$

17. $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{d^3y}{dx^3} - \frac{3dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0$

18. $y = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$

19. $y = \tan^{-1}(x+2) + C$

20. $x = \tan^{-1}(y+1) + C$

21. $y + 2x^2 = e^x + C$

22. $y = (x^2 - 2x + 2)e^x + C$

23. $y = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + \log|\log x| + C$

24. $y = \log|\tan^2 x + 2\tan x + 5| + C$

25. $y = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + (x-1)e^x + C$

26. $\sin x \sin y = C$

27. $x + y + \log(xy) = C$

28. $x\tan x - y\tan y = \log|\sec x| - \log|\sec y| + C$

29. $[\log(\sec x + \tan x)]^2 = [\log(\sec y + \tan y)]^2 + C$

30. $\frac{1}{2}(\log|y|)^2 + (2 - x^2)\cos x + 2x\sin x = C$

31. $(y-1)^2|1 - x^2| = C^2y^2$

32. $y^2\log y = x\sin x + C$

33. $e^y + e^{-y} - \frac{1}{2}x^2 + \log|x| = C$

34. $-1 = (x + C)e^{x+y}$

35. $x + y = \tan(x + C)$

36. $y - C = \tan\left(\frac{x+y}{2}\right)$

37. $y - 2x = Cx^2y$

38. $Cxy = e^{x/y}$

39. $2(y - x) - \log(2x + 2y - 1) = C$

40. $y\operatorname{cosec} x = -\cot x + C$

41. $y\sin x = C - \frac{1}{2}\cos 2x$

42. $y\cos x = e^x + C$

43. $y\cos x = \frac{e^x}{2}(\cos x + \sin x) + C$

44. $xy = -\tan^{-1}x + C$

45. $y = Ce^{-2\tan^{-1}x} + \frac{1}{2}$

46. $y^7\tan x = ke^{-y}$

47. $y + C = \frac{a}{2}[\log x + 3\log(x + 2a)]$

48. $x^3 + y^3 = kx^2$

49. $C = xe^{\tan^{-1}(y/x)}$

50. $y = k(y - b)(1 + bx)$

51. $y = \frac{3}{13}\left(\frac{2}{3}\sin 3x - \cos 3x\right) + Ce^{-2x}$

52. $y = 4(x - 1) + Ce^{-x}$

53. $y = \frac{4}{41}\left(\sin 4x + \frac{5}{4}\cos 4x\right) + C\,e^{-5x}$

54. $\tan\left(\frac{y}{x}\right) = \log x + C$

55. $y\,e^{\tan x} = C + e^{\tan x}(\tan x - 1)$

56. $xy \sec x = C + \tan x$

57. $xe^{\tan^{-1} y} = C + \tan^{-1} y$

58. $x = C\,e^{1/y} + \left(\frac{1}{y} + 1\right)$

59. $y = \cos x - 2\cos^2 x$

60. $xe^{\tan^{-1} y} = C + e^{\tan^{-1} y}(\tan^{-1} y - 1)$

61. $y \sec x = C + \frac{x^{n+1}}{n+1}$

62. $x^2 + y^2 + 2x - 4y + C = 0$

63. $y = \frac{1}{2x^2 + 1}$

64. (i) $y = 2\tan\frac{x}{2} - x + C$

(ii) $y = 2\sin(x + C)$

(iii) $\tan^{-1} y = x + \frac{x^3}{3} + C$

(iv) $y = e^{Cx}$

(v) $y = x\sin^{-1} x + \sqrt{1 - x^2} + C$

(vi) $y = 1 + C\,e^{-x}$

65. (i) $y = \frac{1}{2}\log\left(\frac{x^2 - 1}{x^2}\right) - \frac{1}{2}\log\frac{3}{4}$

(ii) $\cos\left(\frac{y-2}{x}\right) = a$

(iii) $y = \sec x$

66. (i) $\log\left|x^2 + xy + y^2\right| = 2\sqrt{3}\tan^{-1}\left(\frac{x + 2y}{\sqrt{3}\,x}\right) + C$

(ii) $\sin\left(\frac{y}{x}\right) = \log|Cx|$

(iii) $Cy = \log\left|\frac{y}{x}\right| - 1$

(iv) $y = \frac{1}{2}(\sin x - \cos x) + C\,e^x$

(v) $y = \frac{x^2}{4} + Cx^{-2}$

(vi) $y = \frac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$

(vii) $y = e^{-2x} + C\,e^{-3x}$

(viii) $xy = \frac{x^4}{4} + C$

(ix) $y(\sec x + \tan x) = \sec x + \tan x - x + C$

(x) $y = \frac{x^2}{16}(4\log x - 1) + Cx^{-2}$

(xi) $y\log x = -\frac{2}{x}(1 + \log x) + C$

(xii) $y = (1 + x)^{-2}\log\sin x + C(1 + x^2)^{-1}$

(xiii) $x + y + 1 = C\,e^y$

(xiv) $x = \frac{y^2}{3} + \frac{C}{y}$

(xv) $x = 3y^2 + C\,y$

67. (i) $y(1 + x^2) = \tan^{-1} x - \frac{\pi}{4}$

(ii) $\log(x^2 + y^2) + 2\tan^{-1}\frac{y}{x} = \frac{\pi}{2} + \log 2$

(iii) $y + 2x = 3x^2 y$

68. $y = x^2 + \log|x|$

69. $y = (3x^2 + 15)^{1/3}$

70. $2y - 1 = e^x(\sin x - \cos x)$

71. $(x + 4)^2 = y + 3$

73. $y = -1 + 2\,e^{\frac{x^2}{2}}$

74. $x + y + 1 = e^x$

75. $y = 4 - x - 2\,e^x$

76. $y^2 = x + 5$

77. $\frac{1}{k}\log 2$, k is the constant of proportionality

78. 0.04%

79. $\frac{\log 20}{\log 2}$

SUMMARY

1. (i) An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a differential equation.

   (ii) The order of a differential equation is the order of the highest order derivative appearing in the equation.

   The order of a differential equation is a positive integer.

   (iii) The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions.

   In other words, the degree of a differential equation is the power of the highest order derivative occurring in a differential equation when it is written as a polynomial in differential coefficients.

2. A differential equation is a linear differential equation if it is expressible in the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

where $P_0, P_1, P_2, \ldots, P_{n-1}, P_n$ and Q are either constants or functions of independent variable x.

Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. Otherwise, it is a non-linear differential equation.

It follows from the above definition that a differential equation will be non-linear differential equation, if

   (i) its degree is more than one.

   (ii) any of the differential coefficient has exponent more than one.

   (iii) exponent of the dependent variable is more than one.

   (iv) products containing dependent variable and its differential coefficients are present.

3. The solution of a differential equation is a relation between the variables involved which satisfies the differential equation. Such a relation and the derivatives obtained therefrom when substituted in the differential equation, makes left hand, and right hand sides identically equal.

The solution which contains as many arbitrary constants as the order of the differential equation is called the general solution of the differential equation.

Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.

4. A differential equation is said to be in the variable separable form if it is expressible in the form $f(x)\,dx = g(y)\,dy$

The solution of this equation is given by

$$\int f(x)\,dx = \int g(y)\,dy + C, \text{ where } C \text{ is a constant.}$$

5. A differential equation of the form $\frac{dy}{dx}=f(ax+by+c)$ can be reduced to variable separable form by the substitution $ax+by+c=v$.
6. If a first order first degree differential equation is expressible in the form
$$\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)},$$
where $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree, then it is called a homogeneous differential equation. Such type of equations can be reduced to variable separable form by the substitution $y=vx$ or, $x=vy$.
7. If a differential equation is expressible in the form $\frac{dy}{dx}+Py+Q$, where P and Q are functions of x, then it is called a linear differential equation. The solution of this equation is given by
$$y\left(e^{\int P\,dx}\right)=\int\left(Q\,e^{\int P\,dx}\right)dx+C$$
Sometimes a linear differential equation is in the form $\frac{dx}{dy}+Rx=S$, where R and S are functions of y.
The solution of this equation is given by
$$x\left(e^{\int R\,dy}\right)=\int\left(S\,e^{\int R\,dy}\right)dy+C$$

23

ALGEBRA OF VECTORS

23.1 INTRODUCTION

Physical quantities are divided into two categories - scalar quantities and vector quantities. Those quantities which have only magnitude and which are not related to any fixed direction in space are called *scalar quantities*, or briefly *scalars*. Examples of scalars are mass, volume, density work, temperature etc. To represent a scalar quantity we assign a real number to it which gives its magnitude in terms of a certain basic unit of a quantity a of the level type. Throughout this chapter by scalars we shall mean real numbers. Second kind of quantities are those which have both magnitude and direction. Such quantities are called *vectors*. Displacement, velocity, acceleration, momentum, weight, force etc. are examples of vector quantities.

<u>NOTE</u> *It is to note here that in addition to magnitude and direction, two vector quantities of the same kind should be capable of being compounded according to the parallelogram law of addition. Quantities having magnitude and direction but not obeying the parallelogram law of addition will not be treated as vectors. For example, the rotations of a rigid body through finite angles have both magnitudes and directions but do not satisfy the parallelogram law of addition.*

23.2 REPRESENTATION OF VECTORS

Vectors are represented by directed line segments such that the length of the line segment is the magnitude of the vector and the direction of arrow marked at one end emphasizes the direction of the vector. A vector, denoted by $\overrightarrow{PQ}$, is determined by two points P, Q such that the magnitude of the vector is the length of the straight line PQ and its direction is that from P to Q. The point P is called the *initial point* of vector $\overrightarrow{PQ}$ and Q is called the *terminal point or tip*. Vectors are generally denoted by $\vec{a}, \vec{b}, \vec{c}$ etc.

The modulus, or module, or magnitude of a vector $\vec{a}$ is the positive number which is the measure of its length and is denoted by $|\vec{a}|$. The modulus $|\vec{a}|$ of a vector $\vec{a}$ is sometimes written as a.

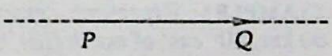


Fig. 23.1

<u>NOTE</u> *Every vector $\overrightarrow{PQ}$ has the following three characteristics:*

LENGTH The length of $\overrightarrow{PQ}$ will be denoted by $|\overrightarrow{PQ}|$ or PQ.

SUPPORT The line of unlimited length of which PQ is segment is called the support of the vector $\overrightarrow{PQ}$.

SENSE The sense of $\overrightarrow{PQ}$ is from P to Q and that of $\overrightarrow{QP}$ is from Q to P. Thus, the sense of a directed line segment is from its initial point to the terminal point.

EQUALITY VECTORS ***Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, written as $\vec{a} = \vec{b}$, if they have (i) the same length (ii) the same or parallel support, and (iii) the same sense.***

23.3 TYPES OF VECTORS

ZERO OR NULL VECTOR *A vector whose initial and terminal points are coincident is called the zero or the null vector.*

Thus, the modulus of the null vector is zero but it can be thought of as having any line as its line of support. The null vector is denoted by $\vec{0}$.

Vectors other than the null vector are called *proper vectors.*

UNIT VECTOR *A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector $\vec{a}$ is denoted by $\hat{a}$, read 'a cap'. Thus, $|\hat{a}| = 1$.*

LIKE AND UNLIKE VECTORS *Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.*

COLLINEAR OR PARALLEL VECTORS *Vectors having the same or parallel supports are called collinear vectors.*

CO-INITIAL VECTORS *Vectors having the same initial point are called co-initial vectors.*

CO-PLANAR VECTORS *A system of vectors is said to be coplanar, if their supports are parallel to the same plane.*

Note that two vectors are always coplanar.

COTERMINOUS VECTORS *Vectors having the same terminal point are called coterminous vectors.*

NEGATIVE OF A VECTOR *The vector which has the same magnitude as the vector $\vec{a}$ but opposite direction, is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$. Thus, if $\vec{PQ} = \vec{a}$, then $\vec{QP} = -\vec{a}$.*

RECIPROCAL OF A VECTOR *A vector having the same direction as that of a given vector $\vec{a}$ but magnitude equal to the reciprocal of the given vector is known as the reciprocal of $\vec{a}$ and is denoted by $\vec{a}^{-1}$. Thus, if $|\vec{a}| = a$, $|\vec{a}^{-1}| = \frac{1}{a}$.*

LOCALIZED AND FREE VECTORS *A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector. For example, a force acting on a rigid body is a localized vector as its effect depends on the line of action of the force. If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector.*

In this chapter we will be dealing with free vectors, unless otherwise stated. Thus a free vector can be taken anywhere in space by choosing an arbitrary initial point.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Represent graphically (i) a displacement of 40 km, 30° west of south. (ii) 60 km, 40° east of north (iii) 50 km south-east.* [NCERT]

SOLUTION (i) The vector $\vec{OP}$ represents the required displacement vector.

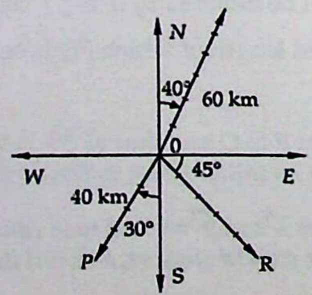


Fig. 23.2

(ii) The vector $\overrightarrow{OQ}$ represents the required vector.

(iii) $\overrightarrow{OR}$ represents the required vector.

EXAMPLE 2 *Classify the following measures as scalars and vectors*

(i) 10 kg (ii) 10 meters noth-west (iii) 10 Newton

(iv) 30 km/hr (v) 50 m/sec towards north (vi) 10^{-19} coloumb. [NCERT]

SOLUTION (i) Mass-scalar (ii) Directed distance-vector (iii) Force-vector (iv) Speed-scalar (v) Velocity-vector (vi) Electric charge-vector.

EXAMPLE 3 *In Fig. 23.3, which of the vectors are:*

(i) *collinear* (ii) *Equal* (iii) *Co-initial* [NCERT]

SOLUTION Clearly,

(i) $\vec{a}$, $\vec{c}$ and $\vec{d}$ are collinear vectors.

(ii) $\vec{a}$ and $\vec{c}$ are equal vectors each of magnitude 2 units.

(iii) $\vec{b}$, $\vec{c}$ and $\vec{d}$ are co-initial vectors.

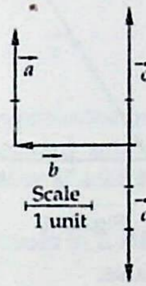


Fig. 23.3

EXAMPLE 4 *In Fig. 23.4 (a square), identify the following vectors:*

(i) Coinitial *(ii) Equal* *(iii) Collinear but not equal.* [NCERT]

SOLUTION Clearly,

(i) $\vec{a}$, $\vec{d}$ are co-initial vectors

(ii) $\vec{d}$ and $\vec{b}$ are equal vectors.

(iii) $\vec{a}$ and $\vec{c}$ are collinear but not equal vectors.

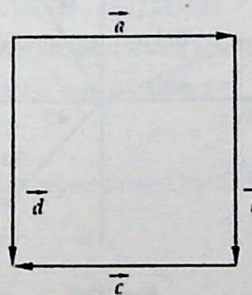


Fig. 23.4

EXERCISE 23.1

1. Represent the following graphically:
   (i) a displacement of 40 km, 30° east of north [NCERT]
   (ii) a displacement of 50 km south-west

(iii) a displacement of 70 km, 40° north of west.

2. Classify the following measures as scalars and vectors:
   (i) 15 kg (ii) 20 kg weight (iii) 45° [NCERT]
   (iv) 10 meters south-east (v) 50 m/sec^2

3. Classify the following as scalars and vector quantities:
   (i) time period (ii) distance (iii) displacement
   (iv) force (v) Work (vi) Velocity [NCERT]
   (vii) Acceleration

4. In Fig. 23.5, which vectors are:
   (i) Collinear (ii) Equal (iii) Coinitial
   (iv) Collinear but not equal.

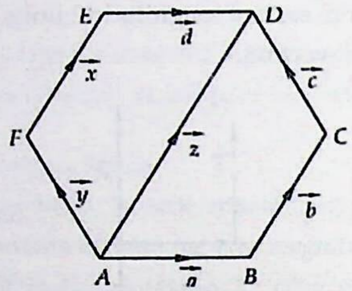


Fig. 23.5

5. Answer the following as true or false:
   (i) $\vec{a}$ and $\vec{a}$ are collinear.
   (ii) Two collinear vectors are always equal in magnitude. [NCERT]
   (iii) Zero vector is unique.
   (iv) Two vectors having same magnitude are collinear.
   (v) Two collinear vectors having the same magnitude are equal.

ANSWERS

1.

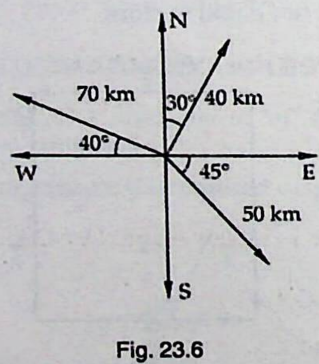


Fig. 23.6

2. (i) Mass — scalar (ii) Weight (Force) —vector (iii) Angle — scalar
   (iv) Directed Distance — vector (v) Magnitude of acceleration —scalar.

3. (i) scalar (ii) scalar (iii) vector (iv) vector
   (v) scalar (vi) vector (vii) vector.

4. (i) $\vec{a}, \vec{d}$; $\vec{b}, \vec{x}, \vec{z}$; $\vec{c}, \vec{y}$ (ii) $\vec{b}, \vec{x}$; $\vec{a}, \vec{d}$; $\vec{c}, \vec{y}$
(iii) $\vec{a}, \vec{y}, \vec{z}$; $\vec{x}, \vec{d}$ (iv) $\vec{b}, \vec{z}$; $\vec{x}, \vec{z}$

5. (i) T (ii) F (iii) F (iv) F (v) F

23.4 PARALLELOGRAM LAW OF ADDITION OF VECTORS

If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two adjacent sides of a parallelogram, then their sum $\vec{c}$ is represented by the diagonal of the parallelogram which is coinitial with the given vectors.

Symbolically, we have

$$\vec{OP} + \vec{OQ} = \vec{OR} \quad \text{or,} \quad \vec{a} + \vec{b} = \vec{c}$$

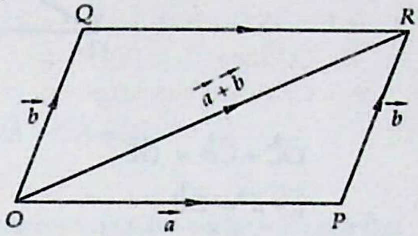


Fig. 23.7

From Fig. 23.7, we have

$$\vec{PR} = \vec{OQ} = \vec{b}.$$

Therefore, in triangle OPR,

$$\vec{OP} + \vec{PR} = \vec{OR}.$$

Thus, it follows that if two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order. This is called the triangle law of addition of vectors.

Using triangle law of addition of vectors in ΔABC, we have

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\vec{BC} + \vec{CA} = \vec{BA}$$

and, $$\vec{BA} + \vec{AC} = \vec{BC}$$

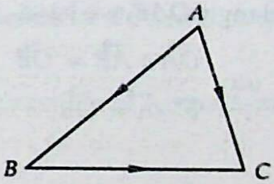


Fig. 23.8

NOTE *It should be noted that the magnitude of $\vec{a} + \vec{b}$ is not equal to the sum of the magnitudes of $\vec{a}$ and $\vec{b}$.*

23.5 PROPERTIES OF ADDITION OF VECTORS

In this section, we shall learn some properties of addition of vectors.

Commutativity : For any two vectors $\vec{a}$ and $\vec{b}$, we have

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

PROOF Let the vectors $\vec{a}$ and $\vec{b}$ be represented by $\vec{OA}$ and $\vec{AB}$ respectively.

Complete the parallelogram $OABC$.

We have, $\vec{CB} = \vec{OA} = \vec{a}$ and $\vec{OC} = \vec{AB} = \vec{b}$

In $\Delta\, OAB$, we have

$$\vec{OA} + \vec{AB} = \vec{OB} \qquad \text{[By triangle law of add.]}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \vec{OB} \qquad \text{...(i)}$$

In $\Delta\, OCB$, we have

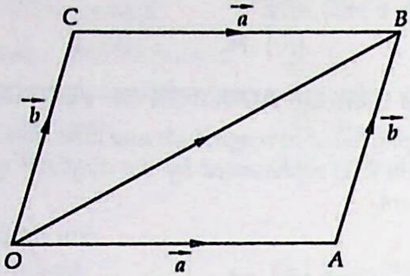


Fig. 23.9

$$\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB} \qquad \text{[By triangle law of add.]}$$

$$\vec{b} + \vec{a} = \overrightarrow{OB} \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

(ii) *Associativity : For any three vectors $\vec{a}, \vec{b}, \vec{c}$, we have*

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

PROOF Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be represented by $\overrightarrow{OA}$, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively. Join O, B; O, C and A, C.

In triangle OAB, we have

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \qquad \text{[By triangle law of add.]}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \overrightarrow{OB} \qquad \text{...(i)}$$

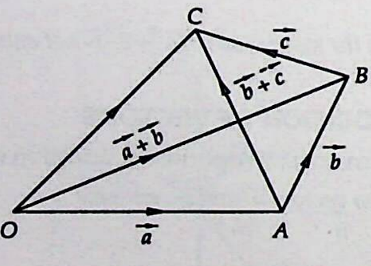


Fig. 23.10

In ΔOBC,

$$\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC} \qquad \text{[By triangle law of add.]}$$

$$\Rightarrow \quad (\vec{a} + \vec{b}) + \vec{c} = \overrightarrow{OC} \qquad \text{[Using (i)]} \qquad \text{...(ii)}$$

In ΔABC, we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\Rightarrow \quad \vec{b} + \vec{c} = \overrightarrow{AC} \qquad \text{...(iii)}$$

In ΔOAC, we have

$$\vec{OA} + \vec{AC} = \vec{OC}$$

$$\Rightarrow \quad \vec{a} + (\vec{b} + \vec{c}) = \vec{OC} \quad \text{...(iv)} \quad \text{[Using (iii)]}$$

From (ii) and (iv), we have

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

(iii) *Existence of additive identity : For every vector $\vec{a}$, we have*

$$\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}, \text{ where } \vec{0} \text{ is the null vector.}$$

PROOF Let $\vec{OA} = \vec{a}$. Then,

$$\vec{a} + \vec{0} = \vec{OA} + \vec{AA} = \vec{OA} \text{ and } \vec{0} + \vec{a} = \vec{OO} + \vec{OA} = \vec{OA} = \vec{a}.$$

Hence, $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$.

(iv) *Existence of additive inverse : For every vector $\vec{a}$, there corresponds a vector $-\vec{a}$ such that*

$$\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$$

PROOF Let $\vec{OA} = \vec{a}$. Then, $\vec{AO} = -\vec{a}$

$$\therefore \quad \vec{a} + (-\vec{a}) = \vec{OA} + \vec{AO} = \vec{OO} = \vec{0} \quad \text{[By triangle law of add.]}$$

and, $$(-\vec{a}) + \vec{a} = \vec{AO} + \vec{OA} = \vec{AA} = \vec{0} \quad \text{[By triangle law of add.]}$$

Hence, $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

23.5 SUBTRACTION OF VECTORS

If $\vec{a}$ and $\vec{b}$ are two vectors, then the subtraction of $\vec{b}$ from $\vec{a}$ is defined as the vector sum of $\vec{a}$ and $-\vec{b}$ and it is denoted by $\vec{a} - \vec{b}$ i.e., $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

Thus, to subtract vector $\vec{b}$ from vector $\vec{a}$, we reverse the direction of vector $\vec{b}$ and add it to vector $\vec{a}$ as shown in Fig. 23.11.

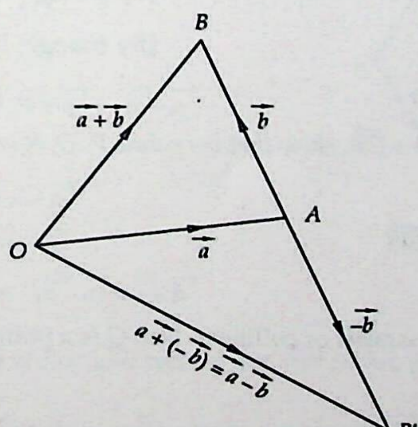


Fig. 23.11

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $\vec{a}$, $\vec{b}$, $\vec{c}$ be the vectors represented by the sides of a triangle, taken in order, then prove that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.*

SOLUTION Let ABC be a triangle such that $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$. Then,

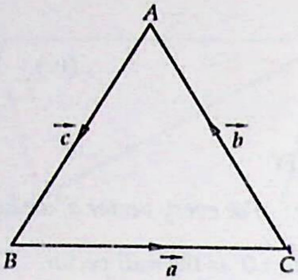


Fig. 23.12

$$\vec{a}+\vec{b}+\vec{c} = \overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}$$

$\Rightarrow \quad \vec{a}+\vec{b}+\vec{c} = \overrightarrow{BA}+\overrightarrow{AB}$ $\quad [\because \overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{BA}]$

$\Rightarrow \quad \vec{a}+\vec{b}+\vec{c} = \overrightarrow{BB}$ $\quad$ [By triangle law]

$\Rightarrow \quad \vec{a}+\vec{b}+\vec{c} = \vec{0}$ $\quad$ [By def. of null vector]

Hence, $\vec{a}+\vec{b}+\vec{c} = \vec{0}$.

EXAMPLE 2 *If P_1, P_2, P_3, P_4 are points in a plane or space and O is the origin of vectors, show that P_4 coincides with O iff $\overrightarrow{OP_1}+\overrightarrow{P_1P_2}+\overrightarrow{P_2P_3}+\overrightarrow{P_3P_4}=\vec{0}$.*

SOLUTION We have,

$$\overrightarrow{OP_1}+\overrightarrow{P_1P_2}+\overrightarrow{P_2P_3}+\overrightarrow{P_3P_4}=\vec{0}$$

$\Rightarrow \quad (\overrightarrow{OP_1}+\overrightarrow{P_1P_2})+\overrightarrow{P_2P_3}+\overrightarrow{P_3P_4}=\vec{0}$ $\quad$ [By assoc. of vector addition]

$\Rightarrow \quad (\overrightarrow{OP_2}+\overrightarrow{P_2P_3})+\overrightarrow{P_3P_4}=\vec{0}$ $\quad$ [By triangle law $\overrightarrow{OP_1}+\overrightarrow{P_1P_2}=\overrightarrow{OP_2}$]

$\Rightarrow \quad \overrightarrow{OP_3}+\overrightarrow{P_3P_4}=\vec{0}$ $\quad$ [By triangle law $\overrightarrow{OP_2}+\overrightarrow{P_2P_3}=\overrightarrow{OP_3}$]

$\Rightarrow \quad \overrightarrow{OP_4}=\vec{0}$ $\quad$ [By triangle law $\overrightarrow{OP_3}+\overrightarrow{P_3P_4}=\overrightarrow{OP_4}$]

$\Rightarrow \quad P_4$ coincides with O $\quad$ [By def. of null vector]

EXAMPLE 3 *If $\overrightarrow{PO}+\overrightarrow{OQ}=\overrightarrow{QO}+\overrightarrow{OR}$, show that the points P, Q, R are collinear.*

SOLUTION We have,

$$\overrightarrow{PO}+\overrightarrow{OQ}=\overrightarrow{QO}+\overrightarrow{OR}$$

$\Rightarrow \quad \overrightarrow{PQ}=\overrightarrow{QR}$ $\quad$ [By triangle law]

Thus, $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are either parallel or collinear. But, Q is a point common to them.

So, $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are collinear.

Hence, points P, Q, R are collinear.

EXAMPLE 4 *If $\vec{a}, \vec{b}$ are any two vectors, then give the geometrical interpretation of the relation*

$$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$$

SOLUTION Let $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{AB}=\vec{b}$

Complete the parallelogram OACB. Then, $\overrightarrow{OC}=\vec{b}$ and $\overrightarrow{CB}=\vec{a}$.

In ΔOAB, we have

$$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$$

$\Rightarrow \quad \vec{a}+\vec{b}=\overrightarrow{OB}$ ...(i)

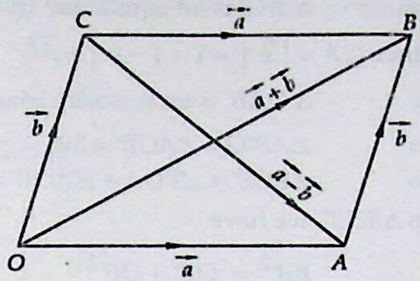


Fig. 23.13

In ΔOCA, we have

$$\overrightarrow{OC}+\overrightarrow{CA}=\overrightarrow{OA}$$

$\Rightarrow \quad \vec{b}+\overrightarrow{CA}=\vec{a}$

$\Rightarrow \quad \overrightarrow{CA}=\vec{a}-\vec{b}$ ...(ii)

Now, $\quad |\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$

$\Rightarrow \quad |\overrightarrow{OB}|=|\overrightarrow{CA}|$

$\Rightarrow \quad OB=CA$

$\Rightarrow$ Diagonals of parallelogram $OACB$ are equal

$\Rightarrow$ $OABC$ is a rectangle

$\Rightarrow \quad OA\perp OC$

$\Rightarrow \quad \vec{a}\perp\vec{b}.$

EXAMPLE 5. *If $\vec{a}$ and $\vec{b}$ are the vectors determined by two adjacent sides of a regular hexagon, what are the vectors determined by the other sides taken in order ?*

SOLUTION Let $ABCDEF$ be a regular hexagon such that $\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{BC}=\vec{b}.$

Since AD is parallel to BC such that $AD=2\,BC$.

$\therefore \quad \overrightarrow{AD}=2\,\overrightarrow{BC}=2\,\vec{b}$

In ΔABC, we have

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC} \qquad \text{[By triangle law of add.]}$$

$\Rightarrow \quad \vec{a}+\vec{b}=\overrightarrow{AC}$

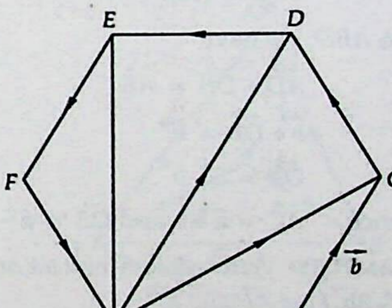


Fig. 23.14

In $\Delta\, ACD$, we have

$$\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$$

$\Rightarrow \quad \overrightarrow{CD}=\overrightarrow{AD}-\overrightarrow{AC}$

$\Rightarrow \quad \overrightarrow{CD}=2\vec{b}-(\vec{a}+\vec{b})=\vec{b}-\vec{a}$

Clearly,

$$\overrightarrow{DE}=-\overrightarrow{AB}=-\vec{a},$$

$$\overrightarrow{EF}=-\overrightarrow{BC}=-\vec{b},$$

and, $\quad \overrightarrow{FA}=-\overrightarrow{CD}=-(\vec{b}-\vec{a})=\vec{a}-\vec{b}$

EXAMPLE 6 *If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.*

SOLUTION Let $\hat{a}$ and $\hat{b}$ be two unit vectors represented by sides OA and AB of a triangle OAB. Then, $\overrightarrow{OA}=\hat{a}$, $\overrightarrow{AB}=\hat{b}$ and

$$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\hat{a}+\hat{b} \qquad \text{[By triangle law of add.]}$$

It is given that

$$|\hat{a}|=|\hat{b}|=|\hat{a}+\hat{b}|=1$$

$\Rightarrow \quad OA = AB = OB = 1$

$\Rightarrow \quad \Delta OAB$ is an equilateral triangle

Since $OA = |\hat{a}| = 1 = |-\hat{b}| = \overrightarrow{AB'}$

$\therefore \quad \Delta\, OAB'$ is an isosceles triangle

$\Rightarrow \quad \angle AB'O = \angle AOB' = 30^\circ$

$\Rightarrow \quad \angle BOB' = \angle BOA + \angle AOB' = 60^\circ + 30^\circ = 90^\circ$

In $\Delta BOB'$, we have

$$BB'^2 = OB^2 + OB'^2$$

$$\Rightarrow \quad 2^2 = |\hat{a}+\hat{b}|^2 + |\hat{a}-\hat{b}|^2$$

$$\Rightarrow \quad 2^2 = 1^2 + |\hat{a}-\hat{b}|^2$$

$$\Rightarrow \quad |\hat{a}-\hat{b}| = \sqrt{3}.$$

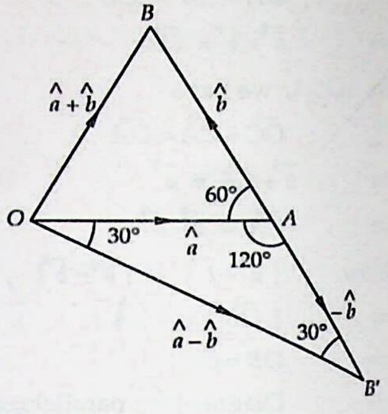


Fig. 23.15

EXAMPLE 7 *If $\vec{a}$ and $\vec{b}$ represent two adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively of a parallelogram ABCD, then show that its diagonals $\overrightarrow{AC}$ and $\overrightarrow{DB}$ are equal to $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ respectively.*

SOLUTION Since $ABCD$ is a parallelogram.

$\therefore \quad AB = DC$ and $AD = BC$

$\Rightarrow \quad \overrightarrow{DC} = \overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \overrightarrow{BC} = \vec{b}$

In $\Delta\, ABC$, we have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\Rightarrow \quad \vec{a}+\vec{b} = \overrightarrow{AC}$$

In $\Delta\, ABD$, we have

$$\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$$

$$\Rightarrow \quad \vec{b} + \overrightarrow{DB} = \vec{a}$$

$$\Rightarrow \quad \overrightarrow{DB} = \vec{a}-\vec{b}$$

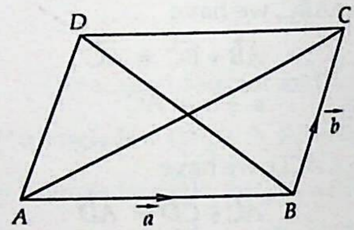


Fig. 23.16

Hence, $\overrightarrow{AC} = \vec{a}+\vec{b}$ and $\overrightarrow{DB} = \vec{a}-\vec{b}$

EXAMPLE 8 *Vectors drawn from the origin O to the points A, B and C are respectively $\vec{a}$, $\vec{b}$ and $4\vec{a}-3\vec{b}$. Find $\overrightarrow{AC}$ and $\overrightarrow{BC}$.*

SOLUTION It is given that $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = 4\vec{a}-3\vec{b}$

In ΔOAC, we have

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}.$$

$$\Rightarrow \quad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$\Rightarrow \quad \overrightarrow{AC} = 4\vec{a}-3\vec{b}-\vec{a} = 3\vec{a}-3\vec{b} = 3(\vec{a}-\vec{b})$$

In ΔOBC, we have

$$\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}$$

$$\Rightarrow \quad \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$\Rightarrow \quad \overrightarrow{BC} = 4\vec{a}-3\vec{b}-\vec{b} = 4(\vec{a}-\vec{b})$$

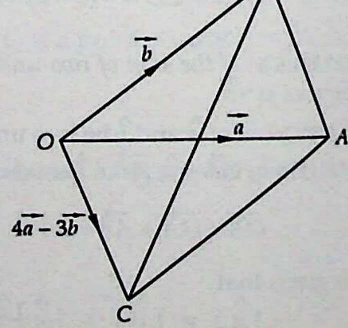


Fig. 23.17

EXAMPLE 9 *If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are distinct non-zero vectors represented by directed line segments from the origin to the points A, B, C and D respectively, and if $\vec{b}-\vec{a}=\vec{c}-\vec{d}$, then prove that ABCD is a parallelogram.*

SOLUTION Let O be the origin.

We have,

$$\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}, \overrightarrow{OC}=\vec{c} \text{ and } \overrightarrow{OD}=\vec{d} \text{ such that}$$

$$\vec{b}-\vec{a}=\vec{c}-\vec{d}$$

$$\Rightarrow \quad \overrightarrow{OB}-\overrightarrow{OA}=\overrightarrow{OC}-\overrightarrow{OD} \quad \text{...(i)}$$

In ΔOAB, we have

$$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$$

$$\Rightarrow \quad \overrightarrow{OB}-\overrightarrow{OA}=\overrightarrow{AB} \quad \text{...(ii)}$$

In ΔOCD, we have

$$\overrightarrow{OC}+\overrightarrow{CD}=\overrightarrow{OD}$$

$$\Rightarrow \quad \overrightarrow{OC}-\overrightarrow{OD}=-\overrightarrow{CD}$$

$$\Rightarrow \quad \overrightarrow{OC}-\overrightarrow{OD}=\overrightarrow{DC} \quad \text{...(iii)}$$

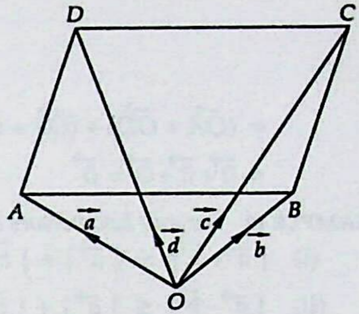


Fig. 23.18

From (i), (ii) and (iii), we get

$$\overrightarrow{AB}=\overrightarrow{DC}$$

Hence, $ABCD$ is a parallelogram.

EXAMPLE 10 *A B, P, Q and R are five points in a plane. Show that the sum of the vectors $\overrightarrow{AP}, \overrightarrow{AQ}, \overrightarrow{AR}, \overrightarrow{PB}, \overrightarrow{QB}$ and $\overrightarrow{RB}$ is 3 $\overrightarrow{AB}$.*

SOLUTION In Δ s, APB, AQB and ARB, we have

$$\overrightarrow{AP}+\overrightarrow{PB}=\overrightarrow{AB}$$

$$\overrightarrow{AQ}+\overrightarrow{QB}=\overrightarrow{AB}$$

and, $$\overrightarrow{AR}+\overrightarrow{RB}=\overrightarrow{AB}$$

Adding all these, we get

$$\overrightarrow{AP}+\overrightarrow{PB}+\overrightarrow{AQ}+\overrightarrow{QB}+\overrightarrow{AR}+\overrightarrow{RB}=3\,\overrightarrow{AB}$$

Hence, the sum of the given vectors is 3 $\overrightarrow{AB}$.

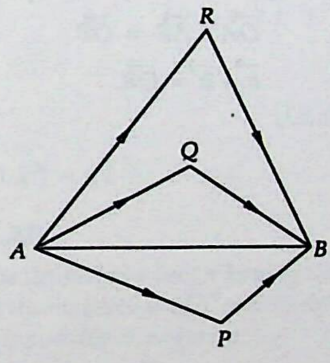


Fig. 23.19

EXAMPLE 11 *Let O be the centre of a regular hexagon ABCDEF. Find the sum of the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}, \overrightarrow{OE}$ and $\overrightarrow{OF}$.*

SOLUTION We know that the centre of a regular hexaong bisects all the diagonals passing through it.

$$\therefore \quad \overrightarrow{OA}=-\overrightarrow{OD}, \overrightarrow{OB}=-\overrightarrow{OE} \text{ and } \overrightarrow{OC}=-\overrightarrow{OF}$$

$$\Rightarrow \quad \overrightarrow{OA}+\overrightarrow{OD}=\vec{0} \text{ and } \overrightarrow{OB}+\overrightarrow{OE}=\vec{0}$$

Hence,

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}+\overrightarrow{OE}+\overrightarrow{OF}$$

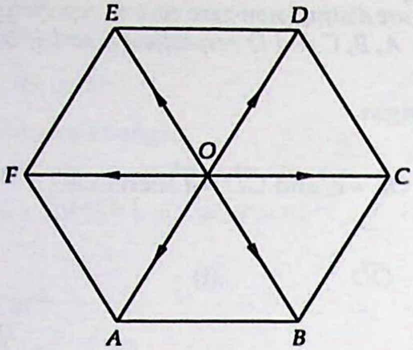


Fig. 23.20

$$= (\vec{OA} + \vec{OD}) + (\vec{OB} + \vec{OE}) + (\vec{OC} + \vec{OF})$$

$$= \vec{0} + \vec{0} + \vec{0} = \vec{0}$$

EXAMPLE 12 *For any two vectors $\vec{a}$ and $\vec{b}$, prove that*

(i) $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

(ii) $|\vec{a} - \vec{b}| \leq |\vec{a}| + |\vec{b}|$

(iii) $|\vec{a} - \vec{b}| \geq |\vec{a}| - |\vec{b}|$

SOLUTION (i) We have following cases:

CASE I *When $\vec{a}, \vec{b}$ are non-collinear vectors:*

Let the vectors $\vec{a}$ and $\vec{b}$ be represented by sides $\vec{OA}$ and $\vec{AB}$ of a triangle OAB. Then,

$$\vec{OA} + \vec{AB} = \vec{OB} \qquad \text{[By triangle law of add.]}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \vec{OB}.$$

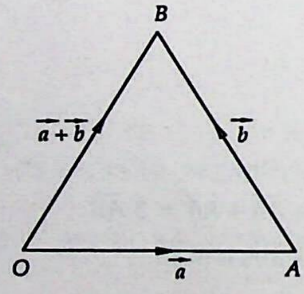


Fig. 23.21

Clearly, $|\vec{a}| = OA$, $|\vec{b}| = AB$ and $|\vec{a} + \vec{b}| = OB$.

Since the sum of two sides of a triangle is always greater than the third side. Therefore, in ΔOAB, we have

$$OA + AB > OB$$

$$\Rightarrow \quad |\vec{OB}| < |\vec{OA}| + |\vec{AB}|$$

$$\Rightarrow \quad |\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}|$$

CASE II *When $\vec{a}, \vec{b}$ are collinear vectors:*

Let $\vec{OA} = \vec{a}$, $\vec{AB} = \vec{b}$. Then,

$$\vec{a} + \vec{b} = \vec{OA} + \vec{AB} = \vec{OB} \qquad \text{[By triangle law of add.]}$$

Clearly, $OA = |\vec{a}|$, $AB = |\vec{b}|$ and, $OB = |\vec{a} + \vec{b}|$

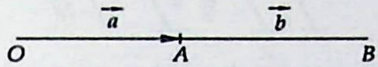


Fig. 23.22

Now, $\quad OB = OA + AB$

$\Rightarrow \quad |\vec{OB}| = |\vec{OA}| + |\vec{AB}|$

$\Rightarrow \quad |\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$

Hence, in general, $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

(ii) We have,

$$|\vec{a} - \vec{b}| = |\vec{a} + (-\vec{b})| \le |\vec{a}| + |-\vec{b}| \qquad \text{[Using (i)]}$$

$\Rightarrow \quad |\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}| \qquad [\because |-\vec{b}| = |\vec{b}|]$

(iii) We have,

$$|\vec{a}| = |\vec{a} - \vec{b} + \vec{b}| \le |\vec{a} - \vec{b}| + |\vec{b}| \qquad \text{[Using (i)]}$$

$\Rightarrow \quad |\vec{a}| - |\vec{b}| \le |\vec{a} - \vec{b}|$ or, $|\vec{a} - \vec{b}| \ge |\vec{a}| - |\vec{b}|$

23.7 MULTIPLICATION OF A VECTOR BY A SCALAR

DEFINITION *Let m be a scalar and $\vec{a}$ be a vector, then $m\vec{a}$ is defined as a vector having the same support as that of $\vec{a}$ such that its magnitude is $|m|$ times the magnitude of $\vec{a}$ and its direction is same as or opposite to the direction of $\vec{a}$ according as m is positive or negative.*

From the above definition it is evident that

$$\vec{a} = |\vec{a}|\,\hat{a} \Rightarrow \hat{a} = \frac{1}{|\vec{a}|}\vec{a}$$

It is also evident that two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel iff $\vec{a} = m\vec{b}$ for some non-zero scalar m.

For any vector $\vec{a}$, we also define

$$1\vec{a} = \vec{a}, (-1)\vec{a} = -\vec{a} \text{ and } 0\vec{a} = \vec{0}$$

REMARK *If $\vec{a}$ is a vector, then $3\vec{a}$ is a vector whose magnitude is 3 times the magnitude of $\vec{a}$ and whose direction is same as that of $\vec{a}$. Also, $-3\vec{a}$ is a vector whose magnitude is 3 times the magnitude of $\vec{a}$ and whose direction is opposite to that of $\vec{a}$.*

A geometric visualization of multiplication of a vector by a scalar is given in Fig. 23.23.

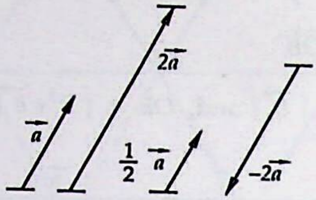


Fig. 23.23

23.7.1 PROPERTIES OF MULTIPLICATION OF VECTORS BY A SCALAR

THEOREM *For vectors $\vec{a}, \vec{b}$ and scalars m, n, we have*

(i) $m(-\vec{a}) = (-m)\vec{a} = -(m\vec{a})$

(ii) $(-m)(-\vec{a}) = m\vec{a}$

(iii) $m(n\vec{a}) = (mn)\vec{a} = n(m\vec{a})$

(iv) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

(v) $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

PROOF (i) Recall that two vectors are equal if their magnitudes are equal and they have the same direction.

CASE I *When* $m > 0$.

We have,

$$| m(-\vec{a}) | = | m | \, | -\vec{a} | = | m | \, | \vec{a} | = m \, | \vec{a} | \quad [\because m > 0 \therefore | m | = m]$$

$$| (-m)\vec{a} | = | -m | \, | \vec{a} | = | m | \, | \vec{a} | = m \, | \vec{a} | \quad [\because m > 0 \therefore | m | = m]$$

and, $$| (-m\vec{a}) | = | m\vec{a} | = | m | \, | \vec{a} | = m \, | \vec{a} | \quad [\because m > 0 \therefore | m | = m]$$

Thus, $m(-\vec{a}), (-m)\vec{a}$ and $-(m\vec{a})$ are vectors of equal magnitude.

Also, they have the same direction which is opposite to that of $\vec{a}$.

Hence, $m(-\vec{a}) = (-m)\vec{a} = -(m\vec{a})$.

CASE II *When* $m < 0$.

We have,

$$| m(-\vec{a}) | = | m | \, | -\vec{a} | = | m | \, | \vec{a} | = -m \, | \vec{a} | \quad \begin{bmatrix} \because m < 0 \\ \therefore | m | = -m \end{bmatrix}$$

$$| (-m)\vec{a} | = | -m | \, | \vec{a} | = | m | \, | \vec{a} | = -m \, | \vec{a} | \quad \begin{bmatrix} \because m < 0 \\ \therefore | m | = -m \end{bmatrix}$$

and,

$$| -(m\vec{a}) | = | m\vec{a} | = | m | \, | \vec{a} | = -m \, | \vec{a} | \quad [\because m < 0 \therefore | m | = -m]$$

Thus, $m(-\vec{a}), (-m)\vec{a}$ and $-(m\vec{a})$ are vectors of equal magnitude.

The direction of $m(-\vec{a})$ is opposite to that of $-\vec{a}$ and therefore, it is same as that of $\vec{a}$.

Similarly, $(-m)\vec{a}$ and $-(m\vec{a})$ have the same direction as that of $\vec{a}$.

Hence, $m(-\vec{a}) = (-m)\vec{a} = -(m\vec{a})$.

(ii) We have,

$$(-m)(-\vec{a}) = (-m)\vec{b}, \text{ where } \vec{b} = -\vec{a}$$

$\Rightarrow \quad (-m)(-\vec{a}) = -(m\vec{b}) \qquad [\because (-m)\vec{b} = -(m\vec{b})]$

$\Rightarrow \quad (-m)(-\vec{a}) = -[m(-\vec{a})] = -[-(m\vec{a})] \qquad [\because m(-\vec{a}) = -(m\vec{a})]$

$\Rightarrow \quad (-m)(-\vec{a}) = m\vec{a} \qquad [\because -(-\vec{a}) = \vec{a}]$

Hence, $(-m)(-\vec{a}) = m\vec{a}$

(iii) <u>CASE I</u> *When* $mn > 0$.

In this case, $m(n\vec{a})$ is a vector of magnitude $mn\,|\vec{a}|$ and its direction is same as that of $\vec{a}$. Also, $(mn)\vec{a}$ is a vector of magnitude $mn\,|\vec{a}|$ and its direction is same as that of $\vec{a}$.

$$\therefore \quad m(n\vec{a}) = (mn)\vec{a}$$

<u>CASE II</u> *When* $mn < 0$.

In this case, $m(n\vec{a})$ is a vector of magnitude $|mn|\,|\vec{a}|$ such that its direction is opposite to that of $\vec{a}$.

Also, $(mn)\vec{a}$ is a vector whose direction is opposite to that of $\vec{a}$ and its magnitude is $|mn|\,|\vec{a}|$.

$$\therefore \quad m(n\vec{a}) = (mn)\vec{a}$$

Thus, $m(n\vec{a}) = (mn)\vec{a}$ in both the cases.

Similarly, $n(m\vec{a}) = (mn)\vec{a}$

Hence, $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iv) <u>CASE I</u> *When* $(m+n) > 0$

In this case, $(m+n)\vec{a}$ is a vector of magnitude $(m+n)\,|\vec{a}|$ and its direction is same as that of $\vec{a}$.

Since $m\vec{a}$ and $n\vec{a}$ are collinear vectors, therefore the magnitude of $m\vec{a} + n\vec{a}$ is $(m+n)$ times that of $\vec{a}$. The direction of $m\vec{a} + n\vec{a}$ is clearly same as that of $\vec{a}$.

$$\therefore \quad (m+n)\vec{a} = m\vec{a} + n\vec{a}$$

<u>CASE II</u> *When* $(m+n) < 0$

In this case, $(m+n)\vec{a}$ is a vector of magnitude $|m+n|\,|\vec{a}|$ and the direction is opposite to that of $\vec{a}$. Also, $m\vec{a}$ and $n\vec{a}$ being collinear vectors, the magnitude of $m\vec{a} + n\vec{a}$ is $|m+n|\,|\vec{a}|$ and the direction is opposite to that of $\vec{a}$.

$$\therefore \quad (m+n)\vec{a} = m\vec{a} + n\vec{a}$$

Hence, in both the cases, we have

$$(m+n)\vec{a} = m\vec{a} + n\vec{a}$$

(v) Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$. Then,

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$$

<u>CASE I</u> *When* $m > 1$.

Produce OA to C such that $OC = m.\,OA$ and draw CD parallel to AB, meeting OB produced at D.

Clearly triangles OAB and OCD are similar.

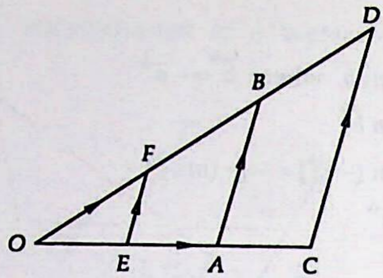


Fig. 23.24

$\therefore \quad \frac{OD}{OB} = \frac{CD}{AB} = \frac{OC}{OA}$

$\Rightarrow \quad \frac{OD}{OB} = \frac{CD}{AB} = m \qquad [\because OC = m\, OA]$

$\Rightarrow \quad OD = m \cdot OB, \; CD = m \cdot AB$ and $OC = m \cdot OA$

$\Rightarrow \quad \vec{OD} = m \cdot (\vec{OB}), \; \vec{CD} = m\,(\vec{AB})$ and $\vec{OC} = m\,(\vec{OA})$

Now, by triangle law of addition of vectors, we have

$$\vec{OD} = \vec{OC} + \vec{CD}$$

$\Rightarrow \quad m\,(\vec{OB}) = m\,(\vec{OA}) + m\,(\vec{AB})$

$\Rightarrow \quad m\,(\vec{a} + \vec{b}) = m\,\vec{a} + m\,\vec{b}.$

<u>CASE II</u> *When* $0 < m < 1$.

Take a point E on OA such that $OE = m\,(OA)$ and draw EF parallel to AB, meeting OB at F. Clearly, triangles OAB and OEF are similar.

$\therefore \quad \frac{OE}{OA} = \frac{EF}{AB} = \frac{OF}{OB}$

$\Rightarrow \quad \frac{OF}{OB} = \frac{EF}{AB} = m \qquad [\because OE = m \cdot OA]$

$\Rightarrow \quad OF = m \cdot OB, \; EF = m \cdot AB$ and $OE = m \cdot OA$

$\Rightarrow \quad \vec{OF} = m\,(\vec{OB}), \; \vec{EF} = m\,(\vec{AB})$ and $\vec{OE} = m\,(\vec{OA})$

Now, $\quad \vec{OF} = \vec{OE} + \vec{EF} \qquad$ [By triangle law of addition]

$\Rightarrow \quad m\,(\vec{OB}) = m\,(\vec{OA}) + m\,(\vec{AB})$

$\Rightarrow \quad m\,(\vec{a} + \vec{b}) = m\,\vec{a} + m\,\vec{b}$

<u>CASE III</u> *When* $m < 0$.

Take a point C on AO produced such that $OC = |\,m\,| \,.\, OA$. From C draw CD parallel to AB but in direction opposite to that of $\vec{AB}$. Now, produce BO to meet CD at D.

Clearly, triangles OAB and OCD are similar.

$\therefore \quad \frac{OD}{OB} = \frac{CD}{AB} = \frac{OC}{OA}$

$\Rightarrow \quad \frac{OD}{OB} = \frac{CD}{AB} = \frac{OC}{OA} = |\,m\,| \qquad [\because OC = m \cdot OA]$

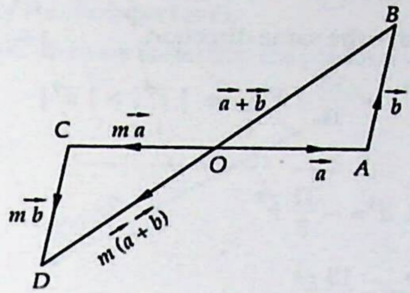


Fig. 23.25

$$\Rightarrow \quad \frac{|\vec{OD}|}{|\vec{OB}|} = \frac{|\vec{CD}|}{|\vec{AB}|} = \frac{|\vec{OC}|}{|\vec{OA}|} = |m|$$

$$\Rightarrow \quad |\vec{OD}| = |m|\,|\vec{OB}|,\ |\vec{CD}| = |m|\,|\vec{AB}|$$

and, $\quad |\vec{OC}| = |m|\,|\vec{OA}|$

$$\Rightarrow \quad \vec{OD} = m\cdot\vec{OB},\ \vec{CD} = m\cdot\vec{AB} \text{ and } \vec{OC} = m\cdot\vec{OA}$$

Now, $\quad \vec{OD} = \vec{OC} + \vec{CD}$ [By triangle law of addition]

$$\Rightarrow \quad m\cdot(\vec{OB}) = m(\vec{OA}) + m(\vec{AB}) \Rightarrow m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}.$$

CASE IV *When $m = 0$.*

In this case,

$$m.(\vec{a}+\vec{b}) = 0(\vec{a}+\vec{b}) = \vec{0} \text{ and } m\vec{a} + m\vec{b} = 0\vec{a} + 0\vec{b} = \vec{0}.$$

$$\therefore \quad m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}.$$

Hence, in all the cases, we have $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

ILLUSTRATION *If $\vec{c} = 3\vec{a} + 4\vec{b}$ and $2\vec{c} = \vec{a} - 3\vec{b}$, show that*

(i) $\vec{c}$ and $\vec{a}$ have the same direction and $|\vec{c}| > |\vec{a}|$

(ii) $\vec{c}$ and $\vec{b}$ have opposite direction and $|\vec{c}| > |\vec{b}|$

SOLUTION We have,

$$\vec{c} = 3\vec{a} + 4\vec{b} \text{ and } 2\vec{c} = \vec{a} - 3\vec{b}$$

$$\Rightarrow \quad 2(3\vec{a} + 4\vec{b}) = \vec{a} - 3\vec{b}$$

$$\Rightarrow \quad 6\vec{a} + 8\vec{b} = \vec{a} - 3\vec{b}$$

$$\Rightarrow \quad 5\vec{a} = -11\vec{b}$$

$$\Rightarrow \quad \vec{a} = -\frac{11}{5}\vec{b} \text{ and } \vec{b} = -\frac{5}{11}\vec{a}$$

(i) $\quad \vec{c} = 3\vec{a} + 4\vec{b}$

$$\Rightarrow \quad \vec{c} = 3\vec{a} + 4\left(-\frac{5}{11}\vec{a}\right) \qquad \left[\because \vec{b} = -\frac{5}{11}\vec{a}\right]$$

$\Rightarrow \qquad \vec{c} = 3\vec{a} - \frac{20}{11}\vec{a} = \frac{13}{11}\vec{a}$

This shows that $\vec{c}$ and $\vec{a}$ have the same direction.

and, $\quad \vec{c} = \frac{13}{11}\vec{a} \Rightarrow |\vec{c}| = \frac{13}{11}|\vec{a}| \Rightarrow |\vec{c}| > |\vec{a}| \qquad \left[\because \frac{13}{11}|\vec{a}| > |\vec{a}|\right]$

(ii) We have,

$$\vec{c} = 3\vec{a} + 4\vec{b} \text{ and } \vec{a} = -\frac{11}{5}\vec{b}$$

$$\therefore \quad \vec{c} = -\frac{33}{5}\vec{b} + 4\vec{b} = -\frac{13}{5}\vec{b}$$

This shows that $\vec{c}$ and $\vec{b}$ have opposite directions.

Also, $\quad |\vec{c}| = \left|-\frac{13}{5}\vec{b}\right| = \frac{13}{5}|\vec{b}| > |\vec{b}| \qquad \left[\because \frac{13}{5}|\vec{b}| > |\vec{b}|\right]$

EXERCISE 23.2

1. If P, Q and R are three collinear points such that $\vec{PQ} = \vec{a}$ and $\vec{QR} = \vec{b}$. Find the vector $\vec{PR}$.
2. Give a condition that three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ form the three sides of a triangle. What are the other possibilities ?
3. If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors having the same initial point. What are the vectors represented by $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.
4. If $\vec{a}$ is a vector and m is a scalar such that $m\vec{a} = \vec{0}$, then what are the alternatives for m and $\vec{a}$.
5. $ABCD$ is a quadrilateral. Find the sum the vectors $\vec{BA}$, $\vec{BC}$, $\vec{CD}$ and $\vec{DA}$.
6. $ABCDE$ is a pentagon, prove that
   (i) $\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{EA} = \vec{0}$
   (ii) $\vec{AB} + \vec{AE} + \vec{BC} + \vec{DC} + \vec{ED} + \vec{AC} = 3\,\vec{AC}$
7. Prove that the sum of all vectors drawn from the centre of a regular octagon to its vertices is the zero vector.
8. If P is a point and $ABCD$ is a quadrilateral and $\vec{AP} + \vec{PB} + \vec{PD} = \vec{PC}$, show that $ABCD$ is a parallelogram.
9. Five forces $\vec{AB}$, $\vec{AC}$, $\vec{AD}$, $\vec{AE}$ and $\vec{AF}$ act at the vertex of a regular hexagon $ABCDEF$. Prove that the resultant is $6\,\vec{AO}$, where O is the centre of hexagon.
10. If $\vec{a}, \vec{b}$ are two vectors, then write the truth value of the following statements :
    (i) $\vec{a} = -\vec{b} \Rightarrow |\vec{a}| = |\vec{b}|$ (ii) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm\vec{b}$
    (ii) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \vec{b}$

ANSWERS

1. $\vec{a} + \vec{b}$
2. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, Other possibilities are $\vec{a} + \vec{b} = \vec{c}$, $\vec{b} + \vec{c} = \vec{a}$ and $\vec{c} + \vec{a} = \vec{b}$
3. Diagonals of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.
4. Either $m = 0$ or, $\vec{a} = \vec{0}$ 5. $2\,\vec{BA}$

10. (i) T (ii) F (iii) F

23.8 POSITION VECTOR

DEFINITION *If a point O is fixed as the origin in space (or plane) and P is any point, then $\vec{OP}$ is called the position vector of P with respect to O.*

If we say that P is the point $\vec{r}$, then we mean that the position vector of P is $\vec{r}$ with respect to some origin O.

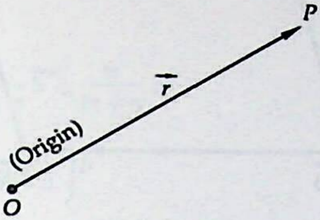


Fig. 23.26

23.8.1 A VECTOR IN TERMS OF POSITION VECTORS OF ITS END POINTS

Let $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B respectively. Then, $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$.

In ΔOAB, we have

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\Rightarrow \quad \vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$$

$\Rightarrow \quad \vec{AB}$ = (Position vector of B) – (Position vector of A)

$\Rightarrow \quad \vec{AB}$ = (Position vector of head) – (Position vector of tail)

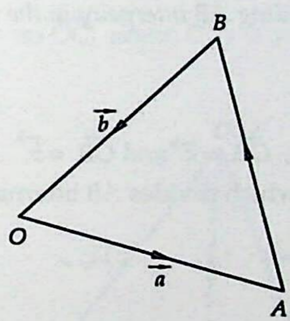


Fig. 23.27

ILLUSTRATION 1 *The position vectors of points A, B, C, D are $\vec{a}, \vec{b}, 2\vec{a}+3\vec{b}$ and $\vec{a}-2\vec{b}$ respectively. Show that $\vec{DB} = 3\vec{b}-\vec{a}$ and $\vec{AC} = \vec{a}+3\vec{b}$.*

SOLUTION We have,

$\vec{DB}$ = Position vector of B – Position vector of D

$$\Rightarrow \quad \vec{DB} = \vec{b} - (\vec{a} - 2\vec{b}) = 3\vec{b} - \vec{a}$$

and, $\vec{AC}$ = Position vector of C – Position vector of A

$$\Rightarrow \quad \vec{AC} = (2\vec{a} + 3\vec{b}) - \vec{a} = \vec{a} + 3\vec{b}$$

ILLUSTRATION 2 *Let ABCD be a parallelogram. If $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B, C respectively with reference to the origin O, find the position vector of D with reference to O.*

SOLUTION We have,

$$\vec{OA} = \vec{a}, \vec{OB} = \vec{b} \text{ and } \vec{OC} = \vec{c}$$

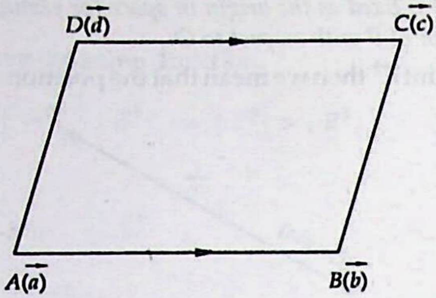


Fig. 23.28

Let $\vec{d}$ be the position vector of point D.

Since opposite sides of a parallelogram are parallel and equal.

$\therefore \quad \vec{AB} = \vec{DC}$

$\Rightarrow \quad \vec{b} - \vec{a} = \vec{c} - \vec{d}$

$\Rightarrow \quad \vec{d} = \vec{c} + \vec{a} - \vec{b}$

Hence, the position vector of D is $(\vec{c} + \vec{a} - \vec{b})$.

23.9 SECTION FORMULAE

THEOREM 1 (INTERNAL DIVISION) *Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively, and let C be a point dividing AB internally in the ration m : n. Then the position vector of C is given by*

$$\vec{OC} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

PROOF Let O be the Origin. Then, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Let $\vec{c}$ be the position vector of C which divides AB internally in the ratio $m : n$. Then,

$$\frac{AC}{CB} = \frac{m}{n}$$

$\Rightarrow \quad n \cdot AC = m \cdot CB$

$\Rightarrow \quad n\,\vec{AC} = m\,\vec{CB}$

$\Rightarrow \quad n(\vec{c} - \vec{a}) = m(\vec{b} - \vec{c})$

$\Rightarrow \quad n\vec{c} + m\vec{c} = m\vec{b} + n\vec{a}$

$\Rightarrow \quad (m+n)\vec{c} = m\vec{b} + n\vec{a}$

$\Rightarrow \quad \vec{c} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

$\Rightarrow \quad \vec{OC} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

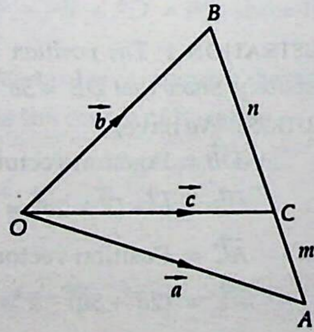


Fig. 23.29

Hence, the position vector of point C is $\dfrac{m\vec{b} + n\vec{a}}{m+n}$

REMARK 1 *If C is the mid point of AB, then it divides AB in the ratio 1 : 1. Therefore, position vector of C is given by*

$$\frac{1\cdot\vec{a}+1\cdot\vec{b}}{1+1}=\frac{\vec{a}+\vec{b}}{2}$$

Thus, the position vector of the mid point of AB is $\frac{1}{2}(\vec{a}+\vec{b})$

REMARK 2 *We have,*

$$\vec{c}=\frac{m\vec{b}+n\vec{a}}{m+n}=\frac{m}{m+n}\vec{b}+\frac{n}{m+n}\vec{a}$$

$$\Rightarrow \quad \vec{c}=\left(\frac{n}{m+n}\right)\vec{a}+\left(\frac{m}{m+n}\right)\vec{b}$$

$$\Rightarrow \quad \vec{c}=\lambda\vec{a}+\mu\vec{b}, \text{ where } \lambda=\frac{n}{m+n}, \mu=\frac{m}{m+n}$$

Thus, position vector of any point C on AB can always be taken as

$$\vec{c}=\lambda\vec{a}+\mu\vec{b}, \text{ where } \lambda+\mu=1$$

REMARK 3 We have

$$\vec{c}=\frac{m\vec{b}+n\vec{a}}{m+n}$$

$$\Rightarrow \quad (m+n)\vec{c}=m\vec{b}+n\vec{a}$$

$$\Rightarrow \quad n\cdot\overrightarrow{OA}+m\cdot\overrightarrow{OB}=(n+m)\overrightarrow{OC}$$

Thus, $n\cdot\overrightarrow{OA}+m\cdot\overrightarrow{OB}=(n+m)\overrightarrow{OC}$*, where C is a point on AB dividing it in the ratio* $m:n$

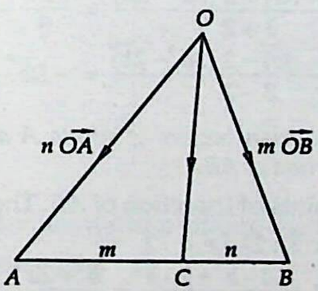


Fig. 23.30

REMARK 4 *It follows from the above remark that, if C is the mid point of AB, then*

$$1\cdot\overrightarrow{OA}+1\cdot\overrightarrow{OB}=(1+1)\overrightarrow{OC}$$

$$\Rightarrow \quad \overrightarrow{OA}+\overrightarrow{OB}=2\cdot\overrightarrow{OC}$$

THEOREM 2 (EXTERNAL DIVISION) *Let A and B be two points with positin vectors* $\vec{a}$ *and* $\vec{b}$ *respectively and let C be a point dividing AB externally in the ratio* $m:n$*. Then, the position vector of C is given by*

$$\overrightarrow{OC}=\frac{m\vec{b}-n\vec{a}}{m-n}$$

PROOF Let O be the origin. Then, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Let $\vec{c}$ be the position vector of a point C dividing AB externally in the ratio $m : n$. Then,

$$\frac{AC}{BC} = \frac{m}{n}$$

$$\Rightarrow \quad n \cdot AC = m \cdot BC$$

$$\Rightarrow \quad n\,\vec{AC} = m\,\vec{BC}$$

$$\Rightarrow \quad n(\vec{c} - \vec{a}) = m(\vec{c} - \vec{b})$$

$$\Rightarrow \quad m\vec{c} - n\vec{c} = m\vec{b} - n\vec{a}$$

$$\Rightarrow \quad (m - n)\,\vec{c} = m\vec{b} - n\vec{a}$$

$$\Rightarrow \quad \vec{c} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

$$\Rightarrow \quad \vec{OC} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

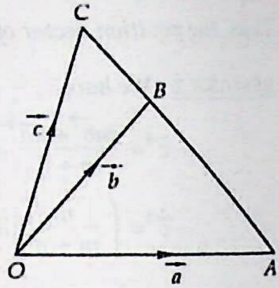


Fig. 23.31

Hence, the position vector of point C is $\dfrac{m\vec{b} - n\vec{a}}{m - n}$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the position vectors of the points which divide the join of the points $2\vec{a} - 3\vec{b}$ and $3\vec{a} - 2\vec{b}$ internally and externally in the ratio 2 : 3.*

SOLUTION Let A and B be the given points with position vectors $2\vec{a} - 3\vec{b}$ and $3\vec{a} - 2\vec{b}$ respectively. Let P and Q be the points dividing AB in the ratio $2 : 3$ internally and externally respectively. Then,

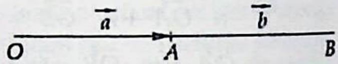


Fig. 23.32

$$\text{Position vector of } P = \frac{3(2\vec{a} - 3\vec{b}) + 2(3\vec{a} - 2\vec{b})}{3 + 2} = \frac{12\,\vec{a}}{5} - \frac{13\,\vec{b}}{5}$$

$$\text{Position vector of } Q = \frac{3(2\vec{a} - 3\vec{b}) - 2(3\vec{a} - 2\vec{b})}{3 - 2} = -5\vec{b}$$

EXAMPLE 2 *If $\vec{a}$ and $\vec{b}$ are position vectors of points A and B respectively, then find the position vector of points of trisection of AB.*

SOLUTION Let P and Q be points of trisection of AB. Then, $AP = PQ = QB = \lambda$ (say).

Since P divides AB in the ratio $\lambda : 2\lambda$ i.e. $1 : 2$.

$$\therefore \quad \text{Position vector of } P = \frac{1 \cdot \vec{b} + 2 \cdot \vec{a}}{1 + 2} = \frac{\vec{b} + 2\vec{a}}{3}$$

Since Q is the mid-point of PB.

$$\therefore \quad \text{Position vector of } Q = \frac{\dfrac{\vec{b} + 2\vec{a}}{3} + \vec{b}}{2} = \frac{4\vec{b} + 2\vec{a}}{6} = \frac{\vec{a} + 2\vec{b}}{3}$$

EXAMPLE 3 *Find the position vector of a point R which divides the line segment joining P and Q whose position vectors are $2\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$, externally in the ratio 1 : 2. Also, show that P is the mid-point of the line segment RQ.* **[CBSE 2010]**

SOLUTION The position vector of R is $\dfrac{1 \times (\vec{a} - 3\vec{b}) - 2(2\vec{a} + \vec{b})}{1 - 2} = 3\vec{a} + 5\vec{b}$

Clearly, Position vector of R + Positive vector of Q

$$= \frac{3\vec{a} + 5\vec{b} + \vec{a} - 3\vec{b}}{2} = 2\vec{a} + \vec{b}$$

= Position vector of P

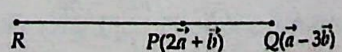


Fig. 23.33

Hence, P is the mid-point of RQ.

EXAMPLE 4 *Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are such that $3\vec{a} - \vec{b} + 2\vec{c} - 4\vec{d} = \vec{0}$. Show that the four points are coplanar. Also, find the position vector of the point of intersection of lines AC and BD.*

SOLUTION We have,

$$3\vec{a} - \vec{b} + 2\vec{c} - 4\vec{d} = \vec{0}$$

$$\Rightarrow \quad 3\vec{a} + 2\vec{c} = \vec{b} + 4\vec{d}$$

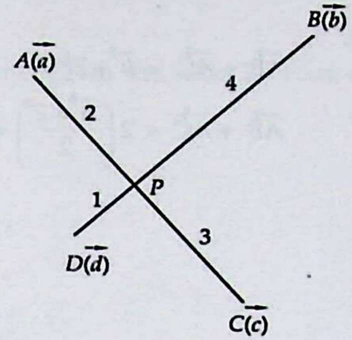


Fig. 23.34

We note that the sum of the coefficients on both sides of the above result is 5. We therefore, divide both the sides by 5 to get

$$\frac{3\vec{a} + 2\vec{c}}{5} = \frac{\vec{b} + 4\vec{d}}{5} \Rightarrow \frac{3\vec{a} + 2\vec{c}}{3+2} = \frac{\vec{b} + 4\vec{d}}{1+4}$$

This shows that the position vector of a point P dividing AC in the ratio 2 : 3 is same as that of a point dividing BD in the ratio 4 : 1. Consequently, point P is common to AC and BD. Therefore, AC and BD intersect. Hence, points A, B ,C and D are coplanar. Since P is the point of intersection of AC and BD. Therefore, the position vector of the point of intersection of AC and BD is

$$\frac{3\vec{a} + 2\vec{c}}{5} \text{ or, } \frac{\vec{b} + 4\vec{d}}{5}$$

EXAMPLE 5 *Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of three distinct points A,B,C. If there exist scalars x, y, z (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ and $x + y + z = 0$, then show that A, B and C lie on a line.*

SOLUTION It is given that x,y,z are not all zero. So, let z be non-zero. Then,

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$$

$$\Rightarrow \quad z\vec{c} = -(x\vec{a} + y\vec{b})$$

$$\Rightarrow \quad \vec{c} = -\frac{(x\vec{a} + y\vec{b})}{z}$$

$$\Rightarrow \quad \vec{c} = \frac{x\vec{a} + y\vec{b}}{x+y} \qquad [\because x + y + z = 0 \therefore z = -(x+y)]$$

This shows that the point C divides the line joining the points A and B in the ratio $y{:}x$. Hence, A,B and C lie on the same line.

23.10 LINEAR COMBINATION OF VECTORS

DEFINITION *A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a}, \vec{b}, \vec{c}$... etc. if there exist scalars x, y, z etc., such that*

$$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots$$

Note that a linear combination of vectors involves two linear compositions of the addition of vectors and the multiplication of vectors with scalars.

In the following sections, the linear combiantions of the form $x\vec{a}, x\vec{a} + y\vec{b}$ and $x\vec{a} + y\vec{b} + z\vec{c}$ will be of special interest to us.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If D is the mid-point of the side BC of a triangle ABC, prove that* $\vec{AB} + \vec{AC} = 2\vec{AD}$.

SOLUTION Let A be the origin and let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then, the position vector of the mid-point of BC is $\dfrac{\vec{b}+\vec{c}}{2}$.

$\therefore$ Position vector of $D = \dfrac{\vec{b}+\vec{c}}{2}$

Now,

$$\overrightarrow{AB}+\overrightarrow{AC}=\vec{b}+\vec{c}$$

$$\Rightarrow \quad \overrightarrow{AB}+\overrightarrow{AC}=2\left(\frac{\vec{b}+\vec{c}}{2}\right)=2\overrightarrow{AD}$$

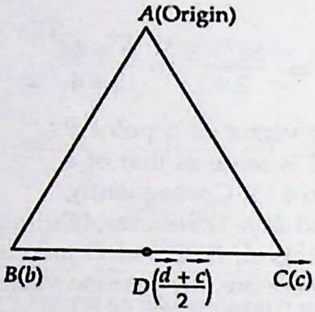


Fig. 23.37

EXAMPLE 2 *Points L, M, N divide the sides BC, CA, AB of ΔABC in the ratios $1:4$, $3:2, 3:7$ respectively. Prove that $\overrightarrow{AL}+\overrightarrow{BM}+\overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, where K divides AB in the ratio $1:3$.*

SOLUTION Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the vertices A, B and C of ΔABC.

Then, the position vectors of L, M and N are $\dfrac{4\vec{b}+\vec{c}}{5}, \dfrac{3\vec{a}+2\vec{c}}{5}$ and $\dfrac{7\vec{a}+3\vec{b}}{10}$ respectively.

The position vector of K is $\dfrac{\vec{b}+3\vec{a}}{4}$

Now,

$$\overrightarrow{AL}+\overrightarrow{BM}+\overrightarrow{CN}$$

$$=\frac{4\vec{b}+\vec{c}}{5}-\vec{a}+\frac{3\vec{a}+2\vec{c}}{5}-\vec{b}+\frac{7\vec{a}+3\vec{b}}{10}-\vec{c}$$

$$=\frac{3\vec{a}+\vec{b}-4\vec{c}}{10}$$

$$=\frac{4}{10}\left(\frac{3\vec{a}+\vec{b}-4\vec{c}}{4}\right)=\frac{4}{10}\overrightarrow{CK}$$

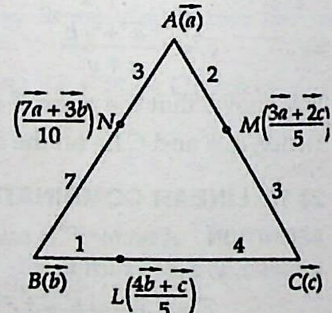


Fig. 23.38

Therefore, $\overrightarrow{AL}+\overrightarrow{BM}+\overrightarrow{CN}$ is parallel to $\overrightarrow{CK}$.

EXAMPLE 3 *Prove using vectors: Medians of a triangle are concurrent.*

SOLUTION Let ABC be a triangle and let D, E, F be the mid-points of its sides BC, CA and AB respectively. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B and C respectively. Then, the position vectors of D, E and F are $\dfrac{\vec{b}+\vec{c}}{2}, \dfrac{\vec{c}+\vec{a}}{2}$ and $\dfrac{\vec{a}+\vec{b}}{2}$ respectively.

The position vector of a point dividing AD in the ratio 2 : 1 is

$$\frac{1\cdot\vec{a}+2\left(\frac{\vec{b}+\vec{c}}{2}\right)}{1+2} \Rightarrow \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

Similarly, position vectors of points dividing BE and CF in the ratio 2 : 1 are each equal to $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$.

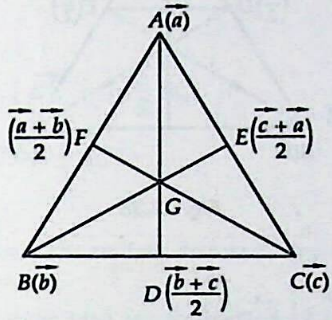


Fig. 23.37

Thus, the point dividing AD in the ratio 2 : 1 also divides BE and CF in the same ratio. Hence, the medians of a triangle are concurrent and the position vector of the centroid is $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$.

NOTE *If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of a triangle, then the position vector of its centroid is* $\frac{(\vec{a}+\vec{b}+\vec{c})}{3}$.

EXAMPLE 4 *If G is the centroid of a triangle ABC, prove that* $\vec{GA}+\vec{GB}+\vec{GC}=\vec{0}$.

SOLUTION Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the vertices A, B and C respectively. Then, the position vector of the centroid G is $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$ (see Example 3).

Now,

$$\vec{GA}+\vec{GB}+\vec{GC}$$

$$=\vec{a}-\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)+\vec{b}-\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)+\vec{c}-\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$$

$$=(\vec{a}+\vec{b}+\vec{c})-3\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$$

$$=(\vec{a}+\vec{b}+\vec{c})-(\vec{a}+\vec{b}+\vec{c})=\vec{0}$$

EXAMPLE 5 *If D and E are the mid-points of sides AB and AC of a triangle ABC respectively, show that* $\vec{BE}+\vec{DC}=\frac{3}{2}\vec{BC}$.

SOLUTION Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Since D and E are the mid-points of AB and AC. Therefore, the position vectors of D and E are $\frac{\vec{b}}{2}$ and $\frac{\vec{c}}{2}$ respectively.

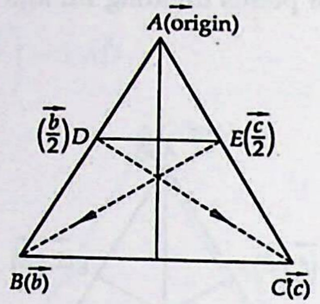


Fig. 23.38

Now,

$$\vec{BE} + \vec{DC} = \left(\frac{\vec{c}}{2} - \vec{b}\right) + \left(\vec{c} - \frac{\vec{b}}{2}\right)$$

$$\Rightarrow \quad \vec{BE} + \vec{DC} = \frac{3}{2}(\vec{c} - \vec{b}) = \frac{3}{2}\vec{BC}$$

EXAMPLE 6 *If ABC and A′B′C′ are two triangles and G, G′ be their centroids, prove that*

$$\vec{AA'} + \vec{BB'} + \vec{CC'} = 3\,\vec{GG'}$$

SOLUTION Let the position vectors of A, B, C and A', B', C' with reference to some origin be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a'}, \vec{b'}, \vec{c'}$ respectively. Then,

$$\vec{AA'} + \vec{BB'} + \vec{CC'} = (\vec{a'} - \vec{a}) + (\vec{b'} - \vec{b}) + (\vec{c'} - \vec{c})$$

$$\Rightarrow \quad \vec{AA'} + \vec{BB'} + \vec{CC'} = (\vec{a'} + \vec{b'} + \vec{c'}) - (\vec{a} + \vec{b} + \vec{c}) \qquad \text{...(i)}$$

The position vectors of the centroids G and G' of triangles ABC and $A'B'C'$ are $\frac{\vec{a} + \vec{b} + \vec{c}}{3}$ and $\frac{\vec{a'} + \vec{b'} + \vec{c'}}{3}$ respectively.

$$\therefore \quad \vec{GG'} = \frac{\vec{a'} + \vec{b'} + \vec{c'}}{3} - \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

$$\Rightarrow \quad \vec{GG'} = \frac{1}{3}\left\{(\vec{a'} + \vec{b'} + \vec{c'}) - (\vec{a} + \vec{b} + \vec{c})\right\}$$

$$\Rightarrow \quad 3\,\vec{GG'} = (\vec{a'} + \vec{b'} + \vec{c'}) - (\vec{a} + \vec{b} + \vec{c}) \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$\vec{AA'} + \vec{BB'} + \vec{CC'} = 3\,\vec{GG'}$$

EXAMPLE 7 *Prove that the line segment joining the mid points of two sides of a triangle is parallel to the third side and equal to half of it.*

SOLUTION Let ABC be a triangle and let D and E be the mid points of its sides AB and AC respectively. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of vertices A, B and C respectively. Then, position vectors of D and E are $\dfrac{\vec{a}+\vec{b}}{2}$ and $\dfrac{\vec{a}+\vec{c}}{2}$ respectively.

Now,

$$\vec{DE} = \left(\frac{\vec{a}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{b}}{2}\right)$$

$$\Rightarrow \quad \vec{DE} = \frac{1}{2}(\vec{c}-\vec{b}) = \frac{1}{2}\vec{BC}$$

$$\therefore \quad \vec{DE} \parallel \vec{BC}$$

Also, $DE = |\vec{DE}| = \dfrac{1}{2}|\vec{BC}| = \dfrac{1}{2}BC$

Hence, $DE \parallel BC$ and $DE = \dfrac{1}{2}BC$

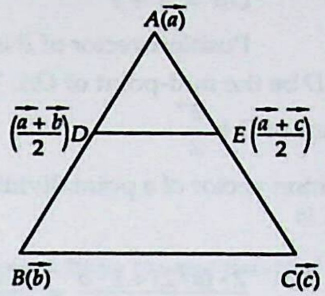


Fig. 23.39

EXAMPLE 8 *Prove that the sum of the vectors directed from the vertices to the mid-points of opposite sides of a triangle is zero.*

SOLUTION Let ABC be a triangle and let the position vectors of vertices A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Let D, E, F be mid points of sides BC, CA and AB respectively. Then, position vectors of D, E and F are $\dfrac{\vec{b}+\vec{c}}{2}, \dfrac{\vec{c}+\vec{a}}{2}$ and $\dfrac{\vec{a}+\vec{b}}{2}$ respectively.

We have to prove that $\vec{AD}+\vec{BE}+\vec{CF}=\vec{0}$.

Now,

$$\vec{AD}+\vec{BE}+\vec{CF}$$

$$= \left(\frac{\vec{b}+\vec{c}}{2}-\vec{a}\right)+\left(\frac{\vec{c}+\vec{a}}{2}-\vec{b}\right)+\left(\frac{\vec{a}+\vec{b}}{2}-\vec{c}\right)$$

$$= \frac{1}{2}(\vec{b}+\vec{c}-2\vec{a}+\vec{c}+\vec{a}-2\vec{b}+\vec{a}+\vec{b}-2\vec{c}) = \vec{0}$$

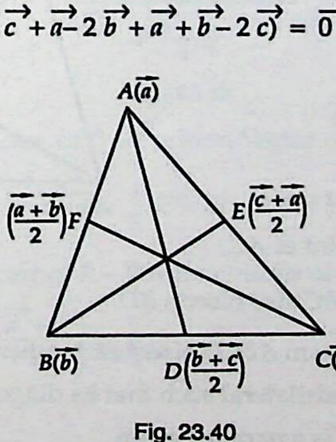


Fig. 23.40

EXAMPLE 9 *Show that the line joining one vertex of a parallelogram to the mid-point of an opposite side trisects the diagonal and is trisected thereat.*

SOLUTION Let $OABC$ be a parallelogram. Taking O as the origin let the position vectors of A and C be $\vec{a}$ and $\vec{b}$ respectively.

.ı ΔOAB, we have

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\Rightarrow \quad \vec{OA} + \vec{OC} = \vec{OB}$$

$$\Rightarrow \quad \vec{OB} = \vec{a} + \vec{b}$$

$\Rightarrow$ Position vector of B is $\vec{a} + \vec{b}$.

Let D be the mid-point of OA. Then, the position vector of D is $\frac{\vec{a}}{2}$.

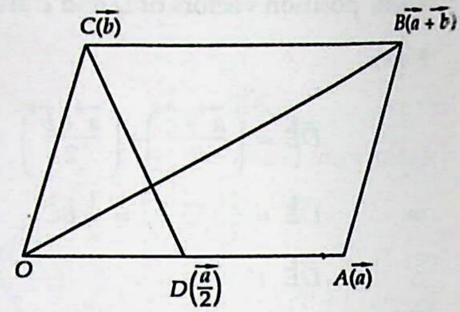


Fig. 23.41

Position vector of a point dividing CD in the ratio $2:1$ is

$$\frac{2\cdot(\vec{a}/2) + 1\cdot\vec{b}}{2+1} = \frac{\vec{a}+\vec{b}}{3}$$

Position vector of a point dividing OB in the ratio $1:2$ is

$$\frac{1\cdot(\vec{a}+\vec{b}) + 2\cdot\vec{0}}{1+2} = \frac{\vec{a}+\vec{b}}{3}$$

ıs, the position vectors of points trisecting OB and DC are same.

ıce, DC trisects OB and DC is trisected thereat.

MPLE 10 *Prove using vectors: The diagonals of a quadrilateral bisect each other iff it is a ıllelogram.*

LUTION First, let us assume that $ABCD$ is a parallelogram. Then, we have to prove ıat its diagonals bisect each other. Let the position vectors of A, B, C and D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

Since $ABCD$ is a parallelogram.

$$\therefore \quad \vec{AB} = \vec{DC}$$

$$\Rightarrow \quad \vec{b} - \vec{a} = \vec{c} - \vec{d}$$

$$\Rightarrow \quad \vec{b} + \vec{d} = \vec{a} + \vec{c}$$

$$\Rightarrow \quad \frac{1}{2}(\vec{b} + \vec{d}) = \frac{1}{2}(\vec{a} + \vec{c})$$

$\Rightarrow$ P.V. of the mid-point of BD

= P.V. of the mid-point of AC.

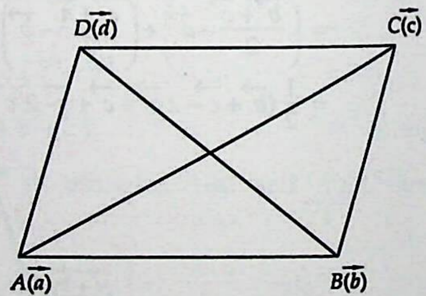


Fig. 23.42

Thus, the point which bisects AC also bisects BD.

Hence, diagonals of parallelogram $ABCD$ bisect each other.

Conversely, let $ABCD$ be a quadrilateral such that its diagonals bisect each other. Then, we have to prove that it is a parallelogram.

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of its vertices A, B, C and D respectively.

Since diagonals AC and BD bisect each other.

$\therefore$ P.V. of the mid-point of AC = P.V. of the mid-point of BD

$$\Rightarrow \quad \frac{1}{2}(\vec{a}+\vec{c}) = \frac{1}{2}(\vec{b}+\vec{d})$$

$$\Rightarrow \quad \vec{a}+\vec{c} = \vec{b}+\vec{d} \quad \text{...(i)}$$

$$\Rightarrow \quad \vec{b}-\vec{a} = \vec{c}-\vec{d}$$

$$\Rightarrow \quad \vec{AB} = \vec{DC}$$

Again, from (i), we have

$$\vec{d}-\vec{a} = \vec{c}-\vec{b}$$

$$\Rightarrow \quad \vec{AD} = \vec{BC}$$

Hence, $ABCD$ is a parallelogram.

EXAMPLE 11 *Using vector method, prove that the line segments joining the mid-points of the adjacent sides of a quadrilateral taken in order form a parallelogram.*

SOLUTION Let $ABCD$ be a quadrilateral and let P, Q, R, S be the mid-points of the sides AB, BC, CD and DA respectively. Then, the position vectors of P, Q, R, S are $\frac{\vec{a}+\vec{b}}{2}, \frac{\vec{b}+\vec{c}}{2}, \frac{\vec{c}+\vec{d}}{2}$ and $\frac{\vec{d}+\vec{a}}{2}$ respectively.

In order to prove that $PQRS$ is a parallelogram, it is sufficient to show that $\vec{PQ} = \vec{SR}$.

We have,

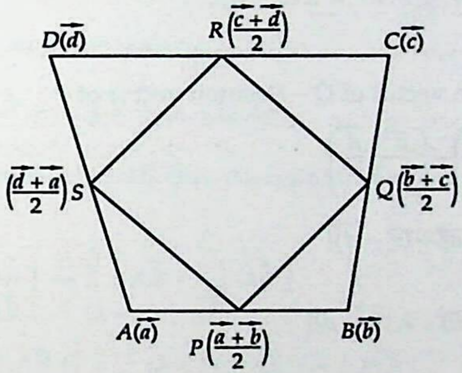


Fig. 23.43

$$\vec{PQ} = \text{Position Vector of } Q - \text{Position Vector of } P$$

$$\Rightarrow \quad \vec{PQ} = \left(\frac{\vec{b}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{b}}{2}\right) = \frac{1}{2}(\vec{c}-\vec{a})$$

and, $\vec{SR} = $ Position vector of R – Position vector of S

$$\Rightarrow \quad \vec{SR} = \left(\frac{\vec{c}+\vec{d}}{2}\right) - \left(\frac{\vec{d}+\vec{a}}{2}\right) = \frac{1}{2}(\vec{c}-\vec{a})$$

So, $\vec{PQ} = \vec{SR}$

Consequently, $PQ \parallel SR$ and $PQ = SR$.

Hence, $PQRS$ is a parallelogram.

EXAMPLE 12 *Prove that the segment joining the middle points of two non-parallel sides of a trapezium is parallel to the parallel sides and half of their sum.*

SOLUTION Let $ABCD$ be the given trapezium. Let the position vectors of A, B, C and D with reference to some origin O be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively.

Let P and Q be the mid-points of AD and BC respectively. Then, the position vectors of P and Q are $\frac{\vec{a}+\vec{d}}{2}$ and $\frac{\vec{b}+\vec{c}}{2}$ respectively.

We have, $\vec{AB} = \vec{b} - \vec{a}$ and $\vec{DC} = \vec{c} - \vec{d}$

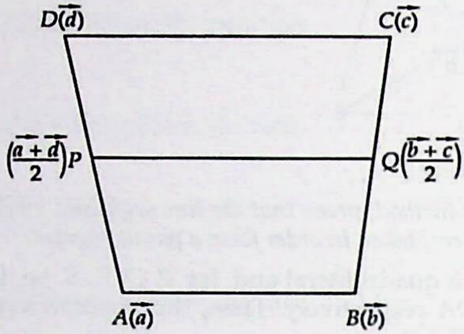


Fig. 23.44

Since $\vec{DC}$ is parallel to $\vec{AB}$. Therefore, there exists a scalar λ such that

$$\vec{DC} = \lambda \vec{AB} \Rightarrow (\vec{c} - \vec{d}) = \lambda (\vec{b} - \vec{a}) \quad \text{...(i)}$$

Now,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$\Rightarrow \quad \vec{PQ} = \left(\frac{\vec{b}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{d}}{2}\right)$$

$$\Rightarrow \quad \vec{PQ} = \frac{1}{2}[(\vec{b} - \vec{a}) + (\vec{c} - \vec{d})]$$

$$\Rightarrow \quad \vec{PQ} = \frac{1}{2}[(\vec{b} - \vec{a}) + \lambda (\vec{b} - \vec{a})] \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \vec{PQ} = \frac{1}{2}(\lambda + 1)(\vec{b} - \vec{a}) = \frac{1}{2}(\lambda + 1)\vec{AB} \quad \text{...(ii)}$$

This shows that PQ is parallel to AB. But, AB is parallel to CD. Therefore, PQ is parallel to CD.

We also have

$$|\vec{AB}| + |\vec{DC}| = |\vec{AB}| + |\lambda \vec{AB}| \quad [\because \vec{DC} = \lambda \vec{AB} \text{ from (i)}]$$

$$\Rightarrow \quad |\vec{AB}| + |\vec{DC}| = |\vec{AB}| + \lambda |\vec{AB}| \quad [\because \lambda > 0]$$

$$\Rightarrow \quad |\vec{AB}| + |\vec{DC}| = (1 + \lambda) |\vec{AB}|$$

$$\therefore \quad \frac{1}{2}\Big[|\vec{AB}| + |\vec{DC}|\Big] = \frac{1}{2}(\lambda + 1) |\vec{AB}|$$

$$\Rightarrow \quad \frac{1}{2}\Big[|\vec{AB}| + |\vec{DC}|\Big] = |\vec{PQ}| \quad \text{[From (ii)]}$$

This shows that PQ is half of the sum of the lengths of AB and CD.

EXAMPLE 13 *Prove by vector method that the line segment joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides and equal to half of their difference.*

SOLUTION Let $ABCD$ be a trapezium and let the position vectors of A, B, C and D with reference to some origin O be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively.

Let P and Q be the mid-points of diagonals BD and AC respectively. Then, the position vectors of P and Q are $\frac{\vec{b}+\vec{d}}{2}$ and $\frac{\vec{a}+\vec{c}}{2}$ respectively.

$\therefore \quad \vec{PQ}$ = Position vector of Q – Position vector of P

$$\Rightarrow \quad \vec{PQ} = \left(\frac{\vec{a}+\vec{c}}{2}\right) - \left(\frac{\vec{b}+\vec{d}}{2}\right)$$

$$\Rightarrow \quad \vec{PQ} = \frac{1}{2}[(\vec{c}-\vec{d}) + (\vec{a}-\vec{b})]$$

$$\Rightarrow \quad \vec{PQ} = \frac{1}{2}[(\vec{c}-\vec{d}) - (\vec{b}-\vec{a})]$$

$$\Rightarrow \quad \vec{PQ} = \frac{1}{2}[\vec{DC} - \vec{AB}]$$

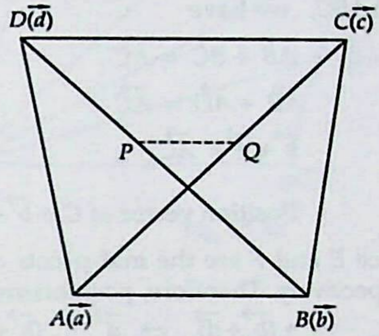


Fig. 23.45

Now,

$$\vec{DC} \parallel \vec{AB}$$

$$\Rightarrow \quad \vec{DC} = \lambda \vec{AB} \text{ for some scalar } \lambda \quad \text{...(i)}$$

$$\therefore \quad \vec{PQ} = \frac{1}{2}[\lambda \vec{AB} - \vec{AB}] = \frac{1}{2}(\lambda - 1)\vec{AB} \quad \text{...(ii)}$$

This shows that PQ is parallel to AB. But, AB is parallel to DC. Therefore, PQ is parallel to DC.

Now,

$$|\vec{DC}| - |\vec{AB}| = \lambda |\vec{AB}| - |\vec{AB}| \quad \text{[Using (i)]}$$

$$\Rightarrow \quad |\vec{DC}| - |\vec{AB}| = (\lambda - 1)|\vec{AB}|$$

$$\Rightarrow \quad \frac{1}{2}\left\{|\vec{DC}| - |\vec{AB}|\right\} = \frac{1}{2}(\lambda - 1)|\vec{AB}| = |\vec{PQ}| \quad \text{[Using (ii)]}$$

This shows that PQ is half of the difference of parallel sides.

EXAMPLE 14 *If ABCD is quadrilateral and E and F are the mid-points of AC and BD respectively, prove that*

$$\vec{AB} + \vec{AD} + \vec{CB} + \vec{CD} = 4\vec{EF}$$

SOLUTION Since F is the mid-point of BD, therefore in triangle ABD, we have

$$1 \cdot \vec{AB} + 1 \cdot \vec{AD} = (1+1)\vec{AF}$$

$$\Rightarrow \quad \vec{AB} + \vec{AD} = 2\vec{AF} \quad \text{...(i)}$$

Similarly, in triangle BCD, we have

$$1 \cdot \vec{CB} + 1 \cdot \vec{CD} = (1+1)\vec{CF}$$

$$\Rightarrow \quad \vec{CB} + \vec{CD} = 2\vec{CF} \quad \text{...(ii)}$$

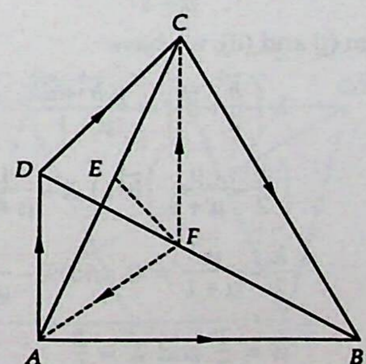


Fig. 23.46

Adding (i) and (ii), we obtain

$$\vec{AB}+\vec{AD}+\vec{CB}+\vec{CD}=2\,\vec{AF}+2\,\vec{CF}$$

$$\Rightarrow \quad \vec{AB}+\vec{AD}+\vec{CB}+\vec{CD}=-2\,\vec{FA}-2\,\vec{FC}$$

$$\Rightarrow \quad \vec{AB}+\vec{AD}+\vec{CB}+\vec{CD}=-2\,(\vec{FA}+\vec{FC})$$

$$\Rightarrow \quad \vec{AB}+\vec{AD}+\vec{CB}+\vec{CD}=-2\,(2\,\vec{FE}) \qquad [\because E \text{ is the mid-point of } AC]$$

$$\Rightarrow \quad \vec{AB}+\vec{AD}+\vec{CB}+\vec{CD}=4\,\vec{EF}$$

EXAMPLE 15 *ABCD is a parallelogram. E, F are mid-points of BC, CD respectively. AE, AF meet the diagonal BD at points Q and P respectively. Show that points P and Q trisect DB.*

SOLUTION Let A be the origin and let the position vectors of B and D be $\vec{b}$ and $\vec{d}$ respectively. Then, $\vec{AB}=\vec{b}$, $\vec{AD}=\vec{d}$.

In ΔABC, we have

$$\vec{AB}+\vec{BC}=\vec{AC}$$

$$\Rightarrow \quad \vec{AB}+\vec{AD}=\vec{AC} \qquad [\because \vec{BC}=\vec{AD}]$$

$$\Rightarrow \quad \vec{b}+\vec{d}=\vec{AC}$$

$\Rightarrow$ Position vector of C is $\vec{b}+\vec{d}$

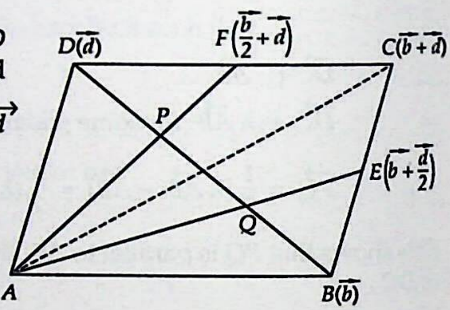


Fig. 23.47

Since E and F are the mid-points of BC and CD respectively. Therefore, position vectors of E and F are $\frac{\vec{b}+(\vec{b}+\vec{d})}{2}=\vec{b}+\frac{\vec{d}}{2}$ and $\frac{(\vec{b}+\vec{d})+\vec{d}}{2}=\frac{\vec{b}}{2}+\vec{d}$ respectively.

Now,

A is the origin and P lies on AF.

$\therefore \quad \vec{AP}=\lambda\,\vec{AF}$ for some scalar λ

$$\Rightarrow \quad \vec{AP}=\lambda\left(\frac{\vec{b}}{2}+\vec{d}\right) \qquad \text{...(i)}$$

Suppose P divides DB in the ratio $\mu:1$. Then,

$$\text{Position vector of } P \text{ is } =\frac{\mu\vec{b}+1\cdot\vec{d}}{\mu+1}$$

$$\Rightarrow \quad \vec{AP}=\frac{\mu\vec{b}+\vec{d}}{\mu+1} \qquad \text{...(ii)} \qquad [\because A \text{ is the origin}]$$

From (i) and (ii), we have

$$\lambda\left(\frac{\vec{b}}{2}+\vec{d}\right)=\frac{\mu\vec{b}+\vec{d}}{\mu+1}$$

$$\Rightarrow \quad \left(\frac{\lambda}{2}-\frac{\mu}{\mu+1}\right)\vec{b}+\left(\lambda-\frac{1}{\mu+1}\right)\vec{d}=0$$

$$\Rightarrow \quad \frac{\lambda}{2}-\frac{\mu}{\mu+1}=0 \text{ and } \lambda-\frac{1}{\mu+1}=0 \qquad [\because \vec{b} \text{ and } \vec{d} \text{ are non-collinear}]$$

$$\Rightarrow \quad \mu=\frac{1}{2} \text{ and } \lambda=\frac{2}{3}$$

Thus, P divides DB in the ratio $\frac{1}{2}:1$ i.e. $1:2$.

Similarly, Q divides DB in the ratio 2 : 1. Hence, P and Q trisect DB.

EXAMPLE 16 *ABCD is a parallelogram. If L and M are the mid-points of BC and DC respectively, then express* $\vec{AL}$ *and* $\vec{AM}$ *in terms of* $\vec{AB}$ *and* $\vec{AD}$. *Also, prove that* $\vec{AL} + \vec{AM} = \frac{3}{2}\vec{AC}$.

SOLUTION Taking A as the origin, let the position vectors of B and D be $\vec{b}$ and $\vec{d}$ respectively. Then, $\vec{AB} = \vec{b}$ and $\vec{AD} = \vec{d}$.

In triangle ABC, we have

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\Rightarrow \quad \vec{AC} = \vec{AB} + \vec{AD} \qquad [\because \vec{BC} = \vec{AD}]$$

$$\Rightarrow \quad \vec{AC} = \vec{b} + \vec{d}$$

$\therefore$ Position vector of C is $\vec{b} + \vec{d}$

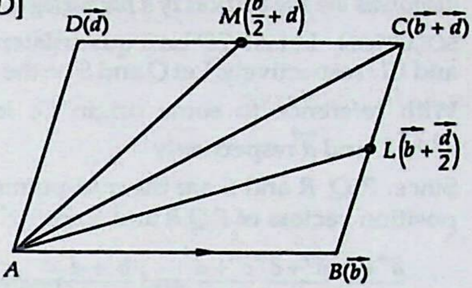


Fig. 23.48

Since L and M are mid-points of B C and CD respectively. Therefore,

$$\text{Position vector of } L = \frac{\vec{b} + (\vec{b} + \vec{d})}{2} = \vec{b} + \frac{1}{2}\vec{d}$$

$$\text{Position vector of } M = \frac{(\vec{b} + \vec{d}) + \vec{d}}{2} = \frac{\vec{b}}{2} + \vec{d}$$

$$\therefore \quad \vec{AL} = \text{Position vector of } L - \text{Position vector of } A$$

$$\Rightarrow \quad \vec{AL} = \vec{b} + \frac{1}{2}\vec{d} - \vec{0} = \vec{b} + \frac{1}{2}\vec{d} = \vec{AB} + \frac{1}{2}\vec{AD}$$

and, $$\vec{AM} = \text{Position vector of } M - \text{Position vector of } A$$

$$\Rightarrow \quad \vec{AM} = \frac{\vec{b}}{2} + \vec{d} - \vec{0} = \frac{\vec{b}}{2} + \vec{d} = \frac{1}{2}\vec{AB} + \vec{AD}$$

Now, $$\vec{AL} + \vec{AM} = \left(\vec{b} + \frac{1}{2}\vec{d}\right) + \left(\frac{1}{2}\vec{b} + \vec{d}\right) = \frac{3}{2}\vec{b} + \frac{3}{2}\vec{d} = \frac{3}{2}(\vec{b} + \vec{d}) = \frac{3}{2}\vec{AC}$$

EXAMPLE 17 *If P and Q are the mid-points of the sides AB and CD of a parallelogram ABCD, prove that DP and BQ cut the diagonal AC in its points of trisection which are also the points of trisection of DP and BQ respectively.*

SOLUTION Taking O as the origin, let the position vector B and D be $\vec{b}$ and $\vec{d}$ respectively.

Then, position vector of C is $\vec{b} + \vec{d}$.

Since P and Q are the mid-points of AB and CD respectively. Therefore, position vectors of P and Q are $\frac{\vec{b}}{2}$ and $\frac{\vec{b}}{2} + \vec{d}$ respectively.

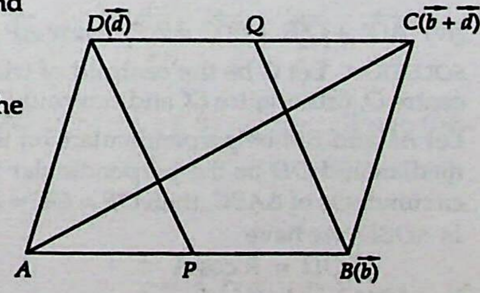


Fig. 23.49

The position vector of a point dividing AC in the ratio 1 : 2 is

$$\frac{1 \cdot (\vec{b} + \vec{d}) + 2 \cdot \vec{0}}{1 + 2} = \frac{\vec{b} + \vec{d}}{3}$$

Also, the position vector of the point dividing DP in the ratio 2 : 1 is

$$\frac{2\left(\frac{\vec{b}}{2}\right)+1\cdot\vec{d}}{2+1}=\frac{\vec{b}+\vec{d}}{3}$$

Thus, the point of trisection of AC coincides with the point of trisection of DP.

Hence, DP cuts the diagonal AC in its point of trisection, which is also the point of trisection of DP. Similarly, BQ cuts the diagonal AC in its point of trisection, which is also the point of trisection of BQ.

EXAMPLE 18 *"The mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals are the vertices of a parallelogram". Prove using vectors.*

SOLUTION Let $ABCD$ be a quadrilateral and let P, R be the mid-points of the sides AB and CD respectively. Let Q and S be the mid-points of diagonals AC and BD respectively. With reference to some origin O, let the position vectors of A, B, C and D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively

Since, P, Q, R and S are the mid-points of AB, AC, CD and BD respectively. Therefore, position vectors of P,Q,R and S are

$$\frac{\vec{a}+\vec{b}}{2}, \frac{\vec{a}+\vec{c}}{2}, \frac{\vec{c}+\vec{d}}{2} \text{ and } \frac{\vec{b}+\vec{d}}{2} \text{ respectively.}$$

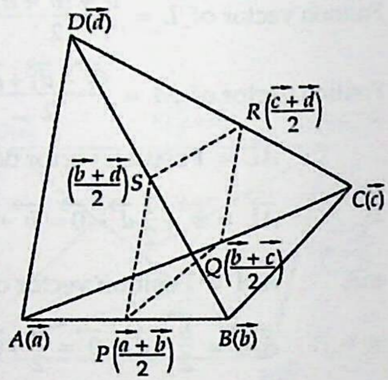


Fig. 23.50

Now,

$\vec{PQ}$ = Position vector of Q − Position vector of P

$$\Rightarrow \vec{PQ}=\left(\frac{\vec{a}+\vec{c}}{2}\right)-\left(\frac{\vec{a}+\vec{b}}{2}\right)=\frac{1}{2}(\vec{c}-\vec{b})$$

and,

$\vec{SR}$ = Position vector of R − Position vector of S

$$\Rightarrow \vec{SR}=\left(\frac{\vec{c}+\vec{d}}{2}\right)-\left(\frac{\vec{b}+\vec{d}}{2}\right)=\frac{1}{2}(\vec{c}-\vec{b})$$

Clearly, $\vec{PQ}=\vec{SR}$.

Hence, $PQRS$ is a parallelogram.

EXAMPLE 19 *If O is the circumcentre and O′ the orthocentre of a triangle ABC, prove that*

(i) $\vec{SA}+\vec{SB}+\vec{SC}=3\,\vec{SG}$, *where S is any point in the plane of triangle ABC whose centroid is at G.*

(ii) $\vec{OA}+\vec{OB}+\vec{OC}=\vec{OO'}$

(iii) $\vec{O'A}+\vec{O'B}+\vec{O'C}=2\,\vec{O'O}$

(iv) $\vec{AO'}+\vec{O'B}+\vec{O'C}=\vec{AP}$, *where $\vec{AP}$ is the diameter of the circumcircle.*

SOLUTION Let G be the centroid of triangle ABC. First we will show that the circumcentre O, orthocentre O' and centroid G are collinear and $O'G=2OG$.

Let AL and BM be perpendiculars on the sides BC and CA respectively. Let AD be the median and OD be the perpendicular from O on side BC. If R is the circum radius of circumcircle of ΔABC, then $OB=OC=R$.

In ΔOBD, we have

$$OD = R\cos A \qquad \text{...(i)}$$

In ΔABM, we have

$$AM = AB\cos A = c\cos A \qquad \text{...(ii)}$$

In $\Delta AO'M$, we have

$$AO' = AM \sec \angle O'AM$$

$\Rightarrow$ $AO' = c \cos A \sec (90° - C)$ [Using (ii)]

$\Rightarrow$ $AO' = c \cos A \operatorname{cosec} C$

$\Rightarrow$ $AO' = \dfrac{c}{\sin C} \cos A = 2R \cos A$ $\left[\because \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{C}{\sin C} = 2R\right]$

$\therefore$ $AO' = 2(OD)$ ...(iii) [Using (i)]

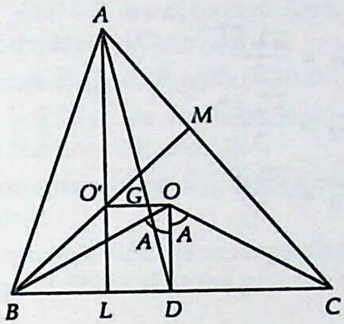


Fig. 23.51

Triangles AGO' and OGD are similar

$\therefore$ $\dfrac{OG}{O'G} = \dfrac{GD}{GA} = \dfrac{OD}{AO'} = \dfrac{1}{2}$ [Using (iii])

$\Rightarrow$ $2 \cdot OG = O'G$ ...(iv)

(i) We have,

$$\vec{SA} + \vec{SB} + \vec{SC} = \vec{SA} + (\vec{SB} + \vec{SC})$$

$\Rightarrow$ $\vec{SA} + \vec{SB} + \vec{SC} = \vec{SA} + 2\vec{SD}$ [$\because$ D is the mid-point of BC]

$\Rightarrow$ $\vec{SA} + \vec{SB} + \vec{SC} = (1+2), \vec{SG}$ [$\because$ G divides AD in the ratio 2 : 1]

$\Rightarrow$ $\vec{SA} + \vec{SB} + \vec{SC} = 3\,\vec{SG}.$

(ii) Replacing S by O in (i), we have

$$\vec{OA} + \vec{OB} + \vec{OC} = 3\vec{OG}$$

$\Rightarrow$ $\vec{OA} + \vec{OB} + \vec{OC} = 2\vec{OG} + \vec{OG} = \vec{GO'} + \vec{OG}$ [$\because$ $2 \cdot OG = GO'$]

$\Rightarrow$ $\vec{OA} + \vec{OB} + \vec{OC} = \vec{OG} + \vec{GO'} = \vec{OO'}$

(iii) We have,

$$\vec{O'A} + \vec{O'B} + \vec{O'C} = 3\vec{O'G} = 2\vec{O'G} + \vec{O'G}$$ [From (i)]

$\Rightarrow$ $\vec{O'A} + \vec{O'B} + \vec{O'C} = 2\vec{O'G} + 2\,\vec{GO}$ [$\because$ $2 \cdot OG = GO'$]

$\Rightarrow$ $\vec{O'A} + \vec{O'B} + \vec{O'C} = 2\vec{O'O}$

(iv) We have,

$$\vec{AO'} + \vec{O'B} + \vec{O'C} = 2\,\vec{AO'} + (\vec{O'A} + \vec{O'B} + \vec{O'C})$$

$\Rightarrow$ $\vec{AO'} + \vec{O'B} + \vec{O'C} = 2\vec{AO'} + 2\,\vec{O'O}$ [From (iii)]

$\Rightarrow \quad \overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2(\overrightarrow{AO'} + \overrightarrow{O'O})$

$\Rightarrow \quad \overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{AO} = \overrightarrow{AP}$ [$\because$ AO is the circum-radius of ΔABC]

EXAMPLE 20 *The lines joining the vertices of a tetrahectron to the centroids of opposite faces are concurrent.*

SOLUTION Let $OABC$ be a tetrahedron. Taking O as the origin, let the position vectors of the vertices A, B, C be $\vec{a}$, $\vec{b}$, and $\vec{c}$ respectively. Let G, G_1, G_2, G_3 be the centroids of the faces ABC, OAB, OBC and OCA respectively. Then,

$$\text{Position vector of } G = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

$$\text{Position vector of } G_1 = \frac{\vec{a}+\vec{b}}{3}$$

$$\text{Position vector of } G_2 = \frac{\vec{b}+\vec{c}}{3}$$

$$\text{Position vector of } G_3 = \frac{\vec{c}+\vec{a}}{3}$$

Now,

$$\text{P.V. of a point dividing } OG \text{ in the ratio } 3:1 = \frac{3\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)+1.\vec{0}}{3+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

$$\text{P.V. of a point dividing } AG_2 \text{ in the ratio } 3:1 = \frac{3\left(\frac{\vec{b}+\vec{c}}{3}\right) 1.\vec{a}}{2+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

$$\text{P.V. of a point dividing } BG_3 \text{ in the ratio } 3:1 = \frac{3\left(\frac{\vec{c}+\vec{a}}{3}\right)+1.\vec{b}}{3+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

$$\text{P.V. of a point dividing } CG_1 \text{ in the ratio } 3:1 = \frac{3\left(\frac{\vec{a}+\vec{b}}{3}\right)+1.\vec{c}}{3+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

Thus, the point having position vector $\frac{\vec{a}+\vec{b}+\vec{c}}{4}$ is common to OG, AG_2, BG_3 and CG_1.

Hence, the line joining the vertices of a tetrahedrun of the centroids of opposite faces are concurrent.

EXERCISE 23.3

1. If O is a point in space, ABC is a triangle and D, E, F are the mid-points of the sides BC, CA and AB respectively of the triangle, prove that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF}$$

2. Show that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.
3. Let $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ be the position vectors of the four distinct points A,B,C,D. If $\vec{b}-\vec{a}=\vec{c}-\vec{d}$, then show that $ABCD$ is a parallelogram.
4. $ABCD$ is a parallelogram and P is the point of intersection of its diagonals. If O is the origin of reference, show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + OD = 4\,\overrightarrow{OP}$$

5. $ABCD$ are four points in a plane and Q is the point of intersection of the lines joining the mid-points of AB and CD; BC and AD. show that

$$\vec{PA} + \vec{PB} + \vec{PC} + \vec{PD} = 4\,\vec{PQ}, \text{ where } P \text{ is any point.}$$

6. If $\vec{a}, \vec{b}$ are the position vectors of A, B respectively, find the position vector of a point C in AB produced such that $AC = 3\,AB$ and that a point D in BA produced such that $BD = 2\,BA$.
7. Prove by vector method that the internal bisectors of the angles of a triangle are concurrent.
8. Show that the four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively such that $3\vec{a} - 2\vec{b} + 5\vec{c} - 6\vec{d} = 0$, are coplanar. Also, find the position vector of the point of intersection of the lines AC and BD.
9. Show that the four points P, Q, R, S with position vectors $\vec{p}, \vec{q}, \vec{r}, \vec{s}$ respectively such that $5\vec{p} - 2\vec{q} + 6\vec{r} - 9\vec{s} = \vec{0}$, are coplanar. Also, find the position vector of the point of intersection of the lines PR and QS.
10. Show that the line segments joining the mid-points of opposite sides of a quadrilateral bisects each other.
11. The vertices A, B, C of triangle ABC have respectively position vectors $\vec{a}, \vec{b}, \vec{c}$ with respect to a given origin O. Show that the point D where the bisector of $\angle A$ meets BC has position vector

$$\vec{d} = \frac{\beta\vec{b} + \gamma\vec{c}}{\beta + \gamma}, \text{ where } \beta = |\vec{c} - \vec{a}| = \gamma = |\vec{a} - \vec{b}|$$

Hence, deduce that the incentre I has position vector $\dfrac{\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}}{\alpha + \beta + \gamma}$, where $\alpha = |\vec{b} - \vec{c}|$

ANSWERS

6. $3\vec{b} - 2\vec{a}, 2\vec{a} - \vec{b}$ 8. $\dfrac{3\vec{a} + 5\vec{c}}{8}$ or, $\dfrac{2\vec{b} + 6\vec{c}}{8}$ 9. $\dfrac{5\vec{p} + 6\vec{q}}{11}$ or, $\dfrac{2\vec{q} + 9\vec{s}}{11}$

23.11 COMPONENTS OF A VECTOR IN TWO DIMENSION

Let $P(x, y)$ be a point in a plane with reference to OX and OY as the coordinate axes as show in Fig. 23.51. Then, $OM = x$ and $PM = y$.

Let $\hat{i}, \hat{j}$ be unit vectors along OX and OY respectively. Then,

$$\vec{OM} = x\hat{i} \text{ and } \vec{MP} = y\hat{j}.$$

Vectors $\vec{OM}$ and $\vec{MP}$ are known as the components of $\vec{OP}$ along x-axis and y-axis respectively. Now,

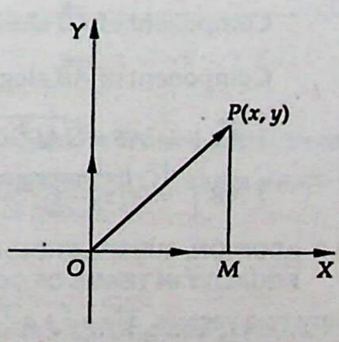


Fig. 23.52

$$\vec{OP} = \vec{OM} + \vec{MP}$$

$\Rightarrow$ $\vec{OP} = x\hat{i} + y\hat{j}$

Let $\vec{OP} = \vec{r}$

Then, $\vec{r} = x\hat{i} + y\hat{j}$

Now,

$$OP^2 = OM^2 + MP^2$$

$$\Rightarrow \quad OP^2 = x^2 + y^2$$

$$\Rightarrow \quad OP = \sqrt{x^2 + y^2}$$

$$\Rightarrow \quad |\vec{r}| = \sqrt{x^2 + y^2}$$

Thus, if a point P in a plane has coordinates (x, y), then

(i) $\overrightarrow{OP} = x\hat{i} + y\hat{j}$

(ii) $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$

(iii) *The components of $\overrightarrow{OP}$ along x-axis is a vector $x\hat{i}$, whose magnitude is $|x|$ and whose direction is along OX or OX' according as x is positive or negative.*

(iv) *The component of $\overrightarrow{OP}$ along y-axis is a vector $y\hat{j}$, whose magnitude is $|y|$ and whose direction is along OY or OY′ according as y is positive or negative.*

23.11.1 COMPONENTS OF A VECTOR IN TERMS OF COORDINATES OF ITS END POINTS

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in XOY plane.

Let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY respectively.

From Fig. 23.53, we have

$$AF = x_2 - x_1,\ BF = y_2 - y_1$$

$$\therefore \quad \overrightarrow{AF} = (x_2 - x_1)\,\hat{i} \text{ and } \overrightarrow{FB} = (y_2 - y_1)\,\hat{j}$$

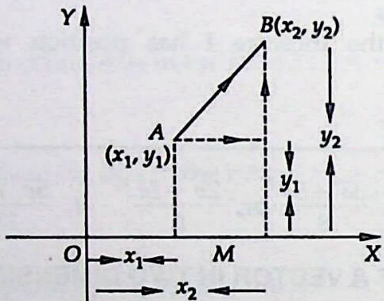


Fig. 23.53

Now, $\quad \overrightarrow{AB} = \overrightarrow{AF} + \overrightarrow{FB}$

$\Rightarrow \quad \overrightarrow{AB} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j}$

$\Rightarrow \quad$ Component of $\overrightarrow{AB}$ along x-axis $= (x_2 - x_1)\,\hat{i}$

and, $\quad$ Component of $\overrightarrow{AB}$ alog y-axis $= (y_2 - y_1)\,\hat{j}$

Also, $\quad |\overrightarrow{AB}| = AB = \sqrt{AF^2 + FB^2}$

$\Rightarrow \quad |\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

23.11.2 ADDITION, SUBTRACTION, MULTIPLICATION OF A VECTOR BY A SCALAR AND EQUALITY IN TERMS OF COMPONENTS

For any two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j}$, we define

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\,\hat{i} + (a_2 + b_2)\,\hat{j}$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1)\,\hat{i} + (a_2 - b_2)\,\hat{j}$

(iii) $m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j}$, where m is a scalar

(iv) $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1$ and $a_2 = b_2$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.*

SOLUTION We know that

$$a_1\hat{i} + b_1\hat{j} = a_2\hat{i} + b_2\hat{j}$$

$$\Leftrightarrow \quad a_1 = a_2 \text{ and } b_1 = b_2$$

$$\therefore \quad 2\hat{i} + 3\hat{j} = x\hat{i} + y\hat{j} \Rightarrow x = 2 \text{ and } y = 3.$$

EXAMPLE 2 *Let O be the origin and let P(– 4, 3) be a point in the xy-plane. Express $\vec{OP}$ in terms of vectors $\hat{i}$ and $\hat{j}$. Also, find $|\vec{OP}|$.*

SOLUTION The position vector of point P is $-4\hat{i} + 3\hat{j}$.

$$\therefore \quad \vec{OP} = -4\hat{i} + 3\hat{j}$$

$$\Rightarrow \quad |\vec{OP}| = \sqrt{(-4)^2 + 3^2} = 5$$

EXAMPLE 3 *Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$. Is $|\vec{a}| = |\vec{b}|$? Are the vectors $\vec{a}$ and $\vec{b}$ equal?*

SOLUTION We have,

$$\vec{a} = \hat{i} + 2\hat{j} \text{ and } \vec{b} = 2\hat{i} + \hat{j}$$

$$\therefore \quad |\vec{a}| = \sqrt{1+4} = \sqrt{5} \text{ and } |\vec{b}| = \sqrt{4+1} = \sqrt{5}$$

So, $|\vec{a}| = |\vec{b}|$.

But, given vectors are not equal as their corresponding components are not equal.

EXAMPLE 4 *If the position vector $\vec{a}$ of a point (12, n) is such that $|\vec{a}| = 13$, find the value of n.*

SOLUTION The position vector of the point $(12, n)$ is $12\hat{i} + n\hat{j}$

$$\therefore \quad \vec{a} = 12\hat{i} + n\hat{j}$$

$$|\vec{a}| = \sqrt{12^2 + n^2}$$

Now, $\quad |\vec{a}| = 13$

$$\Rightarrow \quad 13 = \sqrt{12^2 + n^2} \Rightarrow 169 = 144 + n^2 \Rightarrow n^2 = 25 \Rightarrow n = \pm 5.$$

EXAMPLE 5 *Find the components along the coordinates of the position vector of each of the following points:*

(i) $P(5, 4)$ (ii) $Q(-4, 3)$ (iii) $R(5, -7)$ (iv) $S(-4, -5)$.

SOLUTION Let O be the origin.

(i) We have,

$$\vec{OP} = 5\hat{i} + 4\hat{j}$$

So, component of $\vec{OP}$ along x-axis is a vector of magnitude 5 having its direction along the positive direction of x-axis. Also, the component of $\vec{OP}$ along y-axis is a vector of magnitude 4 along the positive direction of y-axis.

(ii) We have,

$$\vec{OQ} = -4\hat{i} + 3\hat{j}$$

So, its component along x-axis is a vector of magnitude 4, having its direction along the negative direction of x-axis. Also, the component of $\vec{OQ}$ along y-axis is a vector of magnitude 3, having its direction along the positive direction of y-axis.

(iii) We have,

$\overrightarrow{OR} = 5\hat{i} - 7\hat{j}$

So, its component along x-axis is a vector of magnitude 5, having its direction along the positive direction of x-axis. Also, the component of $\overrightarrow{OR}$ along y-axis is a vector of magnitude 7, having its direction along the (–) negative direction of y-axis.

(iv) We have,

$\overrightarrow{OS} = -4\hat{i} - 5\hat{j}$

So, its component along x-axis is a vector of magnitude 4, having its direction along the negative direction of x-axis. Also, the component of $\overrightarrow{OS}$ along y-axis is a vector of magnitude 5, having its direction along the negative direction of y-axis.

EXAMPLE 6 *Find the scalar and vector components of the vector with initial point A (2, 1) and terminal point B (– 5, 7).*

SOLUTION We have, $\overrightarrow{AB}$ = Position Vector of B – Position vector of A

$$\Rightarrow \quad \overrightarrow{AB} = (-5\hat{i} + 7\hat{j}) - (2\hat{i} + \hat{j})$$

$$\Rightarrow \quad \overrightarrow{AB} = -7\hat{i} + 6\hat{j}$$

So, scalar components of $\overrightarrow{AB}$ along OX and OY are – 7 and 6 respectively. The vector components of $\overrightarrow{AB}$ are $-7\hat{i}$ and $6\hat{j}$ along OX and OY respectively.

EXAMPLE 7 *If A = (0, 1) B = (1, 0), C = (1, 2), D = (2, 1), prove that* $\overrightarrow{AB} = \overrightarrow{CD}$.

SOLUTION We have, $\overrightarrow{AB}$ = Position vector of B – Positive vector of A

$$\Rightarrow \quad \overrightarrow{AB} = (\hat{i} + 0\hat{j}) - (0\hat{i} + \hat{j}) = \hat{i} - \hat{j}$$

and, $\overrightarrow{CD}$ = Position vector of D – Position vector of C

$$\Rightarrow \quad \overrightarrow{CD} = (2\hat{i} + \hat{j}) - (\hat{i} + 2\hat{j}) = \hat{i} - \hat{j}.$$

Cleary, $\overrightarrow{AB} = \overrightarrow{CD}$.

EXAMPLE 8 *If the position vector $\vec{a}$ of the point (5, n) is such that $|\vec{a}| = 13$, find the value of n.*

SOLUTION We have, $\vec{a} = 5\hat{i} + n\hat{j}$

$$\therefore \quad |\vec{a}| = \sqrt{25 + n^2}$$

$$\Rightarrow \quad 13 = \sqrt{25 + n^2} \Rightarrow 169 = 25 + n^2 \Rightarrow n^2 = 144 \Rightarrow n = \pm 12$$

EXAMPLE 9 *If $\vec{a}, \vec{b}$ are the position vectors of the points (1, –1), (–2, m), find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear.*

SOLUTION We have, $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = -2\hat{i} + m\hat{j}$

Since $\vec{a}$ and $\vec{b}$ are collinear.

$$\therefore \quad \vec{a} = \lambda \vec{b}, \text{ for some scalar } \lambda$$

$$\Rightarrow \quad \hat{i} - \hat{j} = \lambda(-2\hat{i} + m\hat{j})$$

$$\Rightarrow \quad \hat{i} - \hat{j} = (-2\lambda)\hat{i} + (m\lambda)\hat{j}$$

$$\Rightarrow \quad 1 = -2\lambda \text{ and } -1 = m\lambda$$

$$\Rightarrow \quad \lambda = -\frac{1}{2} \text{ and } \lambda = -\frac{1}{m} \Rightarrow -\frac{1}{2} = -\frac{1}{m} \Rightarrow m = 2$$

EXAMPLE 10 *Using, vectors, show that the points $A(-2, 1)$, $B(-5, -1)$ and $C(1, 3)$ are collinear.*

SOLUTION We have,

$$\vec{AB} = \text{Position vector of } B - \text{Position vector of } A$$

$$\Rightarrow \quad \vec{AB} = (-5\hat{i} - \hat{j}) - (-2\hat{i} + \hat{j}) = -3\hat{i} - 2\hat{j}$$

and,
$$\vec{BC} = \text{Position vector of } C - \text{Position vector of } B$$

$$\Rightarrow \quad \vec{BC} = (\hat{i} + 3\hat{j}) - (-5\hat{i} - \hat{j})$$

$$\Rightarrow \quad \vec{BC} = 6\hat{i} + 4\hat{j}$$

Clearly, $\vec{BC} = -2\,\vec{AB}$.

Therefore, $\vec{AB}$ and $\vec{BC}$ are parallel vectors.

But, B is a common point of $\vec{AB}$ and $\vec{BC}$.

Hence, the points A, B, C are collinear.

EXAMPLE 11 *Find the coordinates of the tip of the position vector which is equivalent to $\vec{AB}$, where the coordinates of A and B are (3, 1) and (5, 0) respectively.*

SOLUTION Let O be the origin and let $P(x, y)$ be the required point. Then, P is the tip of the position vector $\vec{OP}$ of point P.

We have,

$$\vec{OP} = x\hat{i} + y\hat{j}$$

and,
$$\vec{AB} = \text{Position vector of } B - \text{Position vector of } A$$

$$\Rightarrow \quad \vec{AB} = (5\hat{i} + 0\hat{j}) - (3\hat{i} + \hat{j}) = 2\hat{i} - \hat{j}$$

$$\therefore \quad \vec{OP} = \vec{AB}$$

$$\Rightarrow \quad x\hat{i} + y\hat{j} = 2\hat{i} - \hat{j} \Leftrightarrow x = 2 \text{ and } y = -1$$

Hence, the coordinates of the required point are $(2, -1)$.

EXAMPLE 12 *If $\vec{a}$ is a position vector whose tip is $(1, -3)$. Find the coordinates of the point B such that $\vec{AB} = \vec{a}$. If A has coordinates $(-1, 5)$.*

SOLUTION Let O be the origin and let $P(1, -3)$ be the tip of the position vector $\vec{a}$. Then, $\vec{a} = \vec{OP} = \hat{i} - 3\hat{j}$

Let the coordinates of B be (x, y) and A has coordinates $(-1, 5)$

$$\therefore \quad \vec{AB} = \text{Position vector of } B - \text{Position vector of } A$$

$$\Rightarrow \quad \vec{AB} = (x\hat{i} + y\hat{j}) - (-\hat{i} + 5\hat{j})$$

$$\Rightarrow \quad \vec{AB} = (x + 1)\,\hat{i} + (y - 5)\,\hat{j}$$

Now,
$$\vec{AB} = \vec{a}$$

$$\Rightarrow \quad (x + 1)\,\hat{i} + (y - 5)\,\hat{j} = \hat{i} - 3\hat{j}$$

$$\Rightarrow \quad x + 1 = 1 \text{ and } y - 5 = -3 \Rightarrow x = 0 \text{ and } y = 2$$

Hence, the coordinates of B are $(0, 2)$.

EXAMPLE 13 *ABCD is a parallelogram. If the coordinates of A, B, C are (2, 3), (1, 4) and (0, −2) respectively, find the coordinates of D.*

SOLUTION Let the coordinates of D be (x, y). Since $ABCD$ is a parallelogram

$\therefore \quad \vec{AB} = \vec{DC}$

$\Rightarrow \quad (\hat{i} + 4\hat{j}) - (2\hat{i} + 3\hat{j}) = (0\hat{i} - 2\hat{j}) - (x\hat{i} + y\hat{j})$

$\Rightarrow \quad -\hat{i} + \hat{j} = -x\hat{i} - (y+2)\hat{j}$

$\Rightarrow \quad -1 = -x \text{ and } 1 = -(y+2)$

$\Rightarrow \quad x = 1 \text{ and } y = -3$

Hence, the coordinates of D are $(1, -3)$.

EXAMPLE 14 *Find a unit vector parallel to the vector* $-3\hat{i} + 4\hat{j}$.

SOLUTION Let $\vec{a} = -3\hat{i} + 4\hat{j}$. Then,

$$|\vec{a}| = \sqrt{(-3)^2 + (4)^2} = 5$$

$\therefore$ Unit vector parallel to $\vec{a} = \hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{1}{5}(-3\hat{i} + 4\hat{j}) = -\dfrac{3}{5}\hat{i} + \dfrac{4}{5}\hat{j}$

EXAMPLE 15 *Find a vector of magnitude 5 units which is parallel to the vector* $2\hat{i} - \hat{j}$. [NCERT]

SOLUTION Let $\vec{a} = 2\hat{i} - \hat{j}$. Then,

$$|\vec{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$\therefore$ A unit vector parallel to $\vec{a} = \hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$\Rightarrow \quad \hat{a} = \dfrac{1}{\sqrt{5}}(2\hat{i} - \hat{j}) = \dfrac{2}{\sqrt{5}}\hat{i} - \dfrac{1}{\sqrt{5}}\hat{j}$

Hence, required vector $= 5\hat{a} = 5\left(\dfrac{2}{\sqrt{5}}\hat{i} - \dfrac{1}{\sqrt{5}}\hat{j}\right) = 2\sqrt{5}\,\hat{i} - \sqrt{5}\,\hat{j}$.

EXAMPLE 16 *Write all the unit vectors in XY-plane.*

SOLUTION Let $\vec{r} = x\hat{i} + y\hat{j}$ be a unit vector in XY-plane. Then,

$$|\vec{r}| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$$

Clearly, $P(x, y)$ lies on the circle $x^2 + y^2 = 1$ whose centre is at the origin O.

Suppose OP makes an angle θ with OX. Then,

$$x = OL = \cos\theta \text{ and } y = LP = \sin\theta \qquad \left[\because \cos\theta = \frac{OL}{OP} \text{ and } \sin\theta = \frac{LP}{OP}\right]$$

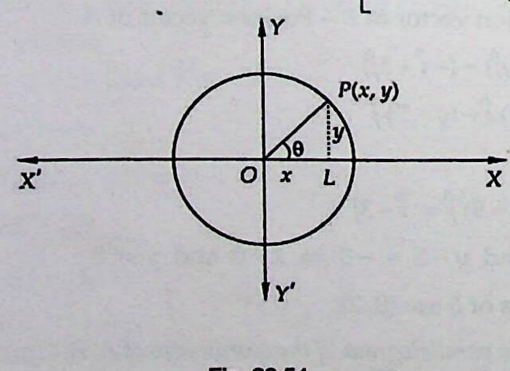


Fig. 23.54

$\therefore \qquad \vec{r} = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$

Clearly, as θ varies from 0 to 2π, the point P traces the circle $x^2 + y^2 = 1$ in counter clockwise sense and this covers all possible directions of $\vec{r}$.

Hence, $\vec{r} = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$ gives every unit vector in XY-plane.

EXAMPLE 17 *Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.* **[NCERT]**

SOLUTION Let $P(x, y)$ be a point in XY-plane such that $OP = 1$ and $\angle XOP = 30°$. Then,

$$x = OP\cos 30° \text{ and } y = OP\sin 30°$$

$$\Rightarrow \qquad x = \frac{\sqrt{3}}{2} \text{ and } y = \frac{1}{2}$$

$$\therefore \qquad \vec{OP} = x\hat{i} + y\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j} \text{ is the required unit vector.}$$

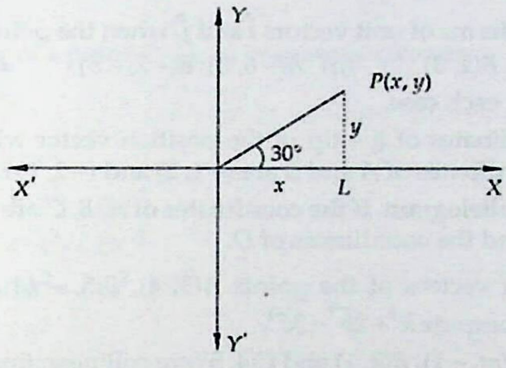


Fig. 23.55

EXAMPLE 18 *A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.* **[NCERT]**

SOLUTION Let $B(x, y)$ be the final position of the girl and O be the initial point of departure. Then,

$$AL = AB\cos 60° = \frac{3}{2} \text{ and, } BL = AB\sin 60° = \frac{3\sqrt{3}}{2}$$

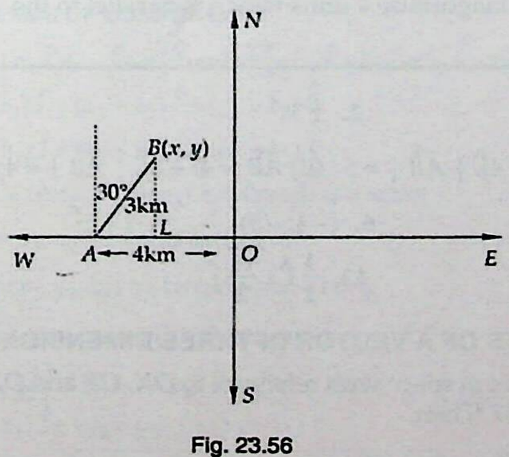


Fig. 23.56

$\therefore \quad OL = OA - AL = \left(4 - \frac{3}{2}\right) = \frac{5}{2}$ and, $BL = \frac{3\sqrt{3}}{2}$

Clearly, $B(x, y)$ lies in second quadrant.

So, coordinates of B are $\left(-\frac{5}{2}, \frac{3\sqrt{3}}{2}\right)$. Hence, position vector of B is $-\frac{5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

EXERCISE 23.4

1. If the position vector of a point $(-4, -3)$ be $\vec{a}$, find $|\vec{a}|$.
2. If the position vector $\vec{a}$ of a point $(12, n)$ is such that $|\vec{a}| = 13$, find the value(s) of n.
3. Find the components along the coordinate axes of the position vector of each of the following points:
   (i) $P(3, 2)$ (ii) $Q(-5, 1)$ (iii) $R(-11, -9)$ (iv) $S(4, -3)$
4. Express $\vec{AB}$ in terms of unit vectors $\hat{i}$ and $\hat{j}$, when the points are:
   (i) $A(4, -1), B(1, 3)$ (ii) $A(-6, 3), B(-2, -5)$
   Find $|\vec{AB}|$ in each case.
5. Find the coordinates of the tip of the position vector which is equivalent to $\vec{AB}$, where the coordinates of A and B are $(-1, 3)$ and $(-2, 1)$ respectively.
6. $ABCD$ is a parallelogram. If the coordinates of A, B, C are $(-2, -1)$ $(3, 0)$ and $(1, -2)$ respectively find the coordinates of D.
7. If the position vectors of the points $A(3, 4)$, $B(5, -6)$ and $C(4, -1)$ are $\vec{a}, \vec{b}, \vec{c}$ respectively, compute $\vec{a} + 2\vec{b} - 3\vec{c}$.
8. If the points $A(m, -1)$, $B(2, 1)$ and $C(4, 5)$ are collinear, find the value of m.
9. Show that the points $(3, 4)$, $(-5, 16)$, $(5, 1)$ are collinear.
10. If the vectors $\vec{a} = 2\hat{i} - 3\hat{j}$ and $\vec{b} = -6\hat{i} + mj$ are collinear, find the value of m.
11. If $\vec{a}$ be the position vector whose tip is $(5, -3)$, find the coordinates of a point B such that $\vec{AB} = \vec{a}$, the coordinates of A being $(4, -1)$.
12. Show that the points $2\hat{i}$, $-\hat{i} - 4\hat{j}$ and $-\hat{i} + 4\hat{j}$ form an isosceles triangle.
13. Find a unit vector parallel to the vector $\hat{i} + \sqrt{3}\,\hat{j}$.
14. Find a vector of magnitude 4 units which is parallel to the vector $\sqrt{3}\hat{i} + \hat{j}$.

ANSWERS

1. 5 2. ± 5 3. $3i$, $2j$

4. (i) $\vec{AB} = -3\hat{i} + 4\hat{j}$, $|\vec{AB}| = 5$ (ii) $\vec{AB} = 4\hat{i} - 8\hat{j}$, $|\vec{AB}| = 4\sqrt{5}$

5. $(-1, -2)$ 6. $(-4, -3)$ 7. $\hat{i} - 5\hat{j}$ 8. 1 10. 9

11. $(9, -4)$ 13. $\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$ 14. $2\sqrt{3}\,\hat{i} + 2\hat{j}$

23.12 COMPONENTS OF A VECTOR IN THREE DIMENSIONS

Let $P(x, y, z)$ be a point in space with reference to OX, OY and OZ as the coordinate axes as shown in Fig. 23.57. Then,

$$OA = x, OB = y \text{ and } OC = z$$

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be unit vectors along OX, OY and OZ respectively. Then,

$$\vec{OA} = x\hat{i},\ \vec{OB} = y\hat{j} \text{ and } \vec{OC} = z\hat{k}.$$

From Fig. 23.53, we have

$$\vec{BC'} = \vec{OA} = x\hat{i},\ \vec{C'P} = \vec{OC} = z\hat{k}$$

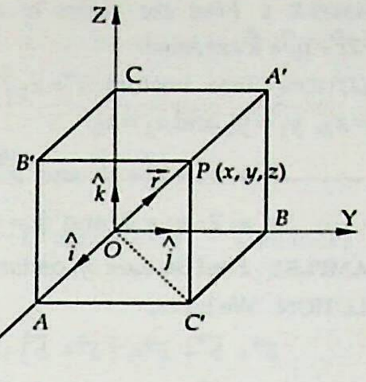


Fig. 23.57

Now, $\vec{OP} = \vec{OC'} + \vec{C'P}$

$\Rightarrow \quad \vec{OP} = \vec{OB} + \vec{BC'} + \vec{C'P}$

$\Rightarrow \quad \vec{OP} = \vec{OB} + \vec{OA} + \vec{OC}$

$\Rightarrow \quad \vec{OP} = \vec{OA} + \vec{OB} + \vec{OC}$

$\Rightarrow \quad \vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

If $\vec{OP} = \vec{r}$. Then, we have

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Thus, the position vector of a point $P(x, y, z)$ in space is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Now,

$OP^2 = OC'^2 + C'P^2$

$\Rightarrow \quad OP^2 = (OB^2 + BC'^2) + C'P^2$

$\Rightarrow \quad OP^2 = (OB^2 + OA^2) + OC^2$

$\Rightarrow \quad OP^2 = OA^2 + OB^2 + OC^2$

$\Rightarrow \quad OP^2 = x^2 + y^2 + z^2$

$\Rightarrow \quad OP = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow \quad |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Thus, if a point P in space has coordinates (x, y, z), then its position vector $\vec{r}$ *is* $x\hat{i} + y\hat{j} + z\hat{k}$ *and* $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

The vectors $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ *are known as the component vectors of* $\vec{r}$ *along x, y and z axes respectively.*

23.12.1 ADDITION, SUBTRACTION AND MULTIPLICATION OF A VECTOR BY A SCALAR AND EQUALITY IN TERMS OF COMPONENTS

For any two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ *and* $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, *we define*

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

(iii) $m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j} + (ma_3)\hat{k}$, *where m is a scalar*

(iv) $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. Then,

$\vec{PQ}$ = Position vector of Q – Position vector of P

$\Rightarrow \quad \vec{PQ} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$\Rightarrow \quad \vec{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$\therefore \quad PQ = |\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the values of x, y and z so that the vectors* $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ *and* $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ *are equal.* **[NCERT]**

SOLUTION Two vectors $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ are equal iff $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.

$\therefore$ $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ and $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ are equal iff.

$$x = 2,\ y = 2 \text{ and } z = 1$$

EXAMPLE 2 *Find the sum of vectors* $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ *and* $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

SOLUTION We have,

$$\vec{a} + \vec{b} + \vec{c} = (\vec{a} + \vec{b}) + \vec{c}$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = \{(\hat{i} - 2\hat{j} + \hat{k}) + (-2\hat{i} + 4\hat{j} + 5\hat{k})\} + (\hat{i} - 6\hat{j} - 7\hat{k})$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = \{(1-2)\hat{i} + (-2+4)\hat{j} + (1+5)\hat{k}\} + (\hat{i} - 6\hat{j} - 7\hat{k})$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = (-\hat{i} + 2\hat{j} + 6\hat{k}) + (\hat{i} - 6\hat{j} - 7\hat{k})$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = (-1+1)\hat{i} + (2-6)\hat{j} + (6-7)\hat{k}$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = 0\hat{i} - 4\hat{j} - \hat{k}$$

EXAMPLE 3 *Find the magnitude of the vector* $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$. **[NCERT]**

SOLUTION We have,

$$|\vec{a}| = \sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{49} = 7$$

EXAMPLE 4 *Find the unit vector in the direction of* $3\hat{i} - 6\hat{j} + 2\hat{k}$.

SOLUTION Let $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$. Then,

$$|\vec{a}| = \sqrt{3^2 + (-6)^2 + 2^2} = 7$$

So, unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}.$$

EXAMPLE 5 *Find the unit vector in the direction of* $\vec{a} + \vec{b}$, *if* $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ *and* $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$. **[NCERT]**

SOLUTION We have,

$$\vec{a} + \vec{b} = (2\hat{i} - \hat{j} + 2\hat{k}) + (-\hat{i} + \hat{j} - \hat{k})$$

$$\Rightarrow \quad \vec{a} + \vec{b} = (2-1)\hat{i} + (-1+1)\hat{j} + (2-1)\hat{k}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \hat{i} + 0\hat{j} + \hat{k}$$

$$\Rightarrow \quad |\vec{a} + \vec{b}| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\therefore \quad \text{Required unit vector} = \frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{1}{\sqrt{2}}(\hat{i} + 0\hat{j} + \hat{k}) = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}.$$

EXAMPLE 6 *Find the unit vector in the direction of* $\vec{PQ}$, *where P and Q are the points* (1, 2, 3) *and* (4, 5, 6) *respectively.* **[NCERT]**

SOLUTION Clearly,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$\Rightarrow \quad \overrightarrow{PQ} = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$\Rightarrow \quad \overrightarrow{PQ} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$\Rightarrow \quad |\overrightarrow{PQ}| = \sqrt{9+9+9} = 3\sqrt{3}$

$\therefore$ The unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{1}{|\overrightarrow{PQ}|}\overrightarrow{PQ} = \frac{1}{3\sqrt{3}}(3\hat{i} + 3\hat{j} + 3\hat{k}) = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

EXAMPLE 7 *Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.* **[NCERT]**

SOLUTION Let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$.

Clearly, $\vec{b} = -2\vec{a}$

$\Rightarrow \quad \vec{a}$ and $\vec{b}$ are collinear.

EXAMPLE 8 *Show that the three points $A(-2, 3, 5)$, $B(1, 2, 3)$ and $C(7, 0, -1)$ are collinear.*

SOLUTION We have,

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$\Rightarrow \quad \overrightarrow{AB} = (\hat{i} + 2\hat{j} + 3\hat{k}) - (-2\hat{i} + 3\hat{j} + 5\hat{k}) = 3\hat{i} - \hat{j} - 2\hat{k}$

$\overrightarrow{BC}$ = Position vector of C – Position vector of B

$\Rightarrow \quad \overrightarrow{BC} = (7\hat{i} + 0\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 6\hat{i} - 2\hat{j} - 4\hat{k}$

$\Rightarrow \quad \overrightarrow{BC} = 2(3\hat{i} - \hat{j} - 2\hat{k})$

Clearly, $\overrightarrow{BC} = 2\,\overrightarrow{AB}$.

This shows that the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel.

But, B is a common point. So, the given points A, B and C are collinear.

EXAMPLE 9 *Find the distance between the points $A(2, 3, 1)$, $B(-1, 2, -3)$, using vector method.*

SOLUTION We have,

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$\overrightarrow{AB} = (-\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} + 3\hat{j} + \hat{k}) = -3\hat{i} - \hat{j} - 4\hat{k}$

$\therefore \quad AB = |\overrightarrow{AB}| = \sqrt{(-3)^2 + (-1)^2 + (-4)^2} = \sqrt{26}$

EXAMPLE 10 *If $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 4\hat{j} - 3\hat{k}$, find $|\vec{a} - 2\vec{b}|$*

SOLUTION We have,

$\vec{a} - 2\vec{b} = (3\hat{i} - 2\hat{j} + \hat{k}) - 2(2\hat{i} - 4\hat{j} - 3\hat{k}) = -\hat{i} + 6\hat{j} + 7\hat{k}$

$\therefore \quad |\vec{a} - 2\vec{b}| = |-\hat{i} + 6\hat{j} + 7\hat{k}| = \sqrt{(-1)^2 + 6^2 + 7^2} = \sqrt{86}$.

EXAMPLE 11 *The position vectors of the points P, Q, R are $\hat{i} + 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} + 5\hat{k}$ and $7\hat{i} - \hat{k}$ respectively. Prove that P, Q and R are collinear.* **[NCERT]**

SOLUTION We have,

$\overrightarrow{PQ}$ = Position vector of Q – Position vector of P

$\Rightarrow \quad \overrightarrow{PQ} = (-2\hat{i} + 3\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = -3\hat{i} + \hat{j} + 2\hat{k}$

and, $\quad \vec{QR}$ = Position vector of R – Position vector of Q

$\Rightarrow \quad \vec{QR} = (7\hat{i} - \hat{k}) - (-2\hat{i} + 3\hat{j} + 5\hat{k}) = 9\hat{i} - 3\hat{j} - 6\hat{k}$

Clearly, $\quad \vec{QR} = -3\,\vec{PQ}$.

This shows that the vectors $\vec{PQ}$ and $\vec{QR}$ are collinear.

But, Q is common point between $\vec{PQ}$ and $\vec{QR}$. Therefore, given points P, Q and R are collinear.

EXAMPLE 12 *If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.*

SOLUTION Let $ABCD$ be a parallelogram such that $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$. Then,

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\Rightarrow \quad \vec{AC} = \vec{a} + \vec{b} = 3\hat{i} + 6\hat{j} - 2\hat{k}$$

and, $\quad \vec{AB} + \vec{BD} = \vec{AD}$

$$\Rightarrow \quad \vec{BD} = \vec{AD} - \vec{AB}$$

$$\Rightarrow \quad \vec{BD} = \vec{b} - \vec{a} = \hat{i} + 2\hat{j} - 8\hat{k}$$

Now, $\quad \vec{AC} = 3\hat{i} + 6\hat{j} - 2\hat{k}$

$$\Rightarrow \quad |\vec{AC}| = \sqrt{9 + 36 + 4} = 7$$

and, $\quad \vec{BD} = \hat{i} + 2\hat{j} - 8\hat{k}$

$$\Rightarrow \quad |\vec{BD}| = \sqrt{1 + 4 + 64} = \sqrt{69}$$

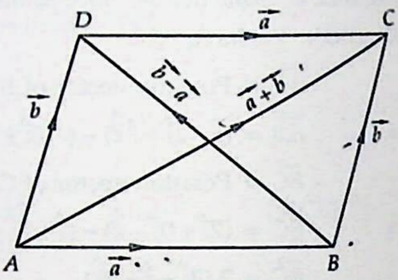


Fig. 23.58

$\therefore \quad$ Unit vector along $\vec{AC} = \dfrac{\vec{AC}}{|\vec{AC}|} = \dfrac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$

and, $\quad$ Unit vector along $\vec{BD} = \dfrac{\vec{BD}}{|\vec{BD}|} = \dfrac{1}{\sqrt{69}}(\hat{i} + 2\hat{j} - 8\hat{k})$.

EXAMPLE 13 *If the position vectors of the points A, B, C, D are $2\hat{i} + 4\hat{k}$, $5\hat{i} + 3\sqrt{3}\,\hat{j} + 4\hat{k}$, $-2\sqrt{3}\,\hat{j} + \hat{k}$ and $2\hat{i} + \hat{k}$ respectively, prove that CD is parallel to AB and $CD = \frac{2}{3}AB$.*

SOLUTION We have,

$\vec{AB}$ = Position vector of B – Position vector of A

$$\Rightarrow \quad \vec{AB} = (5\hat{i} + 3\sqrt{3}\hat{j} + 4\hat{k}) - (2\hat{i} + 4\hat{k}) = 3\hat{i} + 3\sqrt{3}\hat{j} + 0\hat{k} = 3(\hat{i} + \sqrt{3}\hat{j} + 0\hat{k})$$

and, $\quad \vec{CD}$ = Position vector of D – Position vector of C

$$\Rightarrow \quad \vec{CD} = (2\hat{i} + \hat{k}) - (-2\sqrt{3}\hat{j} + \hat{k}) = 2\hat{i} + 2\sqrt{3}\hat{j} + 0\hat{k} = 2(\hat{i} + \sqrt{3}\hat{j} + 0\hat{k})$$

$$\therefore \quad \vec{CD} = 2(\hat{i} + \sqrt{3}\hat{j} + 0\hat{k}) = \frac{2}{3}(3\hat{i} + 3\sqrt{3}\hat{j} + 0\hat{k}) = \frac{2}{3}\vec{AB}$$

Hence, CD is parallel to AB and $CD = \frac{2}{3}AB$.

EXAMPLE 14 *Show that the points A(6, –7, 0), B(16, –19, –4), C(0, 3, –6) and D(2, –5, 10) are such that AB and CD intersect at the point P(1, –1, 2).*

SOLUTION We have,

$$\vec{AP} = \text{Position vector of } P - \text{Position vector of } A$$

$$\Rightarrow \quad \vec{AP} = (\hat{i}-\hat{j}+2\hat{k}) - (6\hat{i}-7\hat{j}+0\hat{k})$$

and, $$\vec{PB} = \text{Position vector of } B - \text{Position vector of } P$$

$$\Rightarrow \quad \vec{PB} = (16\hat{i}-19\hat{j}-4\hat{k}) - (\hat{i}-\hat{j}+2\hat{k})$$

$$\Rightarrow \quad \vec{PB} = 15\hat{i}-18\hat{j}-6\hat{k} = -3(-5\hat{i}+6\hat{j}+2\hat{k})$$

Clearly, $\vec{PB} = -3\,\vec{AP}$.

So, vectors $\vec{AP}$ and $\vec{PB}$ are collinear.

But, P is a point common to $\vec{AP}$ and $\vec{PB}$.

Hence, P, A, B are collinear points.

Similarly, $\vec{CP} = (\hat{i}-\hat{j}+2\hat{k}) - (0\hat{i}+3\hat{j}-6\hat{k}) = \hat{i}-4\hat{j}+8\hat{k}$

and, $$\vec{PD} = (2\hat{i}-5\hat{j}+10\hat{k}) - (\hat{i}-\hat{j}+2\hat{k}) = \hat{i}-4\hat{j}+8\hat{k}$$

$$\therefore \quad \vec{CP} = \vec{PD}$$

So, vectors $\vec{CP}$ and $\vec{PD}$ are collinear.

But, P is a common point to $\vec{CP}$ and $\vec{CD}$.

Hence, C, P, D are collinear points.

Thus, A, B, C, D and P are points such that A, P, B and C, P, D are two sets of collinear points. Hence, AB and CD intersect at the point P.

EXAMPLE 15 *If A, B, C have position vectors (2, 0, 0) (0, 1, 0), (0, 0, 2), show that $\Delta\, ABC$ is isosceles.*

SOLUTION We have,

$$\vec{AB} = \text{Position vector of } B - \text{Position vector of } A$$

$$\Rightarrow \quad \vec{AB} = (0\hat{i}+\hat{j}+0\hat{k}) - (2\hat{i}+0\hat{j}+0\hat{k}),\ = -2\hat{i}+\hat{j}+0\hat{k}$$

$$\Rightarrow \quad AB = |\vec{AB}| = \sqrt{(-2)^2+1^2+0^2} = \sqrt{5}$$

$$\vec{BC} = \text{Position vector of } C - \text{Position vector of } B$$

$$\Rightarrow \quad \vec{BC} = (0\hat{i}+0\hat{j}+2\hat{k}) - (0\hat{i}+\hat{j}+0\hat{k}) = 0\hat{i}-\hat{j}+2\hat{k}$$

$$\Rightarrow \quad BC = |\vec{BC}| = \sqrt{0^2+(-1)^2+2^2} = \sqrt{5}$$

Clearly, $AB = BC$.

Hence, $\Delta\, ABC$ is isosceles.

EXAMPLE 16 *Show that the points A, B and C with position vectors $\vec{a} = 3\hat{i}-4\hat{j}-4\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}-3\hat{j}-5\hat{k}$ respectively, form the vertices of a right angled triangle.* **[NCERT]**

SOLUTION We have,

$$\vec{AB} = \vec{b}-\vec{a} = (2\hat{i}-\hat{j}+\hat{k}) - (3\hat{i}-4\hat{j}-4\hat{k}) = -\hat{i}+3\hat{j}+5\hat{k}$$

$$\vec{BC} = \vec{c}-\vec{b} = (\hat{i}-3\hat{j}-5\hat{k}) - (2\hat{i}-\hat{j}+\hat{k}) = -\hat{i}-2\hat{j}-6\hat{k}$$

and, $$\vec{CA} = \vec{a}-\vec{c} = (3\hat{i}-4\hat{j}-4\hat{k}) - (\hat{i}-3\hat{j}-5\hat{k}) = 2\hat{i}-\hat{j}+\hat{k}$$

Clearly, $\vec{AB}+\vec{BC}+\vec{CA} = \vec{0}$.

So, points A, B and C form a triangle.

Now,

$$|\overrightarrow{AB}| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{35}$$

$$|\overrightarrow{BC}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{41}$$

and, $$|\overrightarrow{CA}| = \sqrt{2^2 + (-1)^2 + (1)^2} = \sqrt{6}$$

Clearly, $|\overrightarrow{BC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2$

Hence, ΔABC is a right-angled triangle.

EXAMPLE 17 *Three vectors of magnitude a, 2a, 3a meet in a point and their directions are along the diagonals of the adjacent faces of a cube. Determine their resultant.*

SOLUTION Consider a unit cube whose one vertex is at the origin and three coterminous edges OA, OB and OC along the coordinate exes OX, OY and OZ respectively. Then, $\overrightarrow{OA} = \hat{i}$, $\overrightarrow{OB} = \hat{j}$ and $\overrightarrow{OC} = \hat{k}$.

Let OD, OE and OF be the diagonals of three adjacent faces of the cube passing through O along which act the vectors of magnitude a, $2a$ and $3a$ respectively.

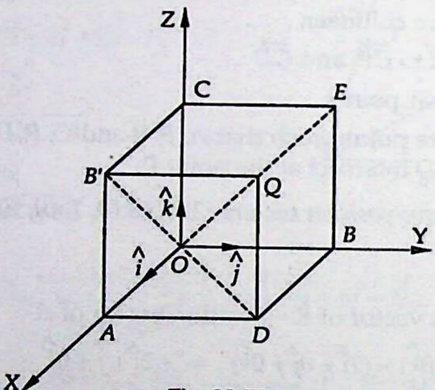


Fig. 23.59

We have,

$$\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = \overrightarrow{OB} + \overrightarrow{OA} = \hat{j} + \hat{i}$$

$\therefore$ $$|\overrightarrow{OD}| = \sqrt{1+1} = \sqrt{2}$$

Thus, the unit vector along $\overrightarrow{OD}$ is $\frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$.

Similarly, unit vectors along $\overrightarrow{OE}$ and $\overrightarrow{OF}$ are $\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})$ and $\frac{1}{\sqrt{2}}(\hat{i}+\hat{k})$ respectively.

A vector of magnitude 'a' along OD is given by

$$\vec{r_1} = a \cdot \frac{1}{\sqrt{2}}(\hat{i}+\hat{j}) = \frac{a}{\sqrt{2}}(\hat{i}+\hat{j})$$

Similarly, vectors of magnitude $2a$ and $3a$ along OE and OF are given by $\vec{r_2} = 2a \cdot \frac{1}{\sqrt{2}}(\hat{j}+\hat{k}) = \frac{2a}{\sqrt{2}}(\hat{j}+\hat{k})$ and, $\vec{r_3} = \frac{3a}{\sqrt{2}}(\hat{i}+\hat{k})$ respectively.

Hence, if $\vec{r}$ is the resultant of $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$. Then,

$$\vec{r} = \vec{r_1} + \vec{r_2} + \vec{r_3}$$

$$\Rightarrow \quad \vec{r} = \frac{a}{\sqrt{2}}(\hat{i}+\hat{j}) + \frac{2a}{\sqrt{2}}(\hat{j}+\hat{k}) + \frac{3a}{\sqrt{2}}(\hat{k}+\hat{i})$$

$$\Rightarrow \quad \vec{r} = \frac{a}{\sqrt{2}}(4\hat{i}+3\hat{j}+5\hat{k})$$

$$\Rightarrow \quad |\vec{r}| = \frac{a}{\sqrt{2}}\sqrt{16+9+25} = 5a.$$

EXERCISE 23.5

1. Find the magnitude of the vector $\vec{a} = 2\hat{i}+3\hat{j}-6\hat{k}$.
2. Find the unit vector in the direction of $3\hat{i}+4\hat{j}-12\hat{k}$.
3. Find a unit vector in the direction of the resultant of the vectors $\hat{i}-\hat{j}+3\hat{k}$, $2\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}+2\hat{j}-2\hat{k}$.
4. The adjacent sides of a parallelogram are represented by the vectors $\vec{a} = \hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = -2\hat{i}+\hat{j}+2\hat{k}$. Find unit vectors parallel to the diagonals of the parallelogram. **[NCERT]**
5. If $\vec{a} = 3\hat{i}-\hat{j}-4\hat{k}$, $\vec{b} = -2\hat{i}+4\hat{j}-3\hat{k}$ and $\vec{c} = \hat{i}+2\hat{j}-\hat{k}$, find $|3\vec{a}-2\vec{b}+4\vec{c}|$.
6. If $\vec{PQ} = 3\hat{i}+2\hat{j}-\hat{k}$ and the coordinates of P are $(1, -1, 2)$, find the coordinates of Q.
7. Prove that the points $\hat{i}-\hat{j}$, $4\hat{i}+3\hat{j}+\hat{k}$ and $2\hat{i}-4\hat{j}+5\hat{k}$ are the vertices of a right-angled triangle.
8. If the vertices of a triangle are the points with position vectors $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, what are the vectors determined by its sides ? Find the length of these vectors.
9. Find the vector from the origin O to the centroid of the triangle whose vertices are $(1, -1, 2)$, $(2, 1, 3)$ and $(-1, 2, -1)$.
10. Find the position vector of a point R which divides the line segment joining points $P(\hat{i}+2\hat{j}+\hat{k})$ and $Q(-\hat{i}+\hat{j}+\hat{k})$ in the ratio 2:1.
    (i) internally (ii) externally **[NCERT]**
11. Find the position vector of the mid-point of the vector joining the points $P(2\hat{i}-3\hat{j}+4\hat{k})$ and $Q(4\hat{i}+\hat{j}-2\hat{k})$.
12. Find the unit vector in the direction of vector $\vec{PQ}$, where P and Q are the points $(1, 2, 3)$ and $(4, 5, 6)$.
13. Show that the points $A(2\hat{i}-\hat{j}+\hat{k})$, $B(\hat{i}-3\hat{j}-5\hat{k})$, $C(3\hat{i}-4\hat{j}-4\hat{k})$ are the vertices of a right angled triangle. **[NCERT]**
14. Find the position vector of the mid-point of the vector joining the points $P(2, 3, 4)$ and $Q(4, 1, -2)$. **[NCERT]**
15. Find the value of x for which $x(\hat{i}+\hat{j}+\hat{k})$ is a unit vector. **[NCERT]**
16. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c} = \hat{i}-2\hat{j}+\hat{k}$, find a unit vector parallel to $2\vec{a}-\vec{b}+3\vec{c}$. **[NCERT]**
17. Show that the points $A(1, -2, -8)$, $B(5, 0, -2)$ and $C(11, 3, 7)$ are collinear, and find the ratio in which B divides AC. **[NCERT]**
18. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 4\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c} = \hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude 6 units which is parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$. **[CBSE 2010]**

ANSWERS

1. 7
2. $\frac{3}{13}\hat{i}+\frac{4}{13}\hat{j}-\frac{12}{13}\hat{k}$
3. $\frac{1}{\sqrt{21}}(4\hat{i}+2\hat{j}-\hat{k})$
4. $\frac{1}{\sqrt{6}}(-\hat{i}+2\hat{j}+\hat{k})$, $\frac{1}{\sqrt{2}}(\hat{i}-\hat{k})$
5. $\sqrt{398}$
6. $(4, 1, 1)$

9. $2\frac{\hat{i}}{3}+\frac{2}{3}\hat{j}+\frac{4}{3}\hat{k}$ 10. (i) $\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$ (ii) $-3\hat{i}+3\hat{k}$

11. $3\hat{i}-\hat{j}+\hat{k}$ 12. $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$ 14. $3\hat{i}+2\hat{j}+\hat{k}$

15. $\pm\frac{1}{\sqrt{3}}$ 16. $\frac{3}{\sqrt{22}}\hat{i}-\frac{3}{\sqrt{22}}\hat{j}+\frac{2}{\sqrt{22}}\hat{k}$ 17. $2:3$ 18. $2\hat{i}-4\hat{j}+4\hat{k}$

23.13 COLLINEARITY

23.13.1 COLLINEARITY OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two collinear or parallel vectors. Then their supports are parallel. Let $\hat{a}$ be the unit vector in the direction of $\vec{a}$. As $\vec{a}$ and $\vec{b}$ may be either like parallel vectors or unlike parallel vectors. So, the unit vector in the direction of $\vec{b}$ is either $\hat{a}$ or $-\hat{a}$.

$\therefore \quad \vec{a} = |\vec{a}|\,\hat{a}$ and $\vec{b} = \pm|\vec{b}|\,\hat{a}$

Now, $\quad \vec{a} = |\vec{a}|\,\hat{a}$

$$\Rightarrow \quad \vec{a} = \pm\frac{|\vec{a}|}{|\vec{b}|}\{\pm|\vec{b}|\,\hat{a}\}$$

$$\Rightarrow \quad \vec{a} = \lambda\vec{b}, \text{ where } \lambda = \pm\frac{|\vec{a}|}{|\vec{b}|}$$

Also, $\quad \vec{b} = \pm|\vec{b}|\,\hat{a}$

$$\Rightarrow \quad \vec{b} = \left\{\pm\frac{|\vec{b}|}{|\vec{a}|}\right\}|\vec{a}|\,\hat{a}$$

$$\Rightarrow \quad \vec{b} = \mu\vec{a}, \text{ where } \mu = \pm\frac{|\vec{b}|}{|\vec{a}|}$$

Thus, if $\vec{a}$ and $\vec{b}$ are two collinear or parallel vectors, then there exists a scalar λ such that $\vec{a}=\lambda\vec{b}$ or, $\vec{b}=\lambda\vec{a}$.

In the following theorem, we prove the general criterion for the coplanarity of two vectors.

THEOREM 1 *Two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear iff there exist scalars x, y not both zero such that $x\vec{a}+y\vec{b}=\vec{0}$.*

<u>PROOF</u> First, let $\vec{a}$ and $\vec{b}$ be two collinear vectors. Then, there exists a scalar λ such that

$$\vec{a} = \lambda\vec{b}$$

$$\Rightarrow \quad 1.\vec{a}+(-\lambda)\vec{b} = \vec{0}$$

$$\Rightarrow \quad x\vec{a}+y\vec{b} = \vec{0}, \text{ where } x = 1 \text{ and } y = -\lambda.$$

Conversely, let $\vec{a}$ and $\vec{b}$ be two non-zero vectors such that $x\vec{a}+y\vec{b}=\vec{0}$ for some scalars x, y not both zero. Then, we have to prove that $\vec{a}$ and $\vec{b}$ are collinear vectors.

We have,

$$x\vec{a}+y\vec{b} = \vec{0}$$

Let $x \neq 0$. Then,

$$x\vec{a}+y\vec{b} = \vec{0}$$

$$\Rightarrow \quad x\vec{a} = -y\vec{b}$$

$$\Rightarrow \quad \vec{a} = \left(-\frac{y}{x}\right)\vec{b}$$

$\Rightarrow \quad \vec{a} = \lambda \vec{b}$, where $\lambda = -\frac{y}{x}$.

$\Rightarrow \quad \vec{a}$ and $\vec{b}$ are collinear vectors.

Q.E.D.

It follows from the above theorem that if two non-zero vectors are non-collinear, we cannot express one in terms of the other. In other words, their linear combination can never be the zero vector.

Following theorem proves that the linear combination of two non-zero vectors is zero if each scalars in the linear combintion is zero.

THEOREM 2 *If $\vec{a}, \vec{b}$ are any two non-zero non-collinear vectors and x, y are scalars, then*

$$x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = y = 0$$

PROOF If possible, let $x \neq 0$. Then,

$$x\vec{a} + y\vec{b} = \vec{0}$$

$\Rightarrow \quad x\vec{a} = -y\vec{b}$

$\Rightarrow \quad \vec{a} = \left(\frac{-y}{x}\right)\vec{b}$

$\Rightarrow \quad \vec{a}$ and $\vec{b}$ are collinear vectos.

This is a contradiction to the hypothesis that $\vec{a}$ and $\vec{b}$ are non-collinear vectors.

Therefore, $x = 0$.

Similarly, we have $y = 0$.

Hence, $\quad x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = y = 0$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $\vec{a}$ and $\vec{b}$ are non-collinear vectors such that $x_1 \vec{a} + y_1 \vec{b} = x_2 \vec{a} + y_2 \vec{b}$, then prove that $x_1 = x_2$ and $y_1 = y_2$.*

SOLUTION We have,

$$x_1 \vec{a} + y_1 \vec{b} = x_2 \vec{a} + y_2 \vec{b}$$

$\Rightarrow \quad (x_1 - x_2)\vec{a} + (y_1 - y_2)\vec{b} = \vec{0}$

$\Rightarrow \quad x_1 - x_2 = 0$ and $y_1 - y_2 = 0 \quad$ [$\because \vec{a}$ and $\vec{b}$ are non-collinear]

$\Rightarrow \quad x_1 = x_2$ and $y_1 = y_2$.

EXAMPLE 2 *If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, find the value of x for which vectors $\vec{\alpha} = (x-2)\vec{a} + \vec{b}$ and $\vec{\beta} = (3+2x)\vec{a} - 2\vec{b}$ are collinear.*

SOLUTION Since vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear. Therefore, there exist scalar λ such that

$$\vec{\alpha} = \lambda \vec{\beta}$$

$\Rightarrow \quad (x-2)\vec{a} + \vec{b} = \lambda \{(3+2x)\vec{a} - 2\vec{b}\}$

$\Rightarrow \quad \{x - 2 - \lambda(3+2x)\}\vec{a} + (1+2\lambda)\vec{b} = \vec{0}$

$\Rightarrow \quad x - 2 - \lambda(3+2x) = 0$ and $1 + 2\lambda = 0$

$\Rightarrow \quad x - 2 - \lambda(3+2x) = 0$ and $\lambda = -\frac{1}{2}$

$$\Rightarrow \quad x - 2 + \frac{1}{2}(3 + 2x) = 0$$

$$\Rightarrow \quad 4x + 1 = 0$$

$$\Rightarrow \quad x = -\frac{1}{4}$$

EXAMPLE 3 *If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, find the value of x for which the vectors $\vec{\alpha} = (2x+1)\vec{a} - \vec{b}$ and $\vec{\beta} = (x-2)\vec{a} + \vec{b}$ are collinear.*

SOLUTION Vectors $\vec{\alpha}$ and $\vec{\beta}$ will be collinear, if

$$\vec{\alpha} = m\,\vec{\beta} \text{ for some scalar } m$$

$$\Rightarrow \quad (2x+1)\vec{a} - \vec{b} = m\{(x-2)\vec{a} + \vec{b}\}$$

$$\Rightarrow \quad \{(2x+1) - m(x-2)\} - (m+1)\vec{b} = \vec{0}$$

$$\Rightarrow \quad (2x+1) - m(x-2)\}\vec{a} - (m+1)\vec{b} = \vec{0}$$

$$\Rightarrow \quad (2x-1) - m(x-2) = 0 \text{ and } -(m+1) = 0$$

$$\Rightarrow \quad m = -1 \text{ and } x = \frac{1}{3}.$$

EXAMPLE 4 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-null vectors such that any two of them are non-collinear. If $\vec{a} + \vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + \vec{c}$ is collinear with $\vec{a}$, then find $\vec{a} + \vec{b} + \vec{c}$.*

SOLUTION It is given that :

$$\vec{a} + \vec{b} \text{ is collinear with } \vec{c}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \lambda_1 \vec{c} \text{ for some scalar } \lambda_1 \qquad ...(i)$$

and, $\quad \vec{b} + \vec{c}$ is collinear with $\vec{a}$

$$\Rightarrow \quad \vec{b} + \vec{c} = \lambda_2 \vec{a} \text{ for some scalar } \lambda_2 \qquad ...(ii)$$

From (i), we have

$$\vec{a} = \lambda_1 \vec{c} - \vec{b}$$

Substituting this value in (ii), we get

$$\vec{b} + \vec{c} = \lambda_2 (\lambda_1 \vec{c} - \vec{b})$$

$$\Rightarrow \quad (1 + \lambda_2)\vec{b} + (1 - \lambda_1 \lambda_2)\vec{c} = \vec{0}$$

$$\Rightarrow \quad 1 + \lambda_2 = 0 \text{ and } 1 - \lambda_1 \lambda_2 = 0 \qquad [\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear}]$$

$$\Rightarrow \quad \lambda_2 = -1, \lambda_1 = -1$$

Substituting the values of λ_1 and λ_2 in (i) and (ii) respectively, we get

$$\vec{a} + \vec{b} = -\vec{c} \Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

EXAMPLE 5 *Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors such that any two of them are non-collinear. If $\vec{a} + 2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 3\vec{c}$ is collinear with $\vec{a}$, then prove that $\vec{a} + 2\vec{b} + 6\vec{c} = \vec{0}$.*

SOLUTION It is given that

$$\vec{a} + 2\vec{b} \text{ is collinear with } \vec{c}$$

$$\Rightarrow \quad \vec{a} + 2\vec{b} = \lambda \vec{c} \text{ for some scalar } \lambda \qquad ...(i)$$

and, $\quad \vec{b} + 3\vec{c}$ is collinear with $\vec{a}$

$\Rightarrow \quad \vec{b} + 3\vec{c} = \mu\vec{a}$ for some scalar μ ...(ii)

From (i), we have

$$\vec{a} = \lambda\vec{c} - 2\vec{b}$$

Substituting this value of $\vec{a}$ in (ii), we get

$$\vec{b} + 3\vec{c} = \mu(\lambda\vec{c} - 2\vec{b})$$

$$\Rightarrow \quad (1 + 2\mu)\vec{b} + (3 - \mu\lambda)\vec{c} = \vec{0}$$

$$\Rightarrow \quad 1 + 2\mu = 0 \text{ and } 3 - \mu\lambda = 0 \qquad [\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear vectors}]$$

$$\Rightarrow \quad \mu = -\frac{1}{2} \text{ and } \lambda = -6$$

Substituting the values of λ and μ in (i) and (ii), respectively, we get

$$\vec{a} + 2\vec{b} + 6\vec{c} = \vec{0}$$

EXAMPLE 6 *If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and vectors $\vec{\alpha} = (x + 4y)\vec{a} + (2x + y + 1)\vec{b}$ and $\vec{\beta} = (-2x + y + 2)\vec{a} + (2x - 3y - 1)\vec{b}$ are connected by the relation $3\vec{\alpha} = 2\vec{\beta}$, find x, y.*

SOLUTION We have,

$$3\vec{\alpha} = 2\vec{\beta}$$

$$\Rightarrow \quad 3\{(x + 4y)\vec{a} + (2x + y + 1)\vec{b}\} = 2\{(-2x + y + 2)\vec{a} + (2x - 3y - 1)\vec{b}\}$$

$$\Rightarrow \quad (3x + 12y + 4x - 2y - 4)\vec{a} + (6x + 3y + 3 - 4x + 6y + 2)\vec{b} = \vec{0}$$

$$\Rightarrow \quad (7x + 10y - 4)\vec{a} + (2x + 9y + 5)\vec{b} = \vec{0}$$

$$\Rightarrow \quad 7x + 10y - 4 = 0 \text{ and } 2x + 9y + 5 = 0$$

$$\Rightarrow \quad x = 2, y = -1.$$

EXAMPLE 7 *Let $\vec{u} = \hat{i} + 2\hat{j}$, $\vec{v} = -2\hat{i} + \hat{j}$ and $\vec{w} = 4\hat{i} + 3\hat{j}$. Find scalars x and y such that $\vec{w} = x\vec{u} + y\vec{v}$?*

SOLUTION We have,

$$\vec{w} = x\vec{u} + y\vec{v}$$

$$\Rightarrow \quad 4\hat{i} + 3\hat{j} = x(\hat{i} + 2\hat{j}) + y(-2\hat{i} + \hat{j})$$

$$\Rightarrow \quad (x - 2y - 4)\hat{i} + (2x + y - 3)\hat{j} = \vec{0}$$

$$\Rightarrow \quad x - 2y - 4 = 0 \text{ and } 2x + y - 3 = 0 \qquad [\because \hat{i} \text{ and } \hat{j} \text{ are non-collinear vectors}]$$

$$\Rightarrow \quad x = 2 \text{ and } y = -1.$$

23.13.2 COLLINEARITY OF POINTS

Let A, B, C be three collinear points. Then, each pair of the vectors $\vec{AB}, \vec{BC}$; $\vec{AB}, \vec{AC}$ and $\vec{BC}, \vec{AC}$ is a pair of collinear vectors. Thus, to check the collinearity of three points, we can check the collinearity of any two vectors obtained with the help of three points. Following theorem provides the general criterion for the collinearity of three points.

THEOREM *Three points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear if and only if there exist three scalars x, y, z not all zero simultaneously such that*

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}, \text{ together with } x + y + z = 0.$$

PROOF First, let three points A, B, C with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively be collinear. Then, vectors $\vec{AB}$ and $\vec{BC}$ are collinear. Therefore, there exist scalar λ such that

$$\vec{AB} = \lambda\, \vec{BC}$$

$$\Rightarrow \quad \vec{b} - \vec{a} = \lambda (\vec{c} - \vec{a})$$

$$\Rightarrow \quad (\lambda - 1)\vec{a} + 1 . \vec{b} + (-\lambda)\vec{c} = \vec{0}$$

$$\Rightarrow \quad x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}, \text{ where } x = \lambda - 1, y = 1 \text{ and } z = -\lambda$$

$$\Rightarrow \quad x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}, x + y + z = \lambda - 1 + 1 + (-\lambda) = 0$$

Conversely, let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of points A, B and C respectively such that

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}, \text{ where } x + y + z = 0$$

$$\Rightarrow \quad x\vec{a} + y\vec{b} = -z\vec{c}$$

$$\Rightarrow \quad x\vec{a} + y\vec{b} = (x + y)\vec{c} \qquad [\because x + y + z = 0 \Rightarrow -z = x + y]$$

$$\Rightarrow \quad \vec{c} = \frac{x\vec{a} + y\vec{b}}{x + y}$$

$\Rightarrow$ Point C divides AB in the ratio $y : x$.

$\Rightarrow$ A, B, C are collinear points.

Q.E.D.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Show that the points with position vectors $\vec{a} - 2\vec{b} + 3\vec{c}, -2\vec{a} + 3\vec{b} - \vec{c}$ and $4\vec{a} - 7\vec{b} + 7\vec{c}$ are collinear.*

SOLUTION Let P, Q, R be the points with position vectors $\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} - \vec{c}$ and $4\vec{a} - 7\vec{b} + 7\vec{c}$ respectively. Then,

$$\vec{PQ} = \text{P.V. of } Q - \text{P.V. of } P = (-2\vec{a} + 3\vec{b} - \vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c})$$

$$\Rightarrow \quad \vec{PQ} = -3\vec{a} + 5\vec{b} - 4\vec{c}$$

and, $$\vec{QR} = \text{P.V. of } R - \text{P.V. of } Q = (4\vec{a} - 7\vec{b} + 7\vec{c}) - (-2\vec{a} + 3\vec{b} - \vec{c})$$

$$\Rightarrow \quad \vec{QR} = 6\vec{a} - 10\vec{b} + 8\vec{c}.$$

Clearly, $\vec{QR} = -2\,\vec{PQ}$.

This shows that $\vec{PQ}$ and $\vec{QR}$ are parallel vectors.

But, Q is a point common to them. So, $\vec{PQ}$ and $\vec{QR}$ are collinear.

Hence, points P, Q and R are collinear.

EXAMPLE 2 *Show that the points with position vectors $\vec{a} - 2\vec{b} + 3\vec{c}, -2\vec{a} + 3\vec{b} + 2\vec{c}$ and $-8\vec{a} + 13\vec{b}$ are collinear whatever be $\vec{a}, \vec{b}, \vec{c}$.*

SOLUTION Let P, Q, R be the points with position vectors $\vec{a} - 2\vec{b} + 3\vec{c}, -2\vec{a} + 3\vec{b} + 2\vec{c}$ and $-8\vec{a} + 13\vec{b}$ respectively. Then,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$\Rightarrow \quad \vec{PQ} = (-2\vec{a} + 3\vec{b} + 2\vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c})$$

$$\Rightarrow \quad \vec{PQ} = -3\vec{a} + 5\vec{b} - \vec{c} \qquad \text{...(i)}$$

and, $\quad \overrightarrow{QR}$ = Position vector of R – Position vector of Q

$\Rightarrow \quad \overrightarrow{QR} = (-8\vec{a} + 13\vec{b}) - (-2\vec{a} + 3\vec{b} + 2\vec{c})$

$\Rightarrow \quad \overrightarrow{QR} = 6\vec{a} + 10\vec{b} - 2\vec{c} = 2(-3\vec{a} + 5\vec{b} - \vec{c})$ ...(ii)

From (i) and (ii), we have

$$\overrightarrow{QR} = 2\,\overrightarrow{PQ}$$

This shows that $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel vectors. But, Q is a point common to them.

So, $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are collinear.

Hence, P, Q and R are collinear points.

EXAMPLE 3 *Show that the points A, B, C with position vectors* $-2\vec{a} + 3\vec{b} + 5\vec{c}$, $\vec{a} + 2\vec{b} + 3\vec{c}$ *and* $7\vec{a} - \vec{c}$ *respectively, are collinear.*

SOLUTION We have,

$\quad \overrightarrow{AB}$ = Position vector of B-Position vector of A

$\Rightarrow \quad \overrightarrow{AB} = (\vec{a} + 2\vec{b} + 3\vec{c}) - (-2\vec{a} + 3\vec{b} + 5\vec{c})$

$\Rightarrow \quad \overrightarrow{AB} = 3\vec{a} - \vec{b} - 2\vec{c}$ ...(i)

and, $\quad \overrightarrow{BC}$ = Position vector of C – Position vector of B

$\Rightarrow \quad \overrightarrow{BC} = (7\vec{a} - \vec{c}) - (\vec{a} + 2\vec{b} + 3\vec{c}) = 6\vec{a} - 2\vec{b} - 4\vec{c}$

$\Rightarrow \quad \overrightarrow{BC} = 2(3\vec{a} - \vec{b} - 2\vec{c})$ ...(ii)

From (i) and (ii), we have

$$2\,\overrightarrow{AB} = \overrightarrow{BC}$$

Therefore, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors.

But, B is a point common to $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Therefore, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear vectors. Hence, points A, B and C are collinear.

EXAMPLE 4 *If the points with position vectors* $60\hat{i} + 3\hat{j}$, $40\hat{i} - 8\hat{j}$ *and* $a\hat{i} - 52\hat{j}$ *are collinear, find the value of a.*

SOLUTION Let the points be A, B and C respectively. Then,

$\quad A, B, C$ are collinear

$\Rightarrow \quad \overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear

$\Rightarrow \quad \overrightarrow{AB} = \lambda\,\overrightarrow{BC}$ for some scalar λ

$\Rightarrow \quad (-20\,\hat{i} - 11\,\hat{j}) = \lambda\,\{(a-40)\,\hat{i} - 44\,\hat{j}\}$

$\Rightarrow \quad \{\lambda(a-40) + 20\}\,\hat{i} - (44\lambda - 11)\,\hat{j} = \vec{0}$

$\Rightarrow \quad \lambda(a-40) + 20 = 0$ and, $44\lambda - 11 = 0$ $\quad [\because \hat{i}, \hat{j}$ are non-collinear]

$\Rightarrow \quad \lambda = \frac{1}{4}$ and $\lambda(a-40) + 20 = 0$

$\Rightarrow \quad \frac{1}{4}(a-40) + 20 = 0 \Rightarrow a = -40.$

Hence, the given points will be collinear, if $a = -40$.

EXAMPLE 5 *If* $\vec{a}, \vec{b}$ *are two non-collinear vectors, show that the points having position vectors* $l_1\vec{a} + m_1\vec{b}$, $l_2\vec{a} + m_2\vec{b}$ *and* $l_3\vec{a} + m_3\vec{b}$ *are collinear, if*

$$\begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$$

SOLUTION If given points are collinear, then there exist scalars x, y, z such that

$$x\,(l_1\,\vec{a} + m_1\,\vec{b}) + y\,(l_2\,\vec{a} + m_2\,\vec{b}) + z\,(l_3\,\vec{a} + m_3\,\vec{b}) = \vec{0}, \text{ where } x + y + z = 0$$

$$\Rightarrow \quad (l_1\,x + l_2\,y + l_3\,z)\,\vec{a} + (m_1\,x + m_2\,y + m_3\,z)\,\vec{b} = \vec{0}, \text{ where } x + y + z = 0$$

$$\Rightarrow \quad l_1\,x + l_2\,y + l_3\,z = 0,\ m_1\,x + m_2\,y + m_3\,z = 0,\ [\because \vec{a}, \vec{b} \text{ are non-collinear vectors}]$$

where $x + y + z = 0$

Thus, we have

$$x + y + z = 0 \qquad \text{...(i)}$$

$$l_1\,x + l_2\,y + l_3\,z = 0 \qquad \text{...(ii)}$$

$$m_1\,x + m_2\,y + m_3\,z = 0 \qquad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we get

$$\frac{x}{l_2\,m_3 - l_3\,m_2} = \frac{y}{l_3\,m_1 - l_1\,m_3} = \frac{z}{l_1\,m_2 - l_2\,m_1} = \lambda \text{ (say)}$$

$$\Rightarrow \quad x = \lambda\,(l_2\,m_3 - l_3\,m_2),\ y = \lambda\,(l_3\,m_1 - l_1\,m_3)\ z = \lambda\,(l_1\,m_2 - l_2\,m_1)$$

Substituting the values of x, y, z in (i), we get

$$(l_2\,m_3 - l_3\,m_2) + (l_3\,m_1 - l_1\,m_3) + (l_1\,m_2 - l_2\,m_1) = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$$

EXERCISE 23.6

1. Show that the points A, B, C with position vectors $\vec{a} - 2\vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$ and $-7\vec{b} + 10\vec{c}$ are collinear.
2. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, prove that the points having the following position vectors are collinear :
   (i) $\vec{a}, \vec{b}, 3\vec{a} - 5\vec{b}$
   (ii) $\vec{a} + \vec{b} + \vec{c}$, $4\vec{a} + 3\vec{b}$, $10\vec{a} + 7\vec{b} - 2\vec{c}$
3. Prove that the points having position vectors $\hat{i} + 2\hat{j} + 3\hat{k}$, $3\hat{i} + 4\hat{j} + 7\hat{k}$, $-3\hat{i} - 2\hat{j} - 5\hat{k}$ ar collinear.
4. If the points with position vectors $10\hat{i} + 3\hat{j}$, $12\hat{i} - 5\hat{j}$ and $a\hat{i} + 11\hat{j}$ are collinear, find the value of a.
5. If $\vec{a}, \vec{b}$ are two non-collinear vectors, prove that the points with position vectors $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$ and $\vec{a} + \lambda\,\vec{b}$ are collinear for all real values of λ.
6. If $\vec{AO} + \vec{OB} = \vec{BO} + \vec{OC}$, prove that A, B, C are collinear points.
7. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

ANSWERS

4. 8

23.14 COPALANARITY

In this section, we shall coplanarity of a system of vectors and also that of four or more points as three points are always coplanar.

A system of vectors is said to be coplanar, if their supports are parallel to the same plane.

We have already seen that any two vectors are always coplanar. But, three or more vectors may or may not be coplanar. The following theorem gives a test of coplanrity of three vectors.

THEOREM 1 *(Test of coplanarity of three vectors) Let $\vec{a}$ and $\vec{b}$ be two given non-zero non-collinear vectors. Then, any vector $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be uniquely expressed as $\vec{r} = x\vec{a} + y\vec{b}$, for some scalars x and y.*

PROOF Let $\vec{a}$ and $\vec{b}$ be two non-collinear non-zero vectors and let $\vec{r}$ be a vector coplanar with $\vec{a}$ and $\vec{b}$.

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OP} = \vec{r}$.

Complete the parallelogram $OLPM$ with OP as diagonal. Since vectors $\vec{OL}$ and $\vec{OM}$ are collinear with $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ respectively.

$\therefore \quad \vec{OL} = x\vec{a}$ and $\vec{OM} = y\vec{b}$ for some scalars x, y.

Now, $\quad \vec{OP} = \vec{OL} + \vec{LP}$ [By triangle law of add.]

$\Rightarrow \quad \vec{OP} = \vec{OL} + \vec{OM}$ $\quad [\because \vec{LP} = \vec{OM}]$

$\Rightarrow \quad \vec{r} = x\vec{a} + y\vec{b}$

Thus, $\vec{r} = x\vec{a} + y\vec{b}$ for some scalars x and y.

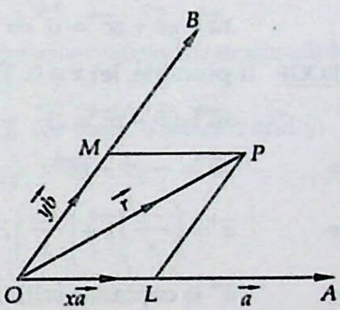


Fig. 23.60

To prove the uniqueness of this representation, let $\vec{r} = x_1\vec{a} + y_1\vec{b}$ for some scalars x_1 and y_1. Then,

$$x\vec{a} + y\vec{b} = x_1\vec{a} + y_1\vec{b}$$

$$\Rightarrow \quad (x - x_1)\vec{a} + (y - y_1)\vec{b} = \vec{0}$$

$$\Rightarrow \quad x - x_1 = 0 \text{ and } y - y_1 = 0 \qquad [\because \vec{a} \text{ and } \vec{b} \text{ are non-collinear}]$$

$$\Rightarrow \quad x = x_1 \text{ and } y = y_1$$

Hence, the representation is unique.

Q.E.D

The above theorem can also be expressed as under:

Three vectors are coplanar if one of them is expressible as a linear combination of the other two.

The following theorem provides an alternative test for the coplanarity of three vectors.

THEOREM 2 *Prove that a necessary and sufficient condition for three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to be coplanar is that there exist scalars l, m, n not all zero simultaneously such that $l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$.*

PROOF *Necessary condition :* First let $\vec{a}, \vec{b}, \vec{c}$ be three coplanar vectors

Then, one of them is expressible as a linear combination of the other two.

Let $\vec{c} = x\vec{a} + y\vec{b}$ for some scalars x, y

$\Rightarrow \quad \vec{c} = x\vec{a} + y\vec{b}$ for some scalars x, y

$\Rightarrow \quad l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$, where $l = x$, $n = y$ and $n = -1$.

Thus, if $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then there exist scalars l, m, n $l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$ such that, where l, m, n are not all zero simultaneously.

Sufficient condition : Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that there exist scalars l, m, n not all zero simultaneously satisfying $l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$.

We have to prove that $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors.

We have,

$$l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$$

$$\Rightarrow \quad n\vec{c} = -l\vec{a} - m\vec{b}$$

$$\Rightarrow \quad \vec{c} = \left(-\frac{l}{n}\right)\vec{a} + \left(-\frac{m}{n}\right)\vec{b}$$

$\Rightarrow$ $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$

$\Rightarrow$ $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$

Hence, $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors.

THEOREM 3 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero non-coplanar vectors and x, y, z are three scalars, then*

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \Rightarrow x = y = z = 0$$

PROOF If possible, let $x \neq 0$. Then,

$$x\vec{a} + y\vec{b} - z\vec{c} = \vec{0}$$

$$\Rightarrow \quad x\vec{a} = -y\vec{b} - z\vec{c}$$

$$\Rightarrow \quad \vec{a} = \left(\frac{-y}{x}\right)\vec{b} + \left(\frac{-z}{x}\right)\vec{c}$$

$\Rightarrow$ $\vec{a}$ is coplanar with $\vec{b}$ and $\vec{c}$.

This is contradiction to the fact that $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar. Therefore, $x = 0$.

Similarly, we have

$$y = z = 0.$$

Hence, $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \Rightarrow x = y = z = 0$

THEOREM 4 *(Test of coplanarity of four points) Four points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar iff there exist scalars x, y, z, u not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$, where $x + y + z + u = 0$.*

PROOF Let A, B, C, and D be four points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

First, let A, B, C, D be four coplanar points. Then $\vec{AB}$, $\vec{AC}$, and $\vec{AD}$ are coplanar vectors. So, there exist scalars λ and μ such that

$$\vec{AB} = \lambda\vec{AC} + \mu\vec{AD}$$

$$\Rightarrow \quad \vec{b} - \vec{a} = \lambda(\vec{c} - \vec{a}) + \mu(\vec{d} - \vec{a})$$

$$\Rightarrow \quad (\lambda + \mu - 1)\vec{a} + \vec{b} + (-\lambda)\vec{c} + (-\mu)\vec{d} = \vec{0}$$

$$\Rightarrow \quad x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}, \text{ where } x = \lambda + \mu - 1, y = 1, z = -\lambda \text{ and } u = -\mu$$

$$\Rightarrow \quad x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}, \text{ where } x + y + z + u = \lambda + \mu - 1 + 1 - \lambda - \mu = 0$$

Conversely, let there be scalars x, y, z, u not all zero such that

$$x\vec{a}+y\vec{b}+z\vec{c}+u\vec{d}=0 \quad ...(i)$$

where $\quad x+y+z+u = 0 \quad ...(ii)$

Putting $x=-(y+z+u)$ from (ii) into (i), we get

$$-(y+z+u)\vec{a}+y\vec{b}+z\vec{c}+u\vec{d}=\vec{0}$$

$$\Rightarrow \quad y(\vec{b}-\vec{a})+z(\vec{c}-\vec{a})+u(\vec{d}-\vec{a})=\vec{0}$$

$$\Rightarrow \quad y\vec{AB}+z\vec{AC}+u\vec{AD}=\vec{0}$$

Let $y \neq 0$. Then,

$$\vec{AB}=\left(\frac{-z}{y}\right)\vec{AC}+\left(\frac{-u}{y}\right)\vec{AD}$$

This shows that $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ are coplanar vectors.

Hence, points A, B, C, D are coplanar points.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Show that the vectors $\vec{a}-2\vec{b}+3\vec{c}$, $\vec{a}-3\vec{b}+5\vec{c}$ and $-2\vec{a}+3\vec{b}-4\vec{c}$ are coplanar, where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar.*

SOLUTION Recall that three vectors are coplanar if one of the given vectors is expressible as a linear combination of the other two. Let6

$$\vec{a}-2\vec{b}+3\vec{c}=x(\vec{a}-3\vec{b}+5\vec{c})+y(-2\vec{a}+3\vec{b}-4\vec{c}), \text{ for some scalars } x \text{ and } y.$$

$$\vec{a}-2\vec{b}+3\vec{c}=(x-2y)\vec{a}+(-3x+3y)\vec{b}+(5x-4y)\vec{c}$$

$$\Rightarrow \quad 1=x-2y,\ -2=-3x+3y \text{ and } 3=5x-4y$$

Solving first two of these equations, we get $x=1/3$, $y=-1/3$.

Clearly, these values of x and y satisfy the third equation.

Hence, the given vectors are coplanar.

EXAMPLE 2 *Show that the vectors $2\vec{a}-\vec{b}+3\vec{c}$, $\vec{a}+\vec{b}-2\vec{c}$ and $\vec{a}+\vec{b}-3\vec{c}$ are non-coplanar vectors.*

SOLUTION Let, if possible, the given vectors be coplanar.

Then one of the given vectors is expressible in terms of the other two.

Let $\quad 2\vec{a}-\vec{b}+3\vec{c}=x(\vec{a}+\vec{b}-2\vec{c})+y(\vec{a}+\vec{b}-3\vec{c})$, for some scalars x and y.

$$\Rightarrow \quad 2\vec{a}-\vec{b}+3\vec{c}=(x+y)\vec{a}+(x+y)\vec{b}+(-2x-3y)\vec{c}$$

$$\Rightarrow \quad 2=x+y,\ -1=x+y \text{ and } 3=-2x-3y$$

Solving, first and third of these equations, we get $x=9$ and $y=-7$.

Clearly, these values do not satisfy the third equation.

Hence, the given vectors are not coplanar.

EXAMPLE 3 *Prove that four points $2\vec{a}+3\vec{b}-\vec{c}$, $\vec{a}-2\vec{b}+3\vec{c}$, $3\vec{a}+4\vec{b}-2\vec{c}$ and $\vec{a}-6\vec{b}+6\vec{c}$ are coplanar.*

SOLUTION Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So, let

$$\vec{PQ} = x\,\vec{PR} + y\,\vec{PS}$$

$\Rightarrow \quad -\vec{a} - 5\vec{b} + 4\vec{c} = x(\vec{a} + \vec{b} - \vec{c}) + y(-\vec{a} - 9\vec{b} + 7\vec{c})$

$\Rightarrow \quad -\vec{a} - 5\vec{b} + 4\vec{c} = (x - y)\vec{a} + (x - 9y)\vec{b} + (-x + 7y)\vec{c}$

$\Rightarrow \quad x - y = -1,\ x - 9y = -5, -x + 7y = 4$ [Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

Solving the first of these three equations, we get $x = -1/2$, $y = 1/2$. These values also satisfy the third equation. Hence the given four points are coplanar.

EXAMPLE 4 *If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors such that $x_1\vec{a} + y_1\vec{b} + z_1\vec{c} = x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$, then prove that $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.*

SOLUTION We have,

$$x_1\vec{a} + y_1\vec{b} + z_1\vec{c} = x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$$

$\Rightarrow \quad (x_1 - x_2)\vec{a} + (y_1 - y_2)\vec{b} + (z_1 - z_2)\vec{c} = \vec{0}$

$\Rightarrow \quad x_1 - x_2 = 0,\ y_1 - y_2 = 0,\ z_1 - z_2 = 0$ [Using Theorem 3]

$\Rightarrow \quad x_1 = x_2,\ y_1 = y_2$ and $z_1 = z_2$.

EXERCISE 23.7

1. Show that the points whose position vectors are given, are collinear:
   (i) $2\hat{i} + \hat{j} - \hat{k}$, $3\hat{i} - 2\hat{j} + \hat{k}$ and $\hat{i} + 4\hat{j} - 3\hat{k}$
   (ii) $3\hat{i} - 2\hat{j} + 4\hat{k}$, $\hat{i} + \hat{j} + \hat{k}$ and $-\hat{i} + 4\hat{j} - 2\hat{k}$
2. Using vector method, prove that the following points are collinear:
   (i) $A(6, -7, -1)$, $B(2, -3, 1)$ and $C(4, -5, 0)$
   (ii) $A(2, -1, 3)$, $B(4, 3, 1)$ and $C(3, 1, 2)$
   (iii) $A(1, 2, 7)$, $B(2, 6, 3)$ and $C(3, 10, -1)$
   (iv) $A(-3, -2, -5)$, $B(1, 2, 3)$ and $C(3, 4, 7)$
   (v) $A(2, -1, 3)$, $B(3, -5, 1)$ and $C(-1, 11, 9)$
3. If $\vec{a}, \vec{b}, \vec{c}$ are non-zero, non-coplanar vectors, prove that the following vectors are coplanar:
   (i) $5\vec{a} + 6\vec{b} + 7\vec{c}$, $7\vec{a} - 8\vec{b} + 9\vec{c}$ and $3\vec{a} + 20\vec{b} + 5\vec{c}$
   (ii) $\vec{a} - 2\vec{b} + 3\vec{c}$, $-3\vec{b} + 5\vec{c}$ and $-2\vec{a} + 3\vec{b} - 4\vec{c}$
4. Show that the four points having position vectors $6\hat{i} - 7\hat{j}$, $16\hat{i} - 19\hat{j} - 4\hat{k}$, $3\hat{j} - 6\hat{k}$, $2\hat{i} - 5\hat{j} + 10\hat{k}$ are coplanar.
5. Prove that the following vectors are coplanar
   (i) $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$
   (ii) $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 3\hat{j} - \hat{k}$ and $-\hat{i} - 2\hat{j} + 2\hat{k}$
6. Prove that the following vectors are non-coplanar:
   (i) $3\hat{i} + \hat{j} - \hat{k}$, $2\hat{i} - \hat{j} + 7\hat{k}$ and $7\hat{i} - \hat{j} + 23\hat{k}$
   (ii) $\hat{i} + 2\hat{j} + 3\hat{k}$, $2\hat{i} + \hat{j} + 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$
7. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, prove that the following vectors are non-coplanar:
   (i) $2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{a} + \vec{b} - 2\vec{c}$ and $\vec{a} + \vec{b} - 3\vec{c}$

(ii) $\vec{a}+2\vec{b}+3\vec{c}$, $2\vec{a}+\vec{b}+3\vec{c}$ and $\vec{a}+\vec{b}+\vec{c}$

8. Prove that a necessary and sufficient condition for three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be coplanar is that there exist scalars l, m, n not all zero simultaneously such that $l\vec{a}+m\vec{b}+n\vec{c}=\vec{0}$.
9. Show that the four points A, B, C and D with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively are coplanar if and only if $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d}=\vec{0}$.
10. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors such that $x_1\vec{a}+y_1\vec{b}+z_1\vec{c}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}$, then prove that $x_1=x_2, y_1=y_2$ and $z_1=z_2$.
11. Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ given by $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ are non-coplanar. Express vector $\vec{d}=2\hat{i}-\hat{j}-3\hat{k}$ as a linear combination of the vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

23.15 DIRECTION COSINES AND DIRECTION RATIOS

23.15.1 DIRECTION COSINES

DEFINITION *If α, β, γ are the angles which a vector $\overrightarrow{OP}$ makes with the positive directions of the coordinate axes OX, OY, OZ respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are known as the direction cosines of $\overrightarrow{OP}$ and are generally denoted by the letters l, m, n respectively.*

$$\therefore \quad l=\cos\alpha,\ m=\cos\beta,\ n=\cos\gamma.$$

The angles α, β, γ are known as the direction angles and satisfy the condition $0 \le \alpha, \beta, \gamma \le \pi$.

In the above definition the vector $\overrightarrow{OP}$ has its initial point at the origin. If a given vector does not have its initial point at the origin, then we can draw a parallel vector of the same magnitude having initial point at the origin.

Clearly, $\overrightarrow{PO}$ makes angles $\pi-\alpha, \pi-\beta, \pi-\gamma$ with OX, OY, OZ, respectively. Therefore, direction cosines of $\overrightarrow{PO}$ are

$$\cos(\pi-\alpha), \cos(\pi-\beta), \cos(\pi-\gamma)$$

or, $\quad -l, -m, -n.$

As the x-axis makes angles $0, \frac{\pi}{2}, \frac{\pi}{2}$ with OX, OY and OZ respectively. Therefore, direction cosines of x-axis are

$$\cos 0,\ \cos\frac{\pi}{2},\ \cos\frac{\pi}{2} \quad \text{i.e., } 1, 0, 0$$

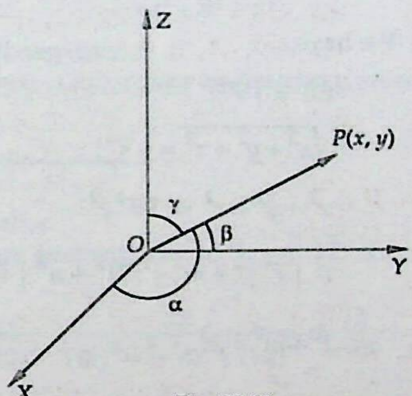


Fig. 23.61

Similarly the direction cosines of Y and Z-axes are 0, 1, 0 and 0, 0, 1 respectively.

THEOREM *Let $P(x, y, z)$ be a point in space such that $\vec{r}=\overrightarrow{OP}$ has direction cosines l, m, n. Then,*

(i) $l\,|\vec{r}|, m\,|\vec{r}|, n\,|\vec{r}|$ are projections of $\vec{r}$ on OX, OY, OZ respectively.

(ii) $x=l\,|\vec{r}|, y=m\,|\vec{r}|, z=n\,|\vec{r}|$

(iii) $\vec{r}=|\vec{r}|\,(l\hat{i}+m\hat{j}+n\hat{k})$ and $\hat{r}=l\hat{i}+m\hat{j}+n\hat{k}$

(iv) $l^2+m^2+n^2=1$

PROOF We have,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

Suppose $\overrightarrow{OP}$ makes angles α, β, γ with OX, OY and OZ respectively. Then, $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

(i) We have,

Projection of $\vec{r}$ on x-axis $= \vec{r} \cdot \hat{i}$ $\quad \left[\because \text{Projection of } \vec{a} \text{ on } \vec{b} = \vec{a} \cdot \hat{b}\right]$

$\Rightarrow$ Projection of $\vec{r}$ on x-axis $= |\vec{r}|\,|\hat{i}|\cos\alpha$ $\quad$ [By def. of dot product]

$\Rightarrow$ Projection of $\vec{r}$ on x-axis $= l\,|\vec{r}|$ $\quad [\because l = \cos\alpha]$

Similarly, projections of $\vec{r}$ on OY and OZ axes are $m\,|\vec{r}|$ and $n\,|\vec{r}|$ respectively.

(ii) We have,

Projection of $\vec{r}$ on x axis $= \vec{r} \cdot \hat{i} = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{i} = x$

Similarly, projections of $\vec{r}$ on OY and OZ axes are y and z respectively.

But, projections of $\vec{r}$ on OX, OY, OZ are $l\,|\vec{r}|$, $m\,|\vec{r}|$ and $n\,|\vec{r}|$ respectively.

$\therefore \quad x = l\,|\vec{r}|,\ y = m\,|\vec{r}|,\ z = n\,|\vec{r}|$

(iii) Putting $x = l\,|\vec{r}|, y = m\,|\vec{r}|, z = n\,|\vec{r}|$ in $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we obtain

$$\vec{r} = |\vec{r}|\,(l\hat{i} + m\hat{j} + n\hat{k})$$

$$\Rightarrow \quad \frac{\vec{r}}{|\vec{r}|} = l\hat{i} + m\hat{j} + n\hat{k}$$

$$\Rightarrow \quad \hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$$

(iv) We have,

$$OP = |\vec{r}|$$

$$\Rightarrow \quad \sqrt{x^2 + y^2 + z^2} = |\vec{r}|$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = |\vec{r}|^2$$

$$\Rightarrow \quad l^2|\vec{r}|^2 + m^2|\vec{r}|^2 + n^2|\vec{r}|^2 = |\vec{r}|^2 \qquad \left[\begin{array}{l}\because x = l\,|\vec{r}|, \\ y = m\,|\vec{r}|, z = n\,|\vec{r}|\end{array}\right]$$

$$\Rightarrow \quad l^2 + m^2 + n^2 = 1$$

Q.E.D.

23.15.2 DIRECTION RATIOS

As we have seen in the previous section that if l, m, n are direction cosines of a vector, then $l^2 + m^2 + n^2 = 1$. Therefore, l, m, n usually involve fractions and radical signs. Also, it is slightly combursome to use direction cosines l, m, n as such. We now introduce the concept of direction ratios of a vector.

DEFINITION *Let l, m, n be direction cosines of a vector $\vec{r}$ and a, b, c, be three numbers such that*

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

Then, we say that the direction ratios or direction numbers of vector $\vec{r}$ are proportional to a, b, c.

For example, if $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$ are direction cosines of a vector $\vec{r}$, then its direction ratios are proportional to 2, – 2, 1 or – 2, 2, – 1 or 4, – 4, 2, because

$$\frac{2/3}{2} = \frac{-2/3}{-2} = \frac{1/3}{1}, \quad \frac{2/3}{-2} = \frac{-2/3}{2} = \frac{1/3}{-1}, \quad \frac{2/3}{4} = \frac{-2/3}{-4} = \frac{1/3}{2}$$

It is evident from the above definition that to obtain direction ratios of a vector from its direction cosines we just multiply them by a common number. This also shows that there can be infinitely many sets of direction ratios for a given vector. But, the direction cosines are unique.

We shall now obtain the direction cosines from the direction ratios.

Let direction ratios of a vector $\vec{r}$ having direction cosines l, m, n. be proportional to a, b, c. Then,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} \qquad \text{[By definition]}$$

Let $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \lambda$. Then, $l = a\lambda, m = b\lambda, n = c\lambda$.

$$\therefore \quad l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \quad a^2\lambda^2 + b^2\lambda^2 + c^2\lambda^2 = 1$$

$$\Rightarrow \quad \lambda = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore \quad l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \; m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \; n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where the signs should be taken all positive or all negative.

Thus, if the direction ratios of a vector are proportional to a, b, c then its direction cosines are given by

$$\pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where the signs should be taken all positive or all negative.

For example, if direction ratios of a vector $\vec{r}$ are proportional to 3, – 4, 12, then its direction cosines are

$$\frac{3}{\sqrt{(3)^2 + (-4)^2 + (12)^2}}, \frac{-4}{\sqrt{(3)^2 + (-4)^2 + (12)^2}}, \frac{12}{\sqrt{(3)^2 + (-4)^2 + (12)^2}} \text{ or } \frac{3}{13}, -\frac{4}{13}, \frac{12}{13}.$$

23.15.3 SOME IMPORTANT RESULTS ON DIRECTION RATIOS AND DIRECTION COSINES

RESULT I *Direction ratios and cosines of a vector given in terms of unit vectors* $\hat{i}, \hat{j}, \hat{k}$. *If* $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$, *then its direction cosines are*

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

and the direction ratios of $\vec{r}$ *are proportional to a, b, c.*

PROOF Let $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ be a vector having direction cosines l, m, n.

If $\vec{r}$ makes angles α, β, γ with OX, OY and OZ respectively. Then, by using $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$, we have

$$\cos\alpha = \frac{\vec{r}\cdot\hat{i}}{|\vec{r}|\,|\hat{i}|}, \cos\beta = \frac{\vec{r}\cdot\hat{j}}{|\vec{r}|\,|\hat{j}|}, \cos\gamma = \frac{\vec{r}\cdot\hat{k}}{|\vec{r}|\,|\hat{k}|}$$

$$\Rightarrow \quad \cos\alpha = \frac{(a\,\hat{i}+b\,\hat{j}+c\,\hat{k})\cdot\hat{i}}{|\vec{r}|}, \cos\beta = \frac{(a\,\hat{i}+b\,\hat{j}+c\,\hat{k})\cdot\hat{j}}{|\vec{r}|}, \cos\gamma = \frac{(a\,\hat{i}+b\,\hat{j}+c\,\hat{k})\cdot\hat{k}}{|\vec{r}|}$$

$$\Rightarrow \quad l = \frac{a}{|\vec{r}|}, m = \frac{b}{|\vec{r}|}, n = \frac{c}{|\vec{r}|} \qquad [\because \cos\alpha = l, \cos\beta = m, \cos\gamma = n]$$

Thus, direction cosines of $\vec{r} = a\,\hat{i}+b\,\hat{j}+c\,\hat{k}$ are

$\frac{a}{|\vec{r}|}, \frac{b}{|\vec{r}|}, \frac{c}{|\vec{r}|}$ and hence its direction ratios are proportional to a, b, c.

For example, $\vec{r} = 2\,\hat{i}+\hat{j}-2\,\hat{k}$ has direction ratios proportional to 2, 1, – 2 and its direction cosines are

$$\frac{2}{\sqrt{2^2+1^2+(-2)^2}}, \frac{1}{\sqrt{2^2+(1)^2+(-2)^2}}, \frac{-2}{\sqrt{2^2+1^2+(-2)^2}} \text{ or, } \frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$$

RESULT II *The direction ratios of the line segment joining points (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$*

PROOF Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. Then,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$\Rightarrow \quad \vec{PQ} = (x_2\,\hat{i}+y_2\,\hat{j}+z_2\,\hat{k}) - (x_1\,\hat{i}+y_1\,\hat{j}+z_1\,\hat{k}) = (x_2-x_1)\,\hat{i}+(y_2-y_1)\,\hat{j}+(z_2-z_1)\,\hat{k}.$$

So, direction ratios of $\vec{PQ}$ are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and its direction cosines are

$$\frac{x_2-x_1}{|\vec{PQ}|}, \frac{y_2-y_1}{|\vec{PQ}|}, \frac{z_2-z_1}{|\vec{PQ}|}.$$

RESULT III *Two parallel vectors have proportional direction rations.*

PROOF Let $\vec{a}$ and $\vec{b}$ be two parallel vectors. Then,

$\vec{b} = \lambda\,\vec{a}$ for some scalar λ.

If $\vec{a} = a_1\,\hat{i}+a_2\,\hat{j}+a_3\,\hat{k}$, then

$$\vec{b} = \lambda\,\vec{a} \Rightarrow \vec{b} = (\lambda\,a_1)\,\hat{i}+(\lambda\,a_2)\,\hat{j}+(\lambda\,a_3)\,\hat{k}.$$

This shows that the direction ratios $\lambda\,\vec{a}$ are proportional to $\lambda\,a_1, \lambda\,a_2, \lambda\,a_3$ or, a_1, a_2, a_3 because

$$\lambda\,a_1 : \lambda\,a_2 : \lambda\,a_3 = a_1 : a_2 : a_3.$$

Thus, $\vec{a}$ and $\vec{b}$ have proportional direction ratios and hence equal direction cosines also.

RESULT IV *If a vector $\vec{r}$ has direction ratios proportional to a, b, c then*

$$\vec{r} = \frac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\,\hat{i}+b\,\hat{j}+c\,\hat{k})$$

PROOF Since direction ratios of $\vec{r}$ are proportional to a, b, c. Therefore, its direction cosines are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

So, $\quad \vec{r} = |\vec{r}|\,(l\,\hat{i}+m\,\hat{j}+n\,\hat{k})$

$$\Rightarrow \quad \vec{r} = |\vec{r}| \left\{ \frac{a}{\sqrt{a^2+b^2+c^2}}\hat{i} + \frac{b}{\sqrt{a^2+b^2+c^2}}\hat{j} + \frac{c}{\sqrt{a^2+b^2+c^2}}\hat{k} \right\}$$

$$\Rightarrow \quad \vec{r} = \frac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\,\hat{i} + b\,\hat{j} + c\,\hat{k})$$

RESULT V *If l, m, n are the direction cosines of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then its projections on the coordinates axes are respectively*

$$l\,|\vec{r}|,\ m\,|\vec{r}|,\ n\,|\vec{r}|$$

PROOF Let l, m, n be the direction cosines of a vectors $\vec{r}$. If $\vec{r}$ makes angels α, β and γ with OX, OY and OZ respectively, then

$$l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma.$$

Now,

$$\text{Projection of } \vec{r} \text{ on } x\text{-axis} = \vec{r}\cdot\hat{i} = |\vec{r}|\,|\hat{i}|\cos\alpha = l\,|\vec{r}|$$

Similarly, we have

$$\text{Projection of } \vec{r} \text{ on } y\text{-axis} = m\,|\vec{r}|$$

and,

$$\text{Projection of } \vec{r} \text{ on } z\text{-axis} = n\,|\vec{r}|$$

Thus, projections of $\vec{r}$ on the coordinate axes are $l\,|\vec{r}|, m\,|\vec{r}|, n\,|\vec{r}|$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *A vector $\overrightarrow{OP}$ is inclined to OX at 45° and OY at 60°. Find the angle at which $\overrightarrow{OP}$ is inclined to OZ.*

SOLUTION Suppose $\overrightarrow{OP}$ is inclined at angle γ to OZ.

Let l, m, n be the direction cosines of $\overrightarrow{OP}$. Then,

$$l = \cos 45^\circ,\ m = \cos 60^\circ,\ n = \cos\gamma \Rightarrow l = \frac{1}{\sqrt{2}},\ m = \frac{1}{2},\ n = \cos\gamma.$$

Now,

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \quad \frac{1}{2} + \frac{1}{4} + n^2 = 1 \Rightarrow n^2 = \frac{1}{4}$$

$$\Rightarrow \quad n = \pm\frac{1}{2} \Rightarrow \cos\gamma = \pm\frac{1}{2} \Rightarrow \gamma = 60^\circ \text{ or, } 120^\circ$$

Hence, $\overrightarrow{OP}$ is inclined to OZ either at 60° or, at 120°.

EXAMPLE 2 *If a vector makes angles α, β, γ with OX, OY and OZ respectively, prove that*

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

SOLUTION Let l, m, n be the direction cosines of the given vector. Then,

$$l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma.$$

Now, $\quad l^2 + m^2 + n^2 = 1$

$$\Rightarrow \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \quad (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$$

$$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

EXAMPLE 3 *Find the direction cosines of a vector $\vec{r}$ which is equally inclined with OX, OY and OZ. If $|\vec{r}|$ is given, find the total number of such vectors.* **[NCERT]**

SOLUTION Let l, m, n be the direction cosines of $\vec{r}$.

Since $\vec{r}$ is equally inclined with OX, OY and OZ.

$\therefore \quad l = m = n \qquad [\because \alpha = \beta = \gamma \; \therefore \; \cos\alpha = \cos\beta = \cos\gamma]$

Now, $\quad l^2 + m^2 + n^2 = 1 \Rightarrow 3l^2 = 1 \Rightarrow l = \pm\frac{1}{\sqrt{3}}$

Hence, direction cosines of $\vec{r}$ are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$.

$\therefore \quad \vec{r} = |\vec{r}|\,(l\hat{i} + m\hat{j} + n\hat{k})$

$\Rightarrow \quad \vec{r} = |\vec{r}|\left\{\pm\frac{1}{\sqrt{3}}\hat{i} \pm \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{3}}\hat{k}\right\}.$

Since + and − signs can be arranged at three places in $2 \times 2 \times 2 = 8$ ways.

Therefore, there are eight vectors of given magnitude which are equally inclined with the coordinate axes.

EXAMPLE 4 *A vector $\vec{r}$ is inclined at equal angles to OX, OY and OZ. If the magnitude of $\vec{r}$ is 6 units, find $\vec{r}$.*

SOLUTION Suppose $\vec{r}$ makes an angle α with each of the axes OX, OY and OZ. Then, its direction cosines are

$$l = \cos\alpha,\; m = \cos\alpha,\; n = \cos\alpha \Rightarrow l = m = n.$$

Now, $\quad l^2 + m^2 + n^2 = 1 \Rightarrow 3l^2 = 1 \Rightarrow l = \pm\frac{1}{\sqrt{3}}$

$\therefore \quad \vec{r} = 6\left(\pm\frac{1}{\sqrt{3}}\hat{i} \pm \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{3}}\hat{k}\right) \Rightarrow \vec{r} = 2\sqrt{3}\,(\pm\hat{i} \pm \hat{j} \pm \hat{k}).$

EXAMPLE 5 *A vector $\vec{r}$ has length 21 and direction ratios 2, − 3, 6. Find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with x-axis.*

SOLUTION Recall that if the direction ratios of a vector are proportional to a, b, c, then its direction cosines are

$$\pm\frac{a}{\sqrt{a^2 + b^2 + c^2}},\; \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}},\; \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Therefore, direction cosines of $\vec{r}$ are

$$\pm\frac{2}{\sqrt{2^2 + (-3)^2 + 6^2}},\; \pm\frac{-3}{\sqrt{2^2 + (-3)^2 + 6^2}},\; \pm\frac{6}{\sqrt{2^2 + (-3)^2 + 6^2}}$$

Since $\vec{r}$ makes an acute angle with x-axis. Therefore, $\cos\alpha > 0$ i.e., $l > 0$.

So, direction cosines of $\vec{r}$ are $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$

$\therefore \quad \vec{r} = 21\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \qquad [\text{Using: } \vec{r} = |\vec{r}|\,(l\hat{i} + m\hat{j} + n\hat{k})]$

$\Rightarrow \quad \vec{r} = 6\hat{i} - 9\hat{j} + 18\hat{k}.$

So, components of $\vec{r}$ along OX, OY and OZ are $6\hat{i}$, $-9\hat{j}$ and $18\hat{k}$ respectively.

EXAMPLE 6 *Find the angles at which the vector $2\hat{i}-\hat{j}+2\hat{k}$ is inclined to each of the coordinate axes.*

SOLUTION Let $\vec{r}$ be the given vector, and let it make angles α, β, γ with OX, OY and OZ respectively. Then, its direction cosine are $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

We have,

$$\vec{r}=2\hat{i}-\hat{j}+2\hat{k}.$$

So, direction ratios of $\vec{r}$ are proportional to $2, -1, 2$.

Therefore, direction cosines of $\vec{r}$ are

$$\frac{2}{\sqrt{2^2+(-1)^2+2^2}}, \frac{-1}{\sqrt{2^2+(-1)^2+2^2}}, \frac{2}{\sqrt{2^2+(-1)^2+2^2}} \text{ i.e. } \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}.$$

$$\therefore \quad \cos\alpha = \frac{2}{3}, \cos\beta = -\frac{1}{3}, \cos\gamma = \frac{2}{3}$$

$$\Rightarrow \quad \alpha = \cos^{-1}\left(\frac{2}{3}\right), \beta = \cos^{-1}\left(-\frac{1}{3}\right), \gamma = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow \quad \alpha = \cos^{-1}\left(\frac{2}{3}\right), \beta = \pi-\cos^{-1}\left(\frac{1}{3}\right), \gamma = \cos^{-1}\left(\frac{2}{3}\right)$$

EXAMPLE 7 *The projection of a vector on the coordinate axes are 6, – 3, 2. Find its length and direction cosines.*

SOLUTION Let l, m, n be the direction cosines of the given vector $\vec{r}$ (say). Then, its projections on the coordinate axes are $l\,|\vec{r}|, m\,|\vec{r}|, n\,|\vec{r}|$.

$$\therefore \quad l\,|\vec{r}| = 6, m\,|\vec{r}| = -3, n\,|\vec{r}| = 2 \qquad \text{...(i)}$$

$$\Rightarrow \quad \{l\,|\vec{r}|\}^2+\{m\,|\vec{r}|\}^2+\{|(n\,|\vec{r}|\}^2 = 6^2+(-3)^2+(2)^2$$

$$\Rightarrow \quad |\vec{r}|^2(l^2+m^2+n^2) = 36+9+4$$

$$\Rightarrow \quad |\vec{r}|^2 = 49 \qquad [\because l^2+m^2+n^2=1]$$

$$\Rightarrow \quad |\vec{r}| = 7$$

Putting $|\vec{r}| = 7$ in (i), we obtain that the direction cosines of $\vec{r}$ are

$$l=\frac{6}{7}, m=-\frac{3}{7}, n=\frac{2}{7}$$

EXAMPLE 8 *Find the direction cosines of the vector joining the points $A(1, 2, -3)$ and $B(-1, -2, 1)$, directed from A to B.*

SOLUTION Clearly,

$$\vec{AB} = \text{Position vector of } B - \text{Position vector of } A$$

$$\Rightarrow \quad \vec{AB} = (-\hat{i}-2\hat{j}+\hat{k})-(\hat{i}+2\hat{j}-3\hat{k})$$

$$\Rightarrow \quad \vec{AB} = -2\hat{i}-4\hat{j}+4\hat{k}$$

The direction ratios of $\vec{AB}$ are proportional to $-2, -4, 4$. So direction cosines of $\vec{AB}$ are

$$\frac{-2}{\sqrt{(-2)^2+(-4)^2+4^2}}, \frac{-4}{\sqrt{(-2)^2+(-4)^2+4^2}}, \frac{4}{\sqrt{(-2)^2+(-4)^2+4^2}} \text{ or, } -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$$

EXERCISE 23.8

1. Can a vector have direction angles $45°, 60°, 120°$?
2. Prove that 1, 1, 1 cannot be direction cosines of a straight line.
3. A vector makes an angle of $\frac{\pi}{4}$ with each of x-axis and y-axis. Find the angle made by it with the z-axis.
4. A vector $\vec{r}$ is inclined at equal acute angles to x-axis, y-axis and z-axis. If $|\vec{r}| = 6$ units, find $\vec{r}$.
5. A vector $\vec{r}$ is inclined to x-axis at $45°$ and y-axis at $60°$. If $|\vec{r}| = 8$ units, find $\vec{r}$.
6. Find the direction cosines of the following vectors :
   (i) $2\hat{i}+2\hat{j}-\hat{k}$ (ii) $6\hat{i}-2\hat{j}-3\hat{k}$ (iii) $3\hat{i}-4\hat{k}$
7. Find the angles at which the following vectors are inclined to each of the coordinate axes :
   (i) $\hat{i}-\hat{j}+\hat{k}$ (ii) $\hat{j}-\hat{k}$ (iii) $4\hat{i}+8\hat{j}+\hat{k}$
8. Find the direction cosines of the vector joining the points $A(1, 2, -3)$ and $B(-1, -2, 1)$, directed from A to B. [NCERT]
9. Show that the vector $\hat{i}+\hat{j}+\hat{k}$ is equally inclined with the axes OX, OY and OZ. [NCERT]
10. Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. [NCERT]
11. If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an accute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$. [NCERT]

ANSWERS

1. Yes 2. No 3. $\frac{\pi}{2}$ 4. $\vec{r} = 2\sqrt{3}\,(\hat{i}+\hat{j}+\hat{k})$ 5. $\vec{r} = 4\,(\sqrt{2}\hat{i}+\hat{j}\pm\hat{k})$

6. (i) $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$ (ii) $\frac{6}{7}, \frac{-2}{7}, \frac{-3}{7}$ (iii) $\frac{3}{5}, 0, \frac{-4}{5}$

7. (i) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right), \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right), \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (ii) $\frac{\pi}{2}, \cos^{-1}\left(\frac{1}{\sqrt{2}}\right), \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

(iii) $\cos^{-1}\left(\frac{4}{9}\right), \cos^{-1}\left(\frac{8}{9}\right), \cos^{-1}\left(\frac{1}{9}\right)$ 8. $\frac{-3}{\sqrt{41}}, \frac{-4}{\sqrt{41}}, \frac{4}{\sqrt{41}}$ 11. $\frac{\pi}{3}; \frac{1}{2}\hat{i}, \frac{1}{\sqrt{2}}\hat{j}, \frac{1}{2}\hat{k}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Define "zero vector".
2. Define unit vector.
3. Define position vector of a point.
4. Write $\overrightarrow{PQ} + \overrightarrow{RP} + \overrightarrow{QR}$ in the simplified form.
5. If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors such that $x\vec{a} + y\vec{b} = \vec{0}$, then write the values of x and y.
6. If $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a parallelogram, then write vectors representing its diagonals.

7. If $\vec{a}, \vec{b}, \vec{c}$ represent the sides of a triangle taken in order, then write the value of $\vec{a} + \vec{b} + \vec{c}$.
8. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the vertices A, B and C respectively, of a triangle ABC, write the value of $\vec{AB} + \vec{BC} + \vec{CA}$.
9. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the points A, B and C respectively, write the value of $\vec{AB} + \vec{BC} + \vec{AC}$.
10. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of a triangle, then write the position vector of its centroid.
11. If G denotes the centroid of ΔABC, then write the value of $\vec{GA} + \vec{GB} + \vec{GC}$.
12. If $\vec{a}$ and $\vec{b}$ denote the position vectors of points A and B respectively and C is a point on AB such that $3\,AC = 2\,AB$, then write the position vector of C.
13. If D is the mid-point of side BC of a triangle ABC such that $\vec{AB} + \vec{AC} = \lambda\,\vec{AD}$, write the value of λ.
14. If D, E, F are the mid-points of the sides BC, CA and AB respectively of a triangle ABC, write the value of $\vec{AD} + \vec{BE} + \vec{CF}$.
15. If $\vec{a}$ is a non-zero vector of modulus a and m is a non-zero scalar such that $m\vec{a}$ is a unit vector, write the value of m.
16. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of an equilateral triangle whose orthocentre is at the origin, then write the value of $\vec{a} + \vec{b} + \vec{c}$.
17. Write a unit vector making equal acute angles with the coordinates axes.
18. If a vector makes angles α, β, γ with OX, OY and OZ respectively, then write the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.
19. Write a vector of magnitude 12 units which makes 45° angle with X-axis, 60°angle with Y-axis and an obtuse angle with Z-axis.
20. Write the length (magnitude) of a vector whose projections on the coordinate axes are 12, 3 and 4 units.
21. Write the position vector of a point dividing the line segment joining points A and B with position vectors $\vec{a}$ and $\vec{b}$ externally in the ratio 1:4, where $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$.
22. Write the direction cosines of the vector $\vec{r} = 6\hat{i} - 2\hat{j} + 3\hat{k}$.
23. If $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$ and $\vec{c} = \hat{k} + \hat{i}$, write unit vectors parallel to $\vec{a} + \vec{b} - 2\vec{c}$.
24. If $\vec{a} = \hat{i} + 2\hat{j}$, $\vec{b} = \hat{j} + 2\hat{k}$, write a unit vector along the vector $3\vec{a} - 2\vec{b}$.
25. Write the position vector of a point dividing the line segment joining points having position vectors $\hat{i} + \hat{j} - 2\hat{k}$ and $2\hat{i} - \hat{j} + 3\hat{k}$ externally in the ratio 2:3.
26. If $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$, $\vec{c} = \hat{k} + \hat{i}$, find the unit vector in the direction of $\vec{a} + \vec{b} + \vec{c}$.
27. If $\vec{a} = 3\hat{i} - \hat{j} - 4\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 3\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$, find $|3\vec{a} - 2\vec{b} + 4\vec{c}|$.
28. A unit vector $\vec{r}$ makes angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$ with $\hat{j}$ and $\hat{k}$ respectively and an acute angle θ with $\hat{i}$. Find θ.
29. Write a unit vector in the direction of $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$. **[CBSE 2008]**
30. If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + 9\hat{k}$, find a unit vector parallel to $\vec{a} + \vec{b}$. **[CBSE 2008]**
31. Write a unit vector in the direction of $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$. **[CBSE 2009]**

32. Find the position vector of the mid-point of the line segment AB, where A is the point $(3, 4, -2)$ and B is the point $(1, 2, 4)$. **[CBSE 2010]**

33. Find a vector in the direction of $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, which has magnitude of 6 units. **[CBSE 2010]**

34. What is the cosine of the angle which the vector $\sqrt{2}\hat{i} + \hat{j} + \hat{k}$ makes with y-axis? **[CBSE 2010]**

35. Write two different vectors having same magnitude.

36. Write two different vectors having same direction.

37. Write a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude of 8 unit.

38. Write the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

ANSWERS

5. $x = 0, y = 0$ 6. $\vec{a} + \vec{b}, \vec{a} - \vec{b}$ 7. $\vec{0}$ 8. $\vec{0}$

9. $2(\vec{c} - \vec{a})$ 10. $\dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$ 11. $\vec{0}$ 12. $\dfrac{\vec{a} + 2\vec{b}}{3}$

13. 2 14. 0 15. $\pm\dfrac{1}{a}$ 16. $\vec{0}$ 17. $\dfrac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

18. 2 19. $6(\sqrt{2}\,\hat{i} + \hat{j} - \hat{k})$ 20. 13 21. $3\hat{i} + \dfrac{11}{3}\hat{j} + 5\hat{k}$

22. $\dfrac{6}{7}, \dfrac{-2}{7}, \dfrac{3}{7}$ 23. $\pm\dfrac{1}{\sqrt{6}}(\hat{i} - 2\hat{j} + \hat{k})$ 24. $\dfrac{1}{\sqrt{41}}(3\hat{i} + 4\hat{j} - 4\hat{k})$

25. $-\hat{i} + 5\hat{j} - 12\hat{k}$ 26. $\dfrac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$ 27. $\sqrt{398}$ 28. $60°$

29. $\hat{a} = \dfrac{3}{7}\hat{i} - \dfrac{2}{7}\hat{j} + \dfrac{6}{7}\hat{k}$ 30. $\dfrac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$ 31. $\dfrac{2}{3}\hat{i} + \dfrac{1}{3}\hat{j} + \dfrac{2}{3}\hat{k}$

32. $(2, 3, 1)$ 33. $4\hat{i} - 2\hat{j} + 4\hat{k}$ 34. $\dfrac{1}{2}$ 35. $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}, \vec{b} = -2\hat{i} + \hat{j} -$

36. $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}, \vec{b} = 4\hat{i} + 2\hat{j} + 4\hat{k}$ 37. $\pm\dfrac{8}{\sqrt{30}}(5\hat{i} - \hat{j} + 2\hat{k})$ 38. $\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If in a ΔABC, $A \equiv (0, 0)$, $B \equiv (3, 3\sqrt{3})$, $C \equiv (-3\sqrt{3}, 3)$, then the vector of magnitude $2\sqrt{2}$ units directed along AO, where O is the circumcentre of ΔABC is
   (a) $(1 - \sqrt{3})\hat{i} + (1 + \sqrt{3})\hat{j}$ (b) $(1 + \sqrt{3})\hat{i} + (1 - \sqrt{3})\hat{j}$
   (c) $(1 + \sqrt{3})\hat{i} + (\sqrt{3} - 1)\hat{j}$ (d) none of these

2. If $\vec{a}, \vec{b}$ are the vectors forming consecutive sides of a regular hexagon $ABCDEF$, then the vector representing side CD is
   (a) $\vec{a} + \vec{b}$ (b) $\vec{a} - \vec{b}$ (c) $\vec{b} - \vec{a}$ (d) $-(\vec{a} + \vec{b})$

3. Forces $3\,\overrightarrow{OA}$, $5\,\overrightarrow{OB}$ act along OA and OB. If their resultant passes through C on AB, then
   (a) C is a mid-point of AB (b) C divides AB in the ratio $2 : 1$
   (c) $3\,AC = 5\,CB$ (d) $2\,AC = 3\,CB$

4. If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors, no two of which are collinear and the vector $\vec{a} + \vec{b}$ is collinear with $\vec{c}$, $\vec{b} + \vec{c}$ is collinear with $\vec{a}$, then $\vec{a} + \vec{b} + \vec{c} =$
   (a) $\vec{a}$ (b) $\vec{b}$ (c) $\vec{c}$ (d) none of these

5. If points $A\,(60\hat{i}+3\hat{j})$, $B\,(40\hat{i}-8\hat{j})$ and $C\,(a\hat{i}-52\hat{j})$ are collinear, then a is equal to
(a) 40 (b) -40 (c) 20 (d) -20

6. If G is the intersection of diagonals of a parallelogram $ABCD$ and O is any point, then $\vec{OA}+\vec{OB}+\vec{OC}+\vec{OD}=$
(a) $2\,\vec{OG}$ (b) $4\,\vec{OG}$ (c) $5\,\vec{OG}$ (d) $3\,\vec{OG}$

7. The vector $\cos\alpha\cos\beta\,\hat{i}+\cos\alpha\sin\beta\,\hat{j}+\sin\alpha\,\hat{k}$ is a
(a) null vector (b) unit vector
(c) constant vector (d) none of these

8. In a regular hexagon $ABCDEF$, $\vec{AB}=a$, $\vec{BC}=\vec{b}$ and $\vec{CD}=\vec{c}$. Then, $\vec{AE}=$
(a) $\vec{a}+\vec{b}+\vec{c}$ (b) $2\vec{a}+\vec{b}+\vec{c}$
(c) $\vec{b}+\vec{c}$ (d) $\vec{a}+2\vec{b}+2\vec{c}$

9. The vector equation of the plane passing through $\vec{a},\vec{b},\vec{c}$ is $\vec{r}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$, provided that
(a) $\alpha+\beta+\gamma=0$ (b) $\alpha+\beta+\gamma=1$
(c) $\alpha+\beta=\gamma$ (d) $\alpha^2+\beta^2+\gamma^2=1$

10. If O and O' are circumcentre and ortho centre of $\Delta\,ABC$, then $\vec{OA}+\vec{OB}+\vec{OC}$ equals
(a) $2\,\vec{OO'}$ (b) $\vec{OO'}$ (c) $\vec{O'O}$ (d) $2\,\vec{O'O}$

11. If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are the position vectors of points A, B, C, D such that no three of them are collinear and $\vec{a}+\vec{c}=\vec{b}+\vec{d}$, then $ABCD$ is a
(a) rhombus (b) rectangle (c) square (d) parallelogram

12. Let G be the centroid of $\Delta\,ABC$. If $\vec{AB}=\vec{a}$, $\vec{AC}=\vec{b}$, then the bisector $\vec{AG}$, in terms of $\vec{a}$ and $\vec{b}$ is
(a) $\frac{2}{3}(\vec{a}+\vec{b})$ (b) $\frac{1}{6}(\vec{a}+\vec{b})$ (c) $\frac{1}{3}(\vec{a}+\vec{b})$ (d) $\frac{1}{2}(\vec{a}+\vec{b})$

13. If $ABCDEF$ is a regular hexagon, then $\vec{AD}+\vec{EB}+\vec{FC}$ equals
(a) $2\vec{AB}$ (b) $\vec{0}$ (c) $3\vec{AB}$ (d) $4\vec{AB}$

14. The position vectors of the points A, B, C are $2\hat{i}+\hat{j}-\hat{k}$, $3\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}+4\hat{j}-3\hat{k}$ respectively. These points
(a) form an isosceles triangle (b) form a right triangle
(c) are collinear (d) form a scalene triangle

15. If three points A, B and C have position vectors $\hat{i}+x\hat{j}+3\hat{k}$, $3\hat{i}+4\hat{j}+7\hat{k}$ and $y\hat{i}-2\hat{j}-5\hat{k}$ respectively are collinear, then $(x, y)=$
(a) $(2,-3)$ (b) $(-2,3)$ (c) $(-2,-3)$ (d) $(2,3)$

16. $ABCD$ is a parallelogram with AC and BD as diagonals. Then, $\vec{AC}-\vec{BD}=$
(a) $4\,\vec{AB}$ (b) $3\,\vec{AB}$ (c) $2\,\vec{AB}$ (d) $\vec{AB}$

17. If $OACB$ is a parallelogram with $\vec{OC}=\vec{a}$ and $\vec{AB}=\vec{b}$, then $\vec{OA}=$
(a) $\vec{a}+\vec{b}$ (b) $\vec{a}-\vec{b}$ (c) $\frac{1}{2}(\vec{b}-\vec{a})$ (d) $\frac{1}{2}(\vec{a}-\vec{b})$

18. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect?
(a) $\vec{b}=\lambda\,\vec{a}$ for some scalar λ (b) $\vec{a}=\pm\vec{b}$
(c) the respective components of $\vec{a}$ and $\vec{b}$ are proportional
(d) both the vectors $\vec{a}$ and $\vec{b}$ have the same direction but different magnitudes

19. In the adjacent figure, which of the following is not true?

(a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$ (b) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(c) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$ (d) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

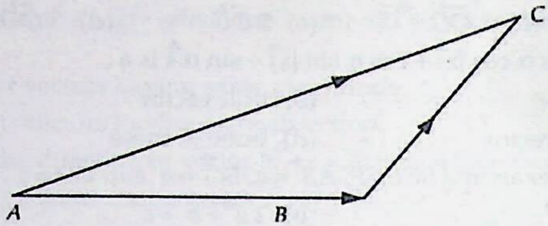


Fig. 23.62

ANSWERS

1. (a)	2. (c)	3. (c)	4. (d)	5. (b)	6. (b)	7. (b)	8 (c)
9. (b)	10. (b)	11. (d)	12. (a)	13. (d)	14. (a)	15. (a)	16. (c)
17. (d)	18. (d)	19. (c)					

SUMMARY

1. A vector is a physical quantity having both magnitude and direction.
2. If $\vec{a}, \vec{b}, \vec{c}$ are the vectors represented by the sides of a triangle taken in order, then $\vec{a} + \vec{b} + \vec{c} = \vec{0}$
   Conversely, if $\vec{a}, \vec{b}, \vec{c}$ are three non-collinear vectors, such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then they form the sides of a triangle taken in order.
3. Two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear iff there exist non-zero scalars x and y such that $x\vec{a} + y\vec{b} = \vec{0}$.
4. If $\vec{a}$ and $\vec{b}$ are two non-zero non-collinear vectors, then
   $$x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = y = 0.$$
5. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors, then any vector $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be uniquely expressed as
   $$\vec{r} = x\vec{a} + y\vec{b}, \text{ where } x, y \text{ are scalars}$$
   or, $$\vec{r} = \{x\,|\vec{a}|\}\hat{a} + \{y\,|\vec{b}|\}\hat{b}$$
6. If $\vec{a}, \vec{b}, \vec{c}$ are three given non-coplanar vectors, then every vector $\vec{r}$ in space can be uniquely expressed as
   $$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} \text{ for some scalars } x, y \text{ and } z.$$
   or, $$\vec{r} = \{x\,|\vec{a}|\}\hat{a} + \{y\,|\vec{b}|\}\hat{b} + \{z\,|\vec{c}|\}\hat{c}$$
   Here, vectors $x\vec{a}$, $y\vec{b}$ and $z\vec{c}$ are called the components of $\vec{r}$ in the directions of $\vec{a}, \vec{b}$ and $\vec{c}$ respectively and the scalars x, y and z are known as the coordinates of $\vec{r}$ relative to the triad of non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$. The triad of non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ relative to which we decompose any vector $\vec{r}$ is called a base.

The scalars $x\,|\vec{a}|, y\,|\vec{b}|, z\,|\vec{c}|$ are known as the projections of $\vec{r}$ in the directions of $\vec{a}, \vec{b}, \vec{c}$ respectively.

7. It $\vec{a}, \vec{b}, \vec{c}$ are three non-zero non-coplanar vectors and x, y, z are three scalars, then

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \Rightarrow x = y = z = 0.$$

8. If l, m, n are direction cosines of a vector $\vec{r}\,(= \vec{OP})$, where O is the origin and the point P has (x, y, z) as its coordinates, then
   (i) $l^2 + m^2 + n^2 = 1$
   (ii) $x = l\,|\vec{r}|, y = m\,|\vec{r}|, z = n\,|\vec{r}|$
   (iii) $\vec{r} = |\vec{r}|\,(l\hat{i} + m\hat{j} + n\hat{k})$
   (iv) $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$

9. If $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$, then a, b, c are proportional to its direction ratios and its direction cosines are

$$\frac{\pm a}{\sqrt{a^2 + b^2 + c^2}}, \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}}, \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}}$$

10. (i) A set of non-zero vectors $\vec{a_1}, \vec{a_2}, \vec{a_3}, \ldots, \vec{a_n}$ is linearly independent, if

$$x_1\vec{a_1} + x_2\vec{a_2} + \ldots + x_n\vec{a_n} = \vec{0} \Rightarrow x_1 = x_2 = \ldots = x_n = 0$$

   (ii) A set of vectors $\vec{a_1}, \vec{a_2}, \vec{a_3}, \ldots, \vec{a_n}$ is linearly dependent, if there exist scalars $x_1, x_2, \ldots, x_n$ not all zero such that

$$x_1\vec{a_1} + x_2\vec{a_2} + \ldots + x_n\vec{a_n} = \vec{0}$$

   (iii) Any two non-zero, non-collinear vectors are linearly independent.
   (iv) Any two collinear vectors are linearly dependent.
   (v) Any three non-coplanar vectors are linearly independent.
   (vi) Any three coplanar vectors are linearly dependent.
   (vii) Any set of four or more vectors in three dimensional space is linearly dependent set.

11. If A and B are two points with position vectors $\vec{a}$ and $\vec{b}$ respectively, then the position vector of a point C dividing AB in the ratio $m : n$
   (i) internally, is $\dfrac{m\vec{b} + n\vec{a}}{m+n}$
   (ii) externally, is $\dfrac{m\vec{b} - n\vec{a}}{m-n}$

12. If A and B are two points with position vectors $\vec{a}$ and $\vec{b}$ respectively and m, n are positive real numbers, then

$$m\,\vec{OA} + n\,\vec{OB} = (m+n)\,\vec{OC},$$

where C is a point on AB dividing it in the ratio $n : m$.

Also, $\vec{OA} + \vec{OB} = 2\vec{OC}$, where C is the mid-point of AB.

13. If S is any point in the plane of a triangle ABC, then

$$\vec{SA} + \vec{SB} + \vec{SC} = 3\,\vec{SG},$$

where G is the centroid of $\Delta\,ABC$.

14. The necessary and sufficient condition for three points with position vectors $\vec{a}, \vec{b}, \vec{c}$ to be collinear is that there exist scalars x, y, z not all zero such that

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}, \text{ where } x + y + z = 0$$

15. The necessary and sufficient condition for four points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ to be coplanar is that there exist scalars x, y, z, t not all zero such that

$$x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = \vec{0}, \text{ where } x + y + z + t = 0$$

24

SCALAR OR DOT PRODUCT

24.1 INTRODUCTION

In the previous chapter, we have introduced the notion of multiplication of a vector by a scalar. In this chapter, we will introduce the notion of product of two vectors. The product of two vectors results in two different ways, viz. a scalar and a vector. Correspondingly, there are two kinds of products, the one a scalar and the other a vector. But, before defining these products we define the angle between two vectors as under.

24.2 ANGLE BETWEEN TWO VECTORS

Let two vectors $\vec{a}$ and $\vec{b}$ be represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Then the angle between $\vec{a}$ and $\vec{b}$ is the angle between their directions when these directions both converge or both diverge from their point of intersection.

It is evident that if θ is the measure of the angle between two vectors, then $0 \le \theta \le \pi$. For $\theta = \frac{\pi}{2}$, the vectors are said to be perpendicular or orthogonal and for $\theta = 0$ or π, the vectors are said to be parallel.

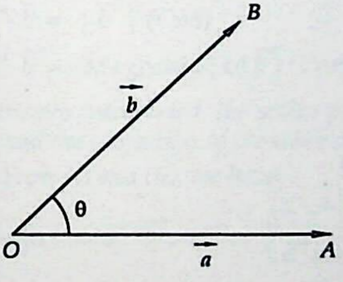


Fig. 24.1

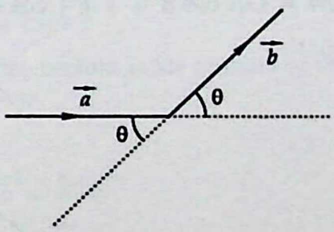


Fig. 24.2

24.3 THE SCALAR OR DOT PRODUCT

DEFINITION *Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ. Then the scalar product of $\vec{a}$ with $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ and is defined as the scalar* $|\vec{a}|\,|\vec{b}|\cos\theta$.

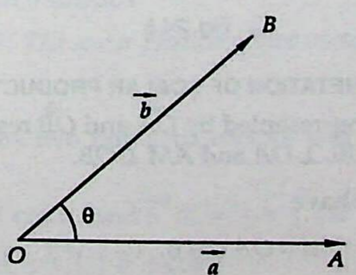


Fig. 24.3

Thus, $\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta$.

Clearly, the scalar product of two vectors is a scalar quantity, due to which this product is called scalar product. Since we are putting a dot between $\vec{a}$ and $\vec{b}$. Therefore, it is also called dot product.

REMARK 1 *If $\vec{a}$ or, $\vec{b}$ or, both is a zero vector, then θ is not defined as $\vec{0}$ has no direction. In this case their dot product $\vec{a} \cdot \vec{b}$ is defined as the scalar zero.*

REMARK 2 *Let $\vec{a} = \vec{AB}$ be a given vector and l be a given line. Let P and Q be the feet of perpendiculars drawn from A and B respectively on line l as shown in Fig. 24.4. Then, PQ is defined as the projection of $\vec{a}$ on line l. Let AM be perpendicular on BQ. Then, $AM = PQ$.*

Let θ be the angle between $\vec{a}$ and line l. Then,

$$\angle BAM = \theta.$$

In Δ AMB, we have

$$\cos\theta = \frac{AM}{AB}$$

$$\Rightarrow \quad \cos\theta = \frac{PQ}{AB} \qquad [\because AM = PQ]$$

$$\Rightarrow \quad PQ = AB\cos\theta.$$

$$PQ = |\vec{a}| \cos\theta$$

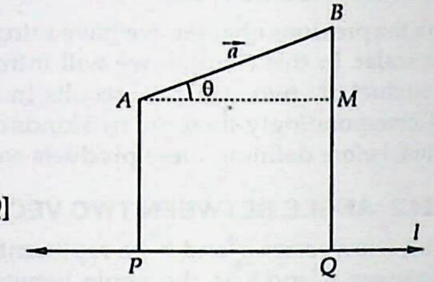


Fig. 24.4

ce, projection of a vector $\vec{a}$ on a line l is $|\vec{a}|\cos\theta$, where θ is the angle between $\vec{a}$ and ine l.

$\vec{OA} = \vec{a}$ and l is a line passing through O as shown in Fig. 24.5. Then, OM is the projection $\vec{a}$ on line l.

Clearly, $OM = OA\cos\theta = |\vec{a}|\cos\theta$.

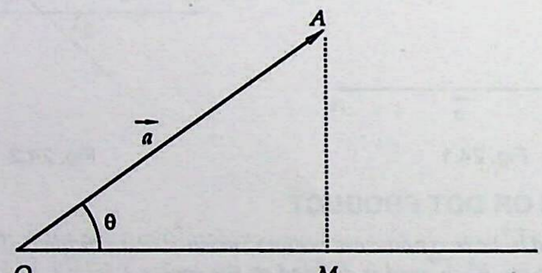


Fig. 24.5

24.3.1 GEOMETRICAL INTERPRETATION OF SCALAR PRODUCT

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\vec{OA}$ and $\vec{OB}$ respectively. Let θ be the angle between $\vec{OA}$ and $\vec{OB}$. Draw $BL \perp OA$ and $AM \perp OB$.

From Δ's OBL and OAM, we have

$$OL = OB\cos\theta \text{ and } OM = OA\cos\theta.$$

Here, OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$ respectively.

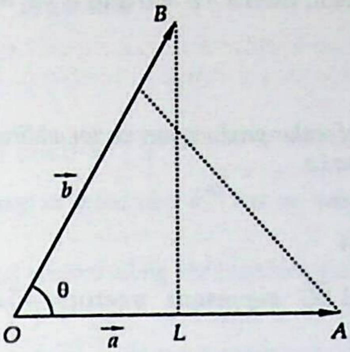


Fig. 24.6

Now, $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = |\vec{a}|\,(OB\cos\theta)$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = |\vec{a}|\,(OL)$

$\Rightarrow \quad \vec{a}\cdot\vec{b} =$ (Magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$) ...(i)

Again,

$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = |\vec{b}|\,(|\vec{a}|\cos\theta)$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = |\vec{b}|\,(OA\cos\theta)$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = |\vec{b}|\,(OM)$

$\Rightarrow \quad \vec{a}\cdot\vec{b} =$ (Magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$) ...(ii)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

REMARK 3 *From (i) and (ii), we have*

$$\text{Projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\vec{a}}{|\vec{a}|}\cdot\vec{b} = \hat{a}\cdot\vec{b}$$

and,
$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\frac{\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$$

Thus, the projection of $\vec{a}$ on $\vec{b}$ is the dot product of $\vec{a}$ with the unit vector along $\vec{b}$ and the projection of $\vec{b}$ on $\vec{a}$ is the dot product of $\vec{b}$ with the unit vector along $\vec{a}$.

24.3.2 PROPERTIES OF SCALAR PRODUCT

PROPERTY I *(Commutativity) The scalar product of two vectors is commutative i.e.*

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}.$$

PROOF CASE I Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors and let θ be the angle between them. Then,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta, \text{ and } \vec{b}\cdot\vec{a} = |\vec{b}|\,|\vec{a}|\cos\theta$$

But, $|\vec{a}|\,|\vec{b}|\cos\theta = |\vec{b}|\,|\vec{a}|\cos\theta$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}.$

CASE II If $\vec{a}$ or $\vec{b}$ is a zero vector, then $\vec{a}\cdot\vec{b}=0$ and $\vec{b}\cdot\vec{a}=0$.

So, $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

Hence, $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$.

PROPERTY II *(Distributivity of scalar product over vector addition) The scalar product of vectors is distributive over vector addition i.e.*

(i) $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ *(Left distributivity)*

(ii) $(\vec{b}+\vec{c})\cdot\vec{a}=\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}$ *(Right distributivity)*

PROOF (i) Let $\vec{OA}$, $\vec{OB}$ and $\vec{BC}$ represent vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then, $\vec{OC}=\vec{OB}+\vec{BC}=\vec{b}+\vec{c}$.

Draw $BL \perp OA$ and $CM \perp OA$.

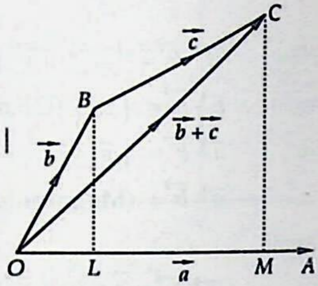


Fig. 24.7

Now,

$$\vec{a}\cdot(\vec{b}+\vec{c})=|\vec{a}|\ (\text{Projection of } \vec{b}+\vec{c} \text{ on } \vec{a})$$

$$\Rightarrow \vec{a}\cdot(\vec{b}+\vec{c})=|\vec{a}|\ (\text{Projection of } \vec{b} \text{ on } \vec{a})+|\vec{a}|$$

$$\Rightarrow \vec{a}\cdot(\vec{b}+\vec{c})=|\vec{a}|\ (\text{Projection of } \vec{OC} \text{ on } \vec{OA})$$

$$\Rightarrow \vec{a}\cdot(\vec{b}+\vec{c})=|\vec{a}|\ (OM)$$

$$\Rightarrow \vec{a}\cdot(\vec{b}+\vec{c})=|\vec{a}|\ (OL+LM)$$

$$\Rightarrow \vec{a}\cdot(\vec{b}+\vec{c})=|\vec{a}|\ (OL)+|\vec{a}|\ (LM)$$

$$\Rightarrow \vec{a}\cdot(\vec{b}+\vec{c})=|\vec{a}|\ (\text{Projection of } \vec{b} \text{ on } \vec{a})+|\vec{a}|\ (\text{Projection of } \vec{c} \text{ on } \vec{a})$$

$$\Rightarrow \vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}.$$

Hence, $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

(ii) We have,

$$(\vec{b}+\vec{c})\cdot\vec{a}=\vec{a}\cdot(\vec{b}+\vec{c}) \quad \text{[By commutativity of scalar product]}$$

$$\Rightarrow (\vec{b}+\vec{c})\cdot\vec{a}=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} \quad \text{[From (i)]}$$

$$\Rightarrow (\vec{b}+\vec{c})\cdot\vec{a}=\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a} \quad \text{[By commutativity of scalar product]}$$

Hence, $(\vec{b}+\vec{c})\cdot\vec{a}=\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}$.

PROPERTY III *Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then,*

$$\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a} \text{ is perpendicular to } \vec{b}.$$

PROOF We have,

$$\vec{a}\cdot\vec{b}=0$$

$$\Leftrightarrow |\vec{a}|\,|\vec{b}|\cos\theta=0$$

$$\Leftrightarrow \cos\theta=0 \quad [\because |\vec{a}|\neq 0, |\vec{b}|\neq 0]$$

$$\Leftrightarrow \theta=\frac{\pi}{2}$$

$$\Leftrightarrow \vec{a} \text{ is perpendicular to } \vec{b}$$

REMARK 1 *Since $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular unit vectors along the coordinate axes.*

$$\therefore \quad \hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{i}=0; \quad \hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{j}=0; \quad \hat{k}\cdot\hat{i}=\hat{i}\cdot\hat{k}=0$$

PROPERTY IV *For any vector* $\vec{a}$, *we have*

$$\vec{a}\cdot\vec{a} = |\vec{a}|^2$$

PROOF We have,

$$\vec{a}\cdot\vec{a} = |\vec{a}|\,|\vec{a}|\cos 0 = |\vec{a}|^2$$

REMARK 2 *Students should keep in mind that* $\vec{a}^2$ *has no meaning but wherever it is used it denotes* $|\vec{a}|^2$.

REMARK 3 *Since* $\hat{i}, \hat{j}, \hat{k}$ *are unit vectors along the coordinate axes. Therefore,*

$$\hat{i}\cdot\hat{i} = |\hat{i}|^2 = 1,\ \hat{j}\cdot\hat{j} = |\hat{j}|^2 = 1 \text{ and } \hat{k}\cdot\hat{k} = |\hat{k}|^2 = 1.$$

PROPERTY V *If m is a scalar and* $\vec{a}, \vec{b}$ *be any two vectors, then*

$$(m\vec{a})\cdot\vec{b} = m(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(m\vec{b})$$

PROOF We have the following cases:

CASE I *When* $m > 0$.

Let θ be the angle between $\vec{a}$ and $\vec{b}$. As $m > 0$, therefore the angle between $m\vec{a}$ and $\vec{b}$ is also θ.

$\therefore\quad (m\vec{a})\cdot\vec{b} = |m\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow\quad (m\vec{a})\cdot\vec{b} = |m|\,|\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow\quad (m\vec{a})\cdot\vec{b} = m\,|\vec{a}|\,|\vec{b}|\cos\theta \quad [\because m > 0 \therefore |m| = m]$

$\Rightarrow\quad (m\vec{a})\cdot\vec{b} = m(\vec{a}\cdot\vec{b})$

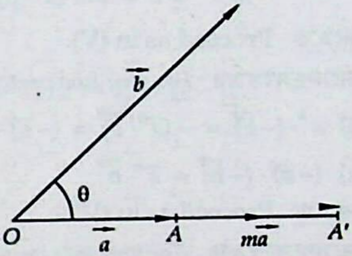


Fig. 24.8

The angle between $\vec{a}$ and $m\vec{b}$ will also be θ.

$\therefore\quad \vec{a}\cdot(m\vec{b}) = |\vec{a}|\,|m\vec{b}|\cos\theta$

$\Rightarrow\quad \vec{a}\cdot(m\vec{b}) = |\vec{a}|\,|m|\,|\vec{b}|\cos\theta$

$\Rightarrow\quad \vec{a}\cdot(m\vec{b}) = m\,|\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow\quad \vec{a}\cdot(m\vec{b}) = m(\vec{a}\cdot\vec{b})$

Hence, $(m\vec{a})\cdot\vec{b} = m(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(m\vec{b})$

CASE II *When* $m < 0$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. As $m < 0$, therefore, the angle between $m\vec{a}$ and $\vec{b}$ is $(\pi - \theta)$

$\therefore\quad (m\vec{a})\cdot\vec{b} = |m\vec{a}|\,|\vec{b}|\cos(\pi - \theta)$

$\Rightarrow\quad (m\vec{a})\cdot\vec{b} = -|m|\,|\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow\quad (m\vec{a})\cdot\vec{b} = m(\vec{a}\cdot\vec{b})$

In this case, the angle between $\vec{a}$ and $m\vec{b}$ is also $(\pi - \theta)$.

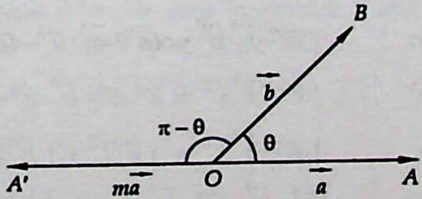


Fig. 24.9

$\therefore\quad \vec{a}\cdot m\vec{b} = |\vec{a}|\,|m\vec{b}|\cos(\pi - \theta)$

$\Rightarrow\quad \vec{a}\cdot m\vec{b} = -|\vec{a}|\,|m|\,|\vec{b}|\cos\theta$

$\Rightarrow\quad \vec{a}\cdot m\vec{b} = m\,|\vec{a}|\,|\vec{b}|\cos\theta)$

$\Rightarrow \quad \vec{a} \cdot m\vec{b} = m(\vec{a} \cdot \vec{b})$

Hence, $\quad (m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (m\vec{b})$

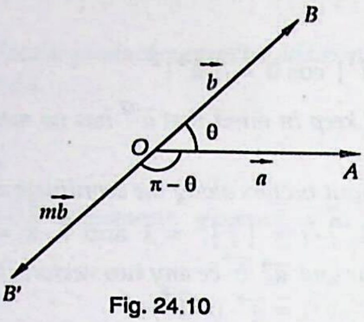


Fig. 24.10

<u>CASE III</u> *If $m = 0$, then the result $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (m\vec{b})$ trivially holds good.*

PROPERTY VI *If m, n are scalars and $\vec{a}, \vec{b}$ be two vectors. Then,*

$$m\vec{a} \cdot n\vec{b} = mn(\vec{a} \cdot \vec{b}) = (mn\vec{a}) \cdot \vec{b} = \vec{a} \cdot (mn\vec{b})$$

<u>PROOF</u> Proceed as in (V)

PROPERTY VII *For any two vectors $\vec{a}$ and $\vec{b}$, we have*

(i) $\vec{a} \cdot (-\vec{b}) = -(\vec{a} \cdot \vec{b}) = (-\vec{a}) \cdot \vec{b}$

(ii) $(-\vec{a}) \cdot (-\vec{b}) = \vec{a} \cdot \vec{b}$

<u>PROOF</u> Proceed as in (VI)

PROPERTY VIII *For any two vectors $\vec{a}$ and $\vec{b}$, we have*

(i) $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}$

(ii) $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$

(iii) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

<u>PROOF</u> We have,

(i) $\quad |\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$ $\quad [\because \vec{a} \cdot \vec{a} = |\vec{a}|^2]$

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}) \cdot \vec{a} + (\vec{a} + \vec{b}) \cdot \vec{b}$ $\quad$ [By left distributivity]

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$ $\quad$ [By right distributivity]

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$ $\quad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}$

(ii) $\quad |\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$ $\quad [\because \vec{a} \cdot \vec{a} = |\vec{a}|^2]$

$\Rightarrow \quad |\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot \vec{a} - (\vec{a} - \vec{b}) \cdot \vec{b}$ $\quad$ [By left distributivity]

$\Rightarrow \quad |\vec{a} - \vec{b}|^2 = \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$ $\quad$ [By right distributivity]

$\Rightarrow \quad |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$ $\quad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$

(iii) $\quad (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (\vec{a} + \vec{b}) \cdot \vec{a} - (\vec{a} + \vec{b}) \cdot \vec{b}$ $\quad$ [By left distributivity]

$\Rightarrow \quad (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{b}$ $\quad$ [By right distributivity]

$\Rightarrow \quad (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$ $\quad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$

24.3.3 SCALAR PRODUCT IN TERMS OF COMPONENTS

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ *and* $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. *Then,*

$$\vec{a}\cdot\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = a_1\hat{i}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

[By right distributivity of scalar product]

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = (a_1 b_1)(\hat{i}\cdot\hat{i}) + (a_1 b_2)(\hat{i}\cdot\hat{j}) + (a_1 b_3)(\hat{i}\cdot\hat{k}) + a_2 b_1(\hat{j}\cdot\hat{i}) + a_2 b_2(\hat{j}\cdot\hat{j}) + a_2 b_3(\hat{j}\cdot\hat{k}) + a_3 b_1(\hat{k}\cdot\hat{i}) + a_3 b_2(\hat{k}\cdot\hat{j}) + a_3 b_3(\hat{k}\cdot\hat{k})$$

[By left distributivity of scalar product]

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \left[\begin{array}{l} \because \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \text{ and} \\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{i}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \end{array}\right]$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Thus, the scalar product of two vectors is equal to the sum of the products of their corresponding components.

ILLUSTRATION *If* $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} + 2\hat{j} + 3\hat{k}$, *find* $\vec{a}\cdot\vec{b}$.

SOLUTION We have,

$$\vec{a}\cdot\vec{b} = (2\hat{i} - \hat{j} + 2\hat{k}) \cdot (3\hat{i} + 2\hat{j} + 3\hat{k}) = (2)(3) + (-1)(2) + (2)(3) = 10.$$

24.3.4 ANGLE BETWEEN TWO VECTORS

Let $\vec{a}, \vec{b}$ *be two vectors inclined at an angle* θ. *Then,*

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right\} \qquad ...(i)$$

This formula is used to find the angle between two given vectors.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,\ |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^3},\ |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

Substituting the values of $\vec{a}\cdot\vec{b}$, $|\vec{a}|$ and $|\vec{b}|$ in (i), we have

$$\theta = \cos^{-1}\left\{\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}\right\}$$

ILLUSTRATION 1 *Find the angle between two vectors* $\vec{a}$ *and* $\vec{b}$ *with magnitude 2 and 1 respectively, and such that* $\vec{a}\cdot\vec{b} = \sqrt{3}$

SOLUTION We have,

$$\vec{a}\cdot\vec{b} = \sqrt{3},\ |\vec{a}| = 2 \text{ and } |\vec{b}| = 1$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\Rightarrow \quad \cos\theta = \frac{\sqrt{3}}{2\times 1} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{6}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$.

ILLUSTRATION 2 *Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.*

SOLUTION Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

We have,

$$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k} \text{ and } \vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = 1\times 3 + (-2)\times(-2) + 3\times 1 = 3 + 4 + 3 = 10$$

$$|\vec{a}| = \sqrt{1+4+9} = \sqrt{14} \text{ and } |\vec{b}| = \sqrt{9+4+1} = \sqrt{14}$$

$$\therefore \quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\Rightarrow \quad \cos\theta = \frac{10}{14}$$

$$\Rightarrow \quad \cos\theta = \frac{5}{7}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Hence, the angle between the given vectors is $\cos^{-1}\left(\frac{5}{7}\right)$.

24.3.5 COMPONENTS OF A VECTOR ALONG AND PERPENDICULAR TO VECTOR

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\vec{OA}$ and $\vec{OB}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$. Draw $BM \perp OA$.

In ΔOBM, we have

$$\vec{OB} = \vec{OM} + \vec{MB} \Rightarrow \vec{b} = \vec{OM} + \vec{MB}$$

Thus, $\vec{OM}$ and $\vec{MB}$ are components of $\vec{b}$ along $\vec{a}$ and perpendicular to $\vec{a}$ respectively.

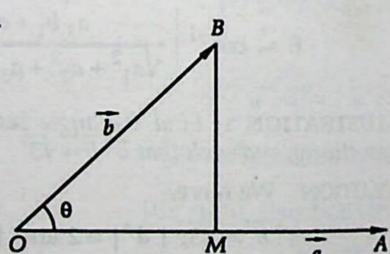


Fig. 24.11

Now,

$$\vec{OM} = (OM)\hat{a}$$

$$\Rightarrow \quad \vec{OM} = (OB\cos\theta)\hat{a}$$

$$\Rightarrow \quad \vec{OM} = \{|\vec{b}|\cos\theta\}\hat{a}$$

$$\Rightarrow \quad \vec{OM} = \left\{|\vec{b}|\,\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|\,|\vec{b}|}\right\}\hat{a} \qquad \left[\because \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right]$$

$$\Rightarrow \quad \vec{OM} = \left\{ \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right\} \hat{a}$$

$$\Rightarrow \quad \vec{OM} = \left\{ \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right\} \frac{\vec{a}}{|\vec{a}|} = \left\{ \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right\} \vec{a}$$

Now,

$$\vec{b} = \vec{OM} + \vec{MB}$$

$$\Rightarrow \quad \vec{MB} = \vec{b} - \vec{OM} = \vec{b} - \left\{ \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right\} \vec{a}$$

Thus, the components of $\vec{b}$ along and perpendicular to $\vec{a}$ are

$$\left\{ \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right\} \vec{a} \text{ and } \vec{b} - \left\{ \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right\} \vec{a} \text{ respectively.}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find $\vec{a} \cdot \vec{b}$ when*

(i) $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$

(ii) $\vec{a} = (1, 1, 2)$ and $\vec{b} = (3, 2, -1)$

SOLUTION We have,

(i) $\quad \vec{a} \cdot \vec{b} = (2\hat{i} + 2\hat{j} - \hat{k}) \cdot (6\hat{i} - 3\hat{j} + 2\hat{k})$

$\Rightarrow \quad \vec{a} \cdot \vec{b} = (2)(6) + (2)(-3) + (-1)(2) = 12 - 6 - 2 = 4.$

(ii) $\quad \vec{a} \cdot \vec{b} = (\hat{i} + \hat{j} + 2\hat{k}) \cdot (3\hat{i} + 2\hat{j} - \hat{k}) = (1)(3) + (1)(2) + (2)(-1) = 3 + 2 - 2 = 3.$

EXAMPLE 2 *Find $(\vec{a} + 3\vec{b}) \cdot (2\vec{a} - \vec{b})$, if $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$.*

[CBSE 2002]

SOLUTION We have,

$\vec{a} + 3\vec{b} = (\hat{i} + \hat{j} + 2\hat{k}) + 3(3\hat{i} + 2\hat{j} - \hat{k}) = 10\hat{i} + 7\hat{j} - \hat{k}$

and, $\quad 2\vec{a} - \vec{b} = 2(\hat{i} + \hat{j} + 2\hat{k}) - (3\hat{i} + 2\hat{j} - \hat{k}) = -\hat{i} + 0\hat{j} + 5\hat{k}$

$\therefore \quad (\vec{a} + 3\vec{b}) \cdot (2\vec{a} - \vec{b}) = (10\hat{i} + 7\hat{j} - \hat{k}) \cdot (-\hat{i} + 0\hat{j} + 5\hat{k})$

$= (10)(-1) + (7)(0) + (-1)(5) = -10 + 0 - 5 = -15$

<u>ALITER</u> We have,

$|\vec{a}| = \sqrt{1+1+4} = \sqrt{6}$, $|\vec{b}| = \sqrt{9+4+1} = \sqrt{14}$ and $\vec{a} \cdot \vec{b} = 3 + 2 - 2 = 3$.

$\therefore \quad (\vec{a} + 3\vec{b}) \cdot (2\vec{a} - \vec{b}) = \vec{a} \cdot 2\vec{a} - \vec{a} \cdot \vec{b} + 3\vec{b} \cdot 2\vec{a} - 3\vec{b} \cdot \vec{b}$

$\Rightarrow \quad (\vec{a} + 3\vec{b}) \cdot (2\vec{a} - \vec{b}) = 2|\vec{a}|^2 - \vec{a} \cdot \vec{b} + 6(\vec{b} \cdot \vec{a}) - 3|\vec{b}|^2$

$\Rightarrow \quad (\vec{a} + 3\vec{b}) \cdot (2\vec{a} - \vec{b}) = 2|\vec{a}|^2 + 5(\vec{a} \cdot \vec{b}) - 3|\vec{b}|^2$

$\Rightarrow \quad (\vec{a} + 3\vec{b}) \cdot (2\vec{a} - \vec{b}) = 2(6) + 5(3) - 3(14) = -15.$

EXAMPLE 3 *For any vector $\vec{r}$, prove that $\vec{r} = (\vec{r} \cdot \hat{i})\hat{i} + (\vec{r} \cdot \hat{j})\hat{j} + (\vec{r} \cdot \hat{k})\hat{k}$.*

SOLUTION Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be an arbitrary vector. Then,

$\vec{r} \cdot \hat{i} = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{i} = x(\hat{i} \cdot \hat{i}) + y(\hat{j} \cdot \hat{i}) + z(\hat{k} \cdot \hat{i}) = x(1) + y(0) + z(0) = x$

$\vec{r} \cdot \hat{j} = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{j} = x(\hat{i} \cdot \hat{j}) + y(\hat{j} \cdot \hat{j}) + z(\hat{k} \cdot \hat{j}) = x(0) + y(1) + z(0) = y$

and, $\vec{r}\cdot\hat{k} = (x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{k} = x(\hat{i}\cdot\hat{k}) + y(\hat{j}\cdot\hat{k}) + z(\hat{k}\cdot\hat{k}) = x(0)+y(0)+z(1) = z$

Putting the values of x, y, z in $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, we obtain

$$\vec{r} = (\vec{r}\cdot\hat{i})\,\hat{i} + (\vec{r}\cdot\hat{j})\,\hat{j} + (\vec{r}\cdot\hat{k})\,\hat{k}.$$

REMARK 1 *In the above example, we have seen that for any vector* $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, *we have* $\vec{r}\cdot\hat{i} = x$, $\vec{r}\cdot\hat{j} = y$ and $\vec{r}\cdot\hat{k} = z$

Also,

$\vec{r}\cdot\hat{i}$ = *Projection of* $\vec{r}$ *on X–axis.*

$\vec{r}\cdot\hat{j}$ = *Projection of* $\vec{r}$ *on Y–axis.*

$\vec{r}\cdot\hat{k}$ = *Projection of* $\vec{r}$ *on Z–axis.*

$\therefore$ x = *Projection of* $\vec{r}$ *on X–axis,* y = *Projection of* $\vec{r}$ *on Y–axis*

z = *Projection of* $\vec{r}$ *on Z–axis*

Thus, if $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, *then x, y and z respectively are the projections of* $\vec{r}$ *on X, Y and Z-axes.*

REMARK 2 *For any vector* $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$. *We call* $x\hat{i}$, $y\hat{j}$ *and* $z\hat{k}$ *as components of* $\vec{r}$ *along x, y and z-axes respectively.*

REMARK 3 *If a vector* $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ *makes angles* α, β *and* γ *with x, y and z-axes respectively, then*

$$\cos\alpha = \frac{\vec{r}\cdot\hat{i}}{|\vec{r}|\,|\hat{i}|}$$

$$\Rightarrow \quad \cos\alpha = \frac{x}{|\vec{r}|}$$

$$\Rightarrow \quad \alpha = \cos^{-1}\left(\frac{x}{|\vec{r}|}\right)$$

Similarly, we have

$$\beta = \cos^{-1}\left(\frac{y}{|\vec{r}|}\right) \text{ and } \gamma = \cos^{-1}\left(\frac{z}{|\vec{r}|}\right)$$

Again,

$$\cos\alpha = \frac{x}{|\vec{r}|},\ \cos\beta = \frac{y}{|\vec{r}|} \text{ and } \cos\gamma = \frac{z}{|\vec{r}|}$$

$$\Rightarrow \quad x = |\vec{r}|\cos\alpha,\ y = |\vec{r}|\cos\beta \text{ and } z = |\vec{r}|\cos\gamma$$

$$\therefore \quad \vec{r} = x\hat{i}+y\hat{j}+z\hat{k} = |\vec{r}|\,\{(\cos\alpha)\,\hat{i} + (\cos\beta)\,\hat{i} + (\cos\gamma)\,\hat{k}\}$$

Also, $|\vec{r}|^2 = x^2+y^2+z^2$

$$\Rightarrow \quad |\vec{r}|^2 = |\vec{r}|^2\cos^2\alpha + |\vec{r}|^2\cos^2\beta + |\vec{r}|^2\cos^2\gamma$$

$$\Rightarrow \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Thus, if a vector $\vec{r}$ makes angles α, β and γ with $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively, then

$$\vec{r} = |\vec{r}|\,\{(\cos\alpha)\,\hat{i} + (\cos\beta)\,\hat{j} + (\cos\gamma)\,\hat{k}\} \text{ and, } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

EXAMPLE 4 *Let* $\vec{a} = 4\hat{i}+5\hat{j}-\hat{k}$, $\vec{b} = \hat{i}-4\hat{j}+5\hat{k}$ *and* $\vec{c} = 3\hat{i}+\hat{j}-\hat{k}$. *Find a vector* $\vec{d}$ *which is perpendicular to both* $\vec{a}$ *and* $\vec{b}$, *and satisfying* $\vec{d}\cdot\vec{c} = 21$.

SOLUTION Let $\vec{d} = x\hat{i}+y\hat{j}+z\hat{k}$.

Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

$\therefore \quad \vec{d}\cdot\vec{a}=0$ and $\vec{d}\cdot\vec{b}=0.$

Now, $\quad \vec{d}\cdot\vec{a}=0 \Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(4\hat{i}+5\hat{j}-\hat{k}) = 0 \Rightarrow 4x+5y-z=0$ ...(i)

$\vec{d}\cdot\vec{b}=0 \Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}-4\hat{j}+5\hat{k}) = 0 \Rightarrow x-4y+5z=0$ ...(ii)

$\vec{d}\cdot\vec{c}=21 \Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}+\hat{j}-\hat{k}) = 21 \Rightarrow 3x+y-z=21$ ...(iii)

Solving (i), (ii) and (iii), we get $\quad x=7, y=z=-7$

Hence, $\quad \vec{d}=7\hat{i}-7\hat{j}-7\hat{k}.$

EXAMPLE 5 *Dot products of a vector with vectors $3\hat{i}-5\hat{k}$, $2\hat{i}+7\hat{j}$ and $\hat{i}+\hat{j}+\hat{k}$ are respectively −1, 6 and 5. Find the vector.*

SOLUTION Let $\vec{a}=3\hat{i}+0\hat{j}-5\hat{k}$, $\vec{b}=2\hat{i}+7\hat{j}+0\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ be three given vectors. Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ be a vector such that its dot products with $\vec{a}, \vec{b}$ and $\vec{c}$ are −1, 6 and 5 respectively. Then,

$\vec{r}\cdot\vec{a}=-1 \Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}+0\hat{j}-5\hat{k})=-1 \Rightarrow 3x+0y-5z=-1$ ...(i)

$\vec{r}\cdot\vec{b}=6 \Rightarrow (x\hat{i}+y\hat{i}+z\hat{k})\cdot(2\hat{i}+7\hat{j}+0\hat{k}) = 6 \Rightarrow 2x+7y+0z=6$ ...(ii)

and, $\quad \vec{r}\cdot\vec{c}=5 \Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k}) = 5 \Rightarrow x+y+z=5$ ...(iii)

Solving (i), (ii) and (iii), we get $x=3, y=0, z=2.$

Hence, the required vector $\vec{r}$ is given by

$$\vec{r}=3\hat{i}+0\hat{j}+2\hat{k}.$$

EXAMPLE 6 *Find the value of λ so that the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ are perpendicular to each other.*

SOLUTION The vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

$\therefore \quad \vec{a}\cdot\vec{b}=0$

$\Rightarrow \quad (2\hat{i}+\lambda\hat{j}+\hat{k})\cdot(\hat{i}-2\hat{j}+3\hat{k}) = 0$

$\Rightarrow \quad (2)(1)+\lambda(-2)+(1)(3)=0$

$\Rightarrow \quad -2\lambda+5=0 \Rightarrow \lambda = 5/2.$

EXAMPLE 7 *Find the value of p for which the vectors $\vec{a}=3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b}=\hat{i}+p\hat{j}+3\hat{k}$ are*

(i) *perpendicular* (ii) *parallel.*

SOLUTION (i) If vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then

$\vec{a}\cdot\vec{b}=0$

$\Rightarrow \quad (3\hat{i}+2\hat{j}+9\hat{k})\cdot(\hat{i}+p\hat{j}+3\hat{k})=0$

$\Rightarrow \quad 3+2p+27=0 \Rightarrow p=-15$

(ii) We know that the vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are parallel iff

$\vec{a}=\lambda\vec{b}$

$\Leftrightarrow \quad (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})=\lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$

$\Leftrightarrow \quad a_1=\lambda b_1, a_2=\lambda b_2, a_3=\lambda b_3$

$\Leftrightarrow \quad \dfrac{a_1}{b_1}=\dfrac{a_2}{b_2}=\dfrac{a_3}{b_3} \;(=\lambda)$

So, given vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ will be parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \Rightarrow 3 = \frac{2}{p} \Rightarrow p = \frac{2}{3}$$

EXAMPLE 8 *Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively, and such that $\vec{a} \cdot \vec{b} = \sqrt{6}$.*

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$.

$3i + 2j + 9k = \lambda(i + pj + 3$

We have,

$$|\vec{a}| = \sqrt{3},\ |\vec{b}| = 2 \text{ and } \vec{a} \cdot \vec{b} = \sqrt{6}$$

$\frac{3}{1} = \frac{2}{p} = \frac{9}{3}$

$\Rightarrow \boxed{p = \frac{2}{3}}$

$$\therefore \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\Rightarrow \quad \cos\theta = \frac{\sqrt{6}}{\sqrt{3} \times 2}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \cos\theta = \cos\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

EXAMPLE 9 *Find the angle between two vectors $\vec{a}$ and $\vec{b}$ having the same length $\sqrt{2}$ and their scalar product is -1.*

SOLUTION Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$.

We have,

$$|\vec{a}| = |\vec{b}| = \sqrt{2} \text{ and } \vec{a} \cdot \vec{b} = -1$$

$$\therefore \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\Rightarrow \quad \cos\theta = \frac{-1}{\sqrt{2} \times \sqrt{2}} = -\frac{1}{2}$$

$$\Rightarrow \quad \cos\theta = \cos 2\pi/3 \qquad [\because 0 \le \theta \le \pi]$$

$$\Rightarrow \quad \theta = 2\pi/3$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $2\pi/3$.

EXAMPLE 10 *Find the angle between the vectors $5\hat{i} + 3\hat{j} + 4\hat{k}$ and $6\hat{i} - 8\hat{j} - \hat{k}$.*

SOLUTION Let $\vec{a} = 5\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = 6\hat{i} - 8\hat{j} - \hat{k}$ be the given vectors and let θ be the angle between them. Then,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

Now, $\vec{a} \cdot \vec{b} = (5\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (6\hat{i} - 8\hat{j} - \hat{k}) = (5)(6) + 3(-8) + 4(-1) = 2$

$$|\vec{a}| = \sqrt{5^2 + 3^2 + 4^2} = \sqrt{50} \text{ and, } |\vec{b}| = \sqrt{6^2 + (-8)^2 + (-1)^2} = \sqrt{101}$$

$$\therefore \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$\Rightarrow \quad \cos\theta = \dfrac{2}{\sqrt{50}\,\sqrt{101}} = \dfrac{2}{5\sqrt{2}\,\sqrt{101}} = \dfrac{\sqrt{2}}{5\,\sqrt{101}}$

$\Rightarrow \quad \theta = \cos^{-1}\left(\dfrac{\sqrt{2}}{5\,\sqrt{101}}\right).$

EXAMPLE 11 *If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3i+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Deduce that $\vec{AB}$ and $\vec{CD}$ are collinear.* **[NCERT]**

SOLUTION Let θ be the angle between the lines AB and CD. Then, θ is also the angle between vectors $\vec{AB}$ and $\vec{CD}$.

We have,

$\vec{AB} = (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k}) = \hat{i}+4\hat{j}-\hat{k}$

and, $\vec{CD} = (\hat{i}-6\hat{j}-\hat{k})-(3\hat{i}+2\hat{j}-3\hat{k}) = -2\hat{i}-8\hat{j}+2\hat{k}$

$\therefore \quad |\vec{AB}| = \sqrt{1+16+1} = 3\sqrt{2}$, $|\vec{CD}| = \sqrt{4+64+4} = 6\sqrt{2}$

and, $(\vec{AB}\cdot\vec{CD}) = (\hat{i}+4\hat{j}-\hat{k})\cdot(-2\hat{i}-8\hat{j}+2\hat{k}) = -2-32-2 = -36$

Now, $\cos\theta = \dfrac{\vec{AB}\cdot\vec{CD}}{|\vec{AB}|\,|\vec{CD}|}$

$\Rightarrow \quad \cos\theta = \dfrac{-36}{3\sqrt{2}\times 6\sqrt{2}}$

$\Rightarrow \quad \cos\theta = -1$

$\Rightarrow \quad \theta = \pi \qquad [\because 0 \le \theta \le \pi]$

$\Rightarrow$ $\vec{AB}$ and $\vec{CD}$ are unlike parallel vectors.

Hence, lines AB and CD are parallel.

EXAMPLE 12 *Find the projection of the vector $7\hat{i}+\hat{j}-4\hat{k}$ on $2\hat{i}+6\hat{j}+3\hat{k}$.*

SOLUTION Let $\vec{a} = 7\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$

We have,

Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Now,

$\vec{a}\cdot\vec{b} = (7\hat{i}+\hat{j}-4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k}) = 14+6-12 = 8$ and $|\vec{b}| = \sqrt{2^2+6^2+3^2} = 7$

$\therefore$ Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{8}{7}$

EXAMPLE 13 *Show that the projection vector of $\vec{a}$ on $\vec{b}\,(\ne \vec{0})$ (component of $\vec{a}$ along $\vec{b}$) is $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$.*

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$.
As shown in Fig. 24.12, length OL is the projection of $\vec{a}$ on $\vec{b}$ and $\vec{OL}$ is the projection vector of $\vec{a}$ on $\vec{b}$.

In ΔOLA, we have

$$\cos\theta = \frac{OL}{OA}$$

$\Rightarrow \quad OL = OA\cos\theta$

$\Rightarrow \quad OL = |\vec{a}|\cos\theta$

$\Rightarrow \quad OL = |\vec{a}|\left\{\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|\,|\vec{b}|}\right\} \qquad \left[\text{Using} : \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right]$

$\Rightarrow \quad OL = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

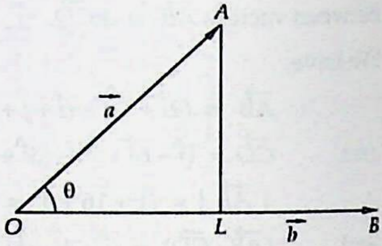


Fig. 24.12

Now, $\quad \vec{OL} = (OL)\,\hat{b}$

$\Rightarrow \quad \vec{OL} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right\}\hat{b}$

$\Rightarrow \quad \vec{OL} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right\}\frac{\vec{b}}{|\vec{b}|} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right\}\vec{b}$

NOTE *Students are advised to use this as a standard result.*

EXAMPLE 14 *Show that the projection vector of* $\vec{b}$ *on* $\vec{a} \neq 0$ *is* $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$.

SOLUTION Proceed as in Example 13.

EXAMPLE 15 *For any two vectors* $\vec{a}$ *and* $\vec{b}$, *prove that:*

(i) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}\cdot\vec{b}$

(ii) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$

(iii) $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2\,(|\vec{a}|^2 + |\vec{b}|^2)$

Interpret the result geometrically.

(iv) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \Leftrightarrow \vec{a}\perp\vec{b}$

Interpret the result geometrically.

(v) $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}| \Leftrightarrow \vec{a}$ *is parallel to* $\vec{b}$

(vi) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \Leftrightarrow \vec{a}, \vec{b}$ *are orthogonal.* [NCERT]

SOLUTION (i) We have,

$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) \qquad [|\vec{x}|^2 = \vec{x}\cdot\vec{x}]$

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot\vec{a} + (\vec{a}+\vec{b})\cdot\vec{b} \qquad$ [By distributivity of dot product over vector add.]

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} \qquad$ [By distributivity of dot product over vector add.]

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2\,\vec{a}\cdot\vec{b} + |\vec{b}|^2 \qquad [\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$

(ii) Proceed as in (i).

(iii) Adding (i) and (ii), we obtain

$$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2\,(|\vec{a}|^2 + |\vec{b}|^2)$$

Geometrical Interpretation: Let $ABCD$ be a parallelogram such that $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$.

In ΔABC, we have

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \vec{AC} \qquad \text{...(i)}$$

In ΔABD, we have

$$\vec{AD} + \vec{DB} = \vec{AB}$$

$$\Rightarrow \quad \vec{AB} - \vec{AD} = \vec{DB}$$

$$\Rightarrow \quad \vec{a} - \vec{b} = \vec{DB} \qquad \text{...(ii)}$$

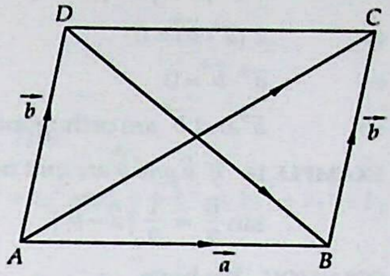


Fig. 24.13

Thus, $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ represent two diagonals of a parallelogram whose two adjacent sides are $\vec{a}$ and $\vec{b}$.

$$\therefore \quad |\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$$

$\Leftrightarrow$ The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals of the parallelogram.

(iv) We have,

$$|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$$

$$\Leftrightarrow \quad |\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2$$

$$\Leftrightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a} \cdot \vec{b})$$

$$\Leftrightarrow \quad 4(\vec{a} \cdot \vec{b}) = 0$$

$$\Leftrightarrow \quad \vec{a} \cdot \vec{b} = 0$$

$$\Leftrightarrow \quad \vec{a} \perp \vec{b}$$

As discussed above that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ represent diagonals of a parallelogram whose two adjacent sides are $\vec{a}$ and $\vec{b}$.

$$\therefore \quad |\vec{a} + \vec{b}| = |\vec{a} - \vec{b}| \Leftrightarrow \vec{a} \perp \vec{b}$$

$\Leftrightarrow$ The diagonals of a parallelogram are equal iff it is a rectangle.

(v) We have,

$$|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$$

$$\Leftrightarrow \quad |\vec{a} + \vec{b}|^2 = (|\vec{a}| + |\vec{b}|)^2$$

$$\Leftrightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|$$

$$\Leftrightarrow \quad \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$$

$$\Leftrightarrow \quad |\vec{a}||\vec{b}| \cos\theta = |\vec{a}||\vec{b}|$$

$$\Leftrightarrow \quad \cos\theta = 1$$

$$\Leftrightarrow \quad \theta = 0$$

$$\Leftrightarrow \quad \vec{a} \parallel \vec{b}$$

(vi). We have,

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$$

$\Leftrightarrow \quad |\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})=|\vec{a}|^2+|\vec{b}|^2$

$\Leftrightarrow \quad 2(\vec{a}\cdot\vec{b})=0$

$\Leftrightarrow \quad \vec{a}\cdot\vec{b}=0$

$\Leftrightarrow \quad \vec{a}$ and $\vec{b}$ are orthogonal.

EXAMPLE 16 *If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that*

$$\sin\frac{\theta}{2}=\frac{1}{2}|\hat{a}-\hat{b}|$$

SOLUTION We have,

$$|\hat{a}-\hat{b}|^2=(\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b}) \qquad [\because \vec{a}\cdot\vec{a}=|\vec{a}|^2]$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2=\hat{a}\cdot\hat{a}-\hat{a}\cdot\hat{b}-\hat{b}\cdot\hat{a}+\hat{b}\cdot\hat{b}$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2=|\hat{a}|^2-2\hat{a}\cdot\hat{b}+|\hat{b}|^2 \qquad [\because \hat{a}\cdot\hat{b}=\hat{b}\cdot\hat{a}]$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2=|\hat{a}|^2-2|\hat{a}||\hat{b}|\cos\theta+|\hat{b}|^2 \qquad [\because \hat{a}\cdot\hat{b}=|\hat{a}||\hat{b}|\cos\theta]$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2=2-2\cos\theta \qquad [\because |\hat{a}|=|\hat{b}|=1]$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2=2(1-\cos\theta)=2\left(2\sin^2\frac{\theta}{2}\right)=4\sin^2\frac{\theta}{2}$$

$$\Rightarrow \quad \sin^2\frac{\theta}{2}=\frac{1}{4}|\hat{a}-\hat{b}|^2$$

$$\Rightarrow \quad \sin\frac{\theta}{2}=\frac{1}{2}|\hat{a}-\hat{b}|.$$

EXAMPLE 17 *For any two vectors $\vec{a}$ and $\vec{b}$, prove that*

$$|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}| \qquad \textit{(Triangle inequality).} \qquad \text{[NCERT]}$$

SOLUTION We have,

$$|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\cos\theta \qquad \text{...(i)}$$

Now, $\cos\theta\le 1$ for all θ

$$\Rightarrow \quad 2|\vec{a}||\vec{b}|\cos\theta\le 2|\vec{a}||\vec{b}|$$

$$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\cos\theta\le|\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2\le\{|\vec{a}|+|\vec{b}|\}^2$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$$

EXAMPLE 18 *Prove Cauchy-Schawarz inequality*

$$(\vec{a}\cdot\vec{b})^2\le|\vec{a}|^2|\vec{b}|^2$$

and hence show that $(a_1b_1+a_2b_2+a_3b_3)^2\le(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$. [NCERT]

SOLUTION We have,

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \quad (\vec{a}\cdot\vec{b})^2=|\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

Now, $\quad \cos^2\theta \leq 1$

$\Rightarrow \quad |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta \leq |\vec{a}|^2 |\vec{b}|^2$

$\Rightarrow \quad (\vec{a}\cdot\vec{b})^2 \leq |\vec{a}|^2 |\vec{b}|^2$

Let $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then,

$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$, $|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2$ and, $|\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2$

$\therefore \quad (\vec{a}\cdot\vec{b})^2 \leq |\vec{a}|^2 |\vec{b}|^2$

$\Rightarrow \quad (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

EXAMPLE 19 *If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a}\cdot\vec{b} = 1$, find $(3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b})$.*

SOLUTION We have,

$(3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b}) = 6\,\vec{a}\cdot\vec{a} + 21\,\vec{a}\cdot\vec{b} - 10\,\vec{b}\cdot\vec{a} - 35\,\vec{b}\cdot\vec{b}$ $\left[\begin{array}{r}\text{Using distributivity}\\ \text{of '}\cdot\text{' over +}\end{array}\right]$

$\Rightarrow \quad (3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b}) = 6\,|\vec{a}|^2 + 21\,(\vec{a}\cdot\vec{b}) - 10\,(\vec{a}\cdot\vec{b}) - 35\,|\vec{b}|^2$

$\Rightarrow \quad (3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b}) = 6\,|\vec{a}|^2 + 11\,(\vec{a}\cdot\vec{b}) - 35\,|\vec{b}|^2$

$\Rightarrow \quad (3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b}) = 6\times 2^2 + 11\times 1 - 35\times 1^2$ $\quad [\because |\vec{a}| = 2, |\vec{b}| = 1, \vec{a}\cdot\vec{b} = 1]$

$\Rightarrow \quad (3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b}) = 24 + 11 - 35 = 0$

EXAMPLE 20 *Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a})\cdot(\vec{x} + \vec{a}) = 15$.*

SOLUTION We have,

$(\vec{x} - \vec{a})\cdot(\vec{x} + \vec{a}) = 15$

$\Rightarrow \quad |\vec{x}|^2 - |\vec{a}|^2 = 15$ $\quad \left[\because (\vec{a} - \vec{b})\cdot(\vec{a} + \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2\right]$

$\Rightarrow \quad |\vec{x}|^2 - 1 = 15$ $\quad [\because \vec{a}$ is a unit vector $\therefore |\vec{a}| = 1]$

$\Rightarrow \quad |\vec{x}|^2 = 16$

$\Rightarrow \quad |\vec{x}| = 4$

EXAMPLE 21 *Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} - \vec{b})\cdot(\vec{a} + \vec{b}) = 27$ and $|\vec{a}| = 2\,|\vec{b}|$.*

SOLUTION We have,

$(\vec{a} - \vec{b})\cdot(\vec{a} + \vec{b}) = 27$

$\Rightarrow \quad |\vec{a}|^2 - |\vec{b}|^2 = 27$

$\Rightarrow \quad 4\,|\vec{b}|^2 - |\vec{b}|^2 = 27$ $\quad [\because |\vec{a}| = 2\,|\vec{b}|$ (Given)$]$

$\Rightarrow \quad 3\,|\vec{b}|^2 = 27$

$\Rightarrow \quad |\vec{b}|^2 = 9$

$\Rightarrow \quad |\vec{b}| = 3$ $\quad [\because |\vec{b}| > 0]$

$\therefore \quad |\vec{a}| = 2\,|\vec{b}| \Rightarrow |\vec{a}| = 2\times 3 = 6$

Thus, $\quad |\vec{a}| = 6$ and $|\vec{b}| = 3$

EXAMPLE 22 *If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 6$, Find $|\vec{a}+\vec{b}|$ and $|\vec{a}-\vec{b}|$.*

SOLUTION We know that

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2 \text{ and } |\vec{a}-\vec{b}|^2 = |\vec{a}|^2 - 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = 9+12+4 \text{ and } |\vec{a}-\vec{b}|^2 = 9-12+4$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = 25 \text{ and } |\vec{a}-\vec{b}|^2 = 1$$

$$\Rightarrow \quad |\vec{a}+\vec{b}| = 5 \text{ and } |\vec{a}-\vec{b}| = 1.$$

EXAMPLE 23 *Let $\vec{a}$ and $\vec{b}$ be two vectors of the same magnitude such that the angle between them is 60° and $\vec{a}\cdot\vec{b} = 8$. Find $|\vec{a}|$ and $|\vec{b}|$.* [NCERT]

SOLUTION We have,

$$\vec{a}\cdot\vec{b} = 8$$

$$\Rightarrow \quad |\vec{a}|\,|\vec{b}| \cos 60^\circ = 8 \qquad [\because \text{ Angle between } \vec{a} \text{ and } \vec{b} \text{ is } 60^\circ]$$

$$\Rightarrow \quad |\vec{a}|^2 \times \frac{1}{2} = 8 \qquad [\because |\vec{a}| = |\vec{b}|]$$

$$\Rightarrow \quad |\vec{a}|^2 = 16$$

$$\Rightarrow \quad |\vec{a}| = 4$$

Hence, $|\vec{a}| = |\vec{b}| = 4$.

EXAMPLE 24 *If $\vec{a}$ makes equal angles with $\hat{i}$, $\hat{j}$ and $\hat{k}$ and has magnitude 3, then prove that the angle between $\vec{a}$ and each of $\hat{i}$, $\hat{j}$ and $\hat{k}$ is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.*

SOLUTION We know that if a vector $\vec{a}$ makes angles α, β and γ with $\hat{i}, \hat{j}$ and $\hat{k}$ respectively, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Here, $\alpha = \beta = \gamma$

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \quad 3\cos^2\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Hence, $\vec{a}$ makes angle $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ with each of $\hat{i}, \hat{j}$ and $\hat{k}$.

EXAMPLE 25 *If a unit vector $\vec{a}$ makes angle $\pi/4$ with $\hat{i}$, $\pi/3$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the component of $\vec{a}$ and the angle θ.*

SOLUTION We know that if a vector $\vec{a}$ makes angles α, β and γ with $\hat{i}, \hat{j}$ and $\hat{k}$ respectively, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Here, we have

$$\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{3} \text{ and } \gamma = \theta \text{ an acute angle.}$$

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \quad \cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{3} + \cos^2\theta = 1$$

$$\Rightarrow \quad \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$$

$$\Rightarrow \quad \cos^2\theta = \frac{1}{4}$$

$$\Rightarrow \quad \cos\theta = \pm\frac{1}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \qquad [\because \theta \text{ is an acute angle } \therefore \cos\theta > 0]$$

$$\Rightarrow \quad \theta = \frac{\pi}{3} \Rightarrow \gamma = \frac{\pi}{3}$$

Now, $\vec{a} = |\vec{a}| \{(\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}\}$

$$\Rightarrow \quad \vec{a} = \left(\cos\frac{\pi}{4}\right)\hat{i} + \left(\cos\frac{\pi}{3}\right)\hat{j} + \left(\cos\frac{\pi}{3}\right)\hat{k}$$

$$\Rightarrow \quad \vec{a} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}.$$

x y z
1 2 -1
3 -1 2

Thus, the components of $\vec{a}$ are $\frac{1}{\sqrt{2}}\hat{i},\ \frac{1}{2}\hat{j},\ \frac{1}{2}\hat{k}$.

EXAMPLE 26 *Find the components of a unit vector which is perpendicular to the vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $3\hat{i} - \hat{j} + 2\hat{k}$.*

SOLUTION Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ be a unit vector perpendicular to the vectors $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$. Then,

$$\vec{a}\cdot\vec{b} = 0 \text{ and } \vec{a}\cdot\vec{c} = 0$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k})\cdot(\hat{i} + 2\hat{j} - \hat{k}) = 0 \text{ and, } (x\hat{i} + y\hat{j} + z\hat{k})\cdot(3\hat{i} - \hat{j} + 2\hat{k}) = 0$$

$$\Rightarrow \quad x + 2y - z = 0 \text{ and, } 3x - y + 2z = 0$$

$$\Rightarrow \quad \frac{x}{4-1} = \frac{y}{-3-2} = \frac{z}{-1-6} \qquad \text{[Using cross–multiplication]}$$

$$\Rightarrow \quad \frac{x}{3} = \frac{y}{-5} = \frac{z}{-7} = \lambda \text{ (say)}$$

$$\Rightarrow \quad x = 3\lambda, y = -5\lambda, z = -7\lambda$$

$\frac{y}{-5}$

$\frac{z}{-1-6} = \frac{z}{-7}$

$\boxed{x = 3\lambda}$
$\boxed{y = -5\lambda}$
$\boxed{z = -7\lambda}$

Since $\vec{a}$ is a unit vector. Therefore,

$$|\vec{a}| = 1$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = 1$$

$$\Rightarrow \quad 9\lambda^2 + 25\lambda^2 + 49\lambda^2 = 1 \Rightarrow 83\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{1}{\sqrt{83}}$$

$$\therefore \quad x = \pm\frac{3}{\sqrt{83}}, y = \mp\frac{5}{\sqrt{83}} \text{ and } z = \mp\frac{7}{\sqrt{83}}$$

Hence, the components of $\vec{a}$ are $\frac{3}{\sqrt{83}}\hat{i}, \frac{-5}{\sqrt{83}}\hat{j}, \frac{-7}{\sqrt{83}}\hat{k}$ or, $\frac{-3}{\sqrt{83}}\hat{i}, \frac{5}{\sqrt{83}}\hat{j}, \frac{7}{\sqrt{83}}\hat{k}$.

ALITER Required components are the components of vectors $\pm (\vec{a} \times \vec{b})$.

EXAMPLE 27 *The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of the vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1. Find the value of λ.* **[NCERT]**

SOLUTION Let $\vec{a} = \hat{i}+\hat{j}+\hat{k}, \vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}$. Then,

$$\vec{b}+\vec{c} = (2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

Let $\hat{r}$ denote the unit vector along $\vec{b}+\vec{c}$. Then,

$$\hat{r} = \frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}$$

$$\Rightarrow \quad \hat{r} = \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+36+4}}$$

$$\Rightarrow \quad \hat{r} = \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+40}}$$

Now, $\quad (\hat{i}+\hat{j}+\hat{k}) \cdot \hat{r} = 1 \qquad$ [Given]

$$\Rightarrow \quad (\hat{i}+\hat{j}+\hat{k}) \cdot \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+40}} = 1$$

$$\Rightarrow \quad (\hat{i}+\hat{j}+\hat{k}) \cdot \{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}\} = \sqrt{(2+\lambda)^2+40}$$

$$\Rightarrow \quad 2+\lambda+6-2 = \sqrt{(2+\lambda)^2+40}$$

$$\Rightarrow \quad (\lambda+6)^2 = (2+\lambda)^2+40 \Rightarrow 8\lambda = 8 \Rightarrow \lambda = 1.$$

EXAMPLE 28 *If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, then show that $\vec{a} = 0$ or, $\vec{b} = \vec{c}$ or, $\vec{a} \perp (\vec{b}-\vec{c})$.*

SOLUTION We have,

$$\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = 0$$

$$\Rightarrow \quad \vec{a}\cdot(\vec{b}-\vec{c}) = 0$$

$$\Rightarrow \quad \vec{a} = \vec{0} \text{ or, } \vec{b}-\vec{c} = \vec{0} \text{ or, } \vec{a} \perp (\vec{b}-\vec{c})$$

$$\Rightarrow \quad \vec{a} = \vec{0} \text{ or, } \vec{b} = \vec{c} \text{ or, } \vec{a} \perp (\vec{b}-\vec{c})$$

EXAMPLE 29 *If $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.* **[CBSE 2008]**

SOLUTION We have,

$$\vec{a}+\vec{b}+\vec{c} = \vec{0}$$

$$\Rightarrow \quad \vec{a}+\vec{b} = -\vec{c}$$

$$\Rightarrow \quad (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = (-\vec{c})\cdot(-\vec{c})$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{c}|^2$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{c}|^2$$

$$|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}|\,|\vec{b}|\cos\theta = |\vec{c}|^2$$

$$\Rightarrow \quad 9 + 25 + 2(3)(5)\cos\theta = 49$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

EXAMPLE 30 *If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$, and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.*

SOLUTION We have,

$$|\vec{a}| = \sqrt{25+1+9} = \sqrt{35} \text{ and } |\vec{b}| = \sqrt{1+9+25} = \sqrt{35}$$

Now, $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

$\Rightarrow \quad (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 35 - 35$

$\Rightarrow \quad (\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular to each other.

ALITER We have,

$$\vec{a} + \vec{b} = 6\hat{i} + 2\hat{j} - 8\hat{k} \text{ and } \vec{a} - \vec{b} = 4\hat{i} - 4\hat{j} + 2\hat{k}$$

$\therefore \quad (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (6\hat{i} + 2\hat{j} - 8\hat{k}) \cdot (4\hat{i} - 4\hat{j} + 2\hat{k})$

$\Rightarrow \quad (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 24 - 8 - 16 = 0$

$\Rightarrow \quad \vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular to each other.

EXAMPLE 31 *Two vectors $\vec{a}$ and $\vec{b}$, prove that the vector $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is orthogonal to the vector $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$.* **[NCERT]**

SOLUTION Let $\vec{\alpha} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ and $\vec{\beta} = |\vec{a}|\vec{b} - |\vec{b}|\vec{a}$. Then,

$$\vec{\alpha} \cdot \vec{\beta} = \{|\vec{a}|\vec{b} + |\vec{b}|\vec{a}\} \cdot \{|\vec{a}|\vec{b} - |\vec{b}|\vec{a}\}$$

$\Rightarrow \quad \vec{\alpha} \cdot \vec{\beta} = |\vec{a}|^2 (\vec{b} \cdot \vec{b}) - |\vec{a}|\,|\vec{b}|(\vec{b} \cdot \vec{a}) + |\vec{b}|\,|\vec{a}|(\vec{a} \cdot \vec{b}) - |\vec{b}|^2 (\vec{a} \cdot \vec{a})$

$\Rightarrow \quad \vec{\alpha} \cdot \vec{\beta} = |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|\,|\vec{b}|(\vec{a} \cdot \vec{b}) + |\vec{a}|\,|\vec{b}|(\vec{a} \cdot \vec{b}) - |\vec{b}|^2 |\vec{a}|^2$

$\Rightarrow \quad \vec{\alpha} \cdot \vec{\beta} = 0$

$\therefore \quad \vec{\alpha}$ is perpendicular (or orthogonal) to $\vec{\beta}$.

EXAMPLE 32 *Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$, $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the sides of a right angled triangle.* **[NCERT]**

SOLUTION Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}$.

Thus, $\vec{a} + \vec{b} = \vec{c}$. Thus, if $\vec{AC} = \vec{b}$, $\vec{CB} = \vec{a}$ and $\vec{AB} = \vec{c}$, then
$\vec{AC} + \vec{CB} = \vec{AB}$

Hence, vectors $\vec{a}, \vec{b}, \vec{c}$ form the sides of a triangle

We have,

$$\vec{a} \cdot \vec{b} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (\hat{i} - 3\hat{j} - 5\hat{k}) = 2 + 3 - 5 = 0$$

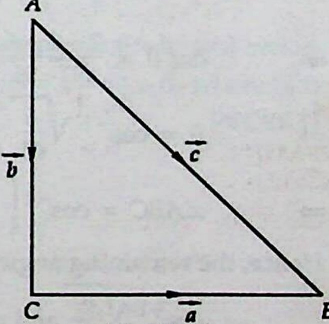


Fig. 24.14

Therefore, $\vec{AC}$ is perpendicular to $\vec{CB}$.

Hence, $\Delta\, ABC$ is a right angled triangle.

EXAMPLE 33 *Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right angled triangle. Also, find the remaining angles of the triangle.* **[NCERT]**

SOLUTION We have,

$$\vec{AB} = \text{Position vector of } B - \text{Position vector of } A$$

$$\Rightarrow \quad \vec{AB} = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\vec{BC} = \text{Position vector of } C - \text{Position vector of } B$$

$$\Rightarrow \quad \vec{BC} = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}$$

and, $\vec{CA}$ = Position vector of A – Position vector of C

$$\vec{CA} = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k}) = -\hat{i} + 3\hat{j} + 5\hat{k}$$

Clearly, $\vec{AB} + \vec{BC} + \vec{CA} = (-\hat{i} - 2\hat{j} - 6\hat{k}) + (2\hat{i} - \hat{j} + \hat{k}) + (-\hat{i} + 3\hat{j} + 5\hat{k}) = \vec{0}$

So, A, B and C are the vertices of a triangle.

Now, $\vec{BC} \cdot \vec{CA} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 3\hat{j} + 5\hat{k}) = -2 - 3 + 5 = 0$

$$\Rightarrow \quad \vec{BC} \perp \vec{CA} \Rightarrow \angle BCA = \frac{\pi}{2}$$

Hence, ΔABC is a right angled triangle.
Since $\angle BAC$ is the angle between the vectors $\vec{AB}$ and $\vec{AC}$. Therefore,

$$\cos A = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}|\,|\vec{AC}|} = \frac{(-\hat{i} - 2\hat{j} - 6\hat{k}) \cdot (\hat{i} - 3\hat{j} - 5\hat{k})}{\sqrt{(-1)^2 + (-2)^2 + (-6)^2}\ \sqrt{1^2 + (-3)^2 + (-5)^2}}$$

$$\Rightarrow \quad \cos A = \frac{-1 + 6 + 30}{\sqrt{1 + 4 + 36}\ \sqrt{1 + 9 + 25}} = \frac{35}{\sqrt{41}\ \sqrt{35}} = \sqrt{\frac{35}{41}}$$

$$\Rightarrow \quad A = \cos^{-1}\sqrt{\frac{35}{41}}$$

$$\Rightarrow \quad \angle BAC = \cos^{-1}\sqrt{\frac{35}{41}}$$

Since $\angle ABC$ is the angle between $\vec{BA}$ and $\vec{BC}$. Therefore,

$$\cos B = \frac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}|\,|\vec{BC}|} = \frac{(\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})}{\sqrt{1^2 + 2^2 + 6^2}\ \sqrt{2^2 + (-1)^2 + (1)^2}}$$

$$\Rightarrow \quad \cos B = \frac{2 - 2 + 6}{\sqrt{41}\ \sqrt{6}} = \sqrt{\frac{6}{41}}$$

$$\Rightarrow \quad B = \cos^{-1}\sqrt{\frac{6}{41}}$$

$$\Rightarrow \quad \angle ABC = \cos^{-1}\left(\sqrt{\frac{6}{41}}\right)$$

Hence, the remaining angles of the triangle are

$$\cos^{-1}\sqrt{\frac{35}{41}} \text{ and } \cos^{-1}\sqrt{\frac{6}{41}}$$

EXAMPLE 34 *Show that the angle between two diagonals of a cube is* $\cos^{-1}\left(\frac{1}{3}\right)$.

SOLUTION Let a be the length of an edge of the cube and let one corner be at the origin as shown in Fig. 24.15. Clearly, OP, AR, BS and CQ are the diagonals of the cube. Consider the diagonals OP and AR.

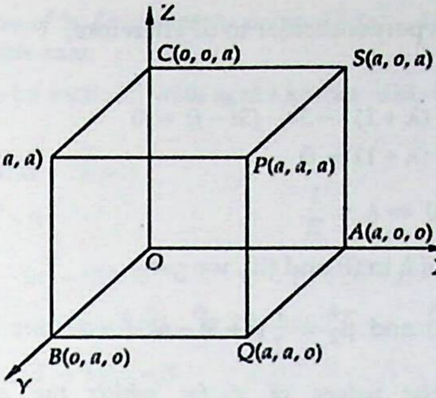


Fig. 24.15

We have, $\vec{OP} = a\hat{i} + a\hat{j} + a\hat{k}$

and, $\vec{AR}$ = Position vector of R – Position vector of A

$\Rightarrow \quad \vec{AR} = (0\hat{i} + a\hat{j} + a\hat{k}) - (a\hat{i} + 0\hat{j} + 0\hat{k}) = -a\hat{i} + a\hat{j} + a\hat{k}.$

Let θ be the angle between $\vec{OP}$ and $\vec{AR}$. Then,

$$\cos\theta = \frac{\vec{OP} \cdot \vec{AR}}{|\vec{OP}|\,|\vec{AR}|}$$

$$\Rightarrow \quad \cos\theta = \frac{(a\hat{i} + a\hat{j} + a\hat{k}) \cdot (-a\hat{i} + a\hat{j} + a\hat{k})}{\sqrt{a^2 + a^2 + a^2}\,\sqrt{(-a)^2 + a^2 + a^2}}$$

$$\Rightarrow \quad \cos\theta = \frac{-a^2 + a^2 + a^2}{\sqrt{3}a\,\sqrt{3}a} = \frac{1}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Similarly, angle θ between the other pairs of diagonals is $\cos^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 35 *If with reference to a right handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$, we have $\vec{\alpha} = 3\hat{i} - \hat{j}$, and $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$. Express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta_1} + \vec{\beta_2}$ where $\vec{\beta_1}$ is parallel to $\vec{\alpha}$ and $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$.* **[NCERT]**

SOLUTION Since $\vec{\beta_1}$ is parallel to $\vec{\alpha}$. Therefore,

$$\vec{\beta_1} = \lambda\vec{\alpha} \text{ for some scalar } \lambda$$

$$\Rightarrow \quad \vec{\beta_1} = \lambda(3\hat{i} - \hat{j}) \quad \text{...(i)}$$

It is given that

$$\vec{\beta} = \vec{\beta_1} + \vec{\beta_2}$$

$$\Rightarrow \quad \vec{\beta_2} = \vec{\beta} - \vec{\beta_1}$$

$$\Rightarrow \quad \vec{\beta_2} = (2\hat{i} + \hat{j} - 3\hat{k}) - \lambda(3\hat{i} - \hat{j})$$

$$\Rightarrow \quad \vec{\beta_2} = (-3\lambda + 2)\hat{i} + (\lambda + 1)\hat{j} - 3\hat{k} \quad \text{...(ii)}$$

It is also given that $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$. Therefore,

$$\vec{\beta_2} \cdot \vec{\alpha} = 0$$

$$\Rightarrow \quad \{(-3\lambda+2)\hat{i}+(\lambda+1)\hat{j}-3\hat{k}\} \cdot (3\hat{i}-\hat{j}) = 0$$

$$\Rightarrow \quad 3(-3\lambda+2)-(\lambda+1) = 0$$

$$\Rightarrow \quad -10\lambda+5 = 0 \Rightarrow \lambda = \frac{1}{2}$$

Substituting the value of λ in (i) and (ii), we get

$$\vec{\beta_1} = \frac{1}{2}(3\hat{i}-\hat{j}) \text{ and } \vec{\beta_2} = \frac{1}{2}(\hat{i}+3\hat{j}-6\hat{k}).$$

EXAMPLE 36 *Find the values of x for which the angle between the vectors $\vec{a} = 2x^2\hat{i}+4x\hat{j}+\hat{k}$ and $\vec{b} = 7\hat{i}-2\hat{j}+x\hat{k}$ is obtuse.*

SOLUTION The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

For the angle θ to be obtuse, we must have

$$\cos\theta < 0$$

$$\Rightarrow \quad \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} < 0$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} < 0 \qquad [\because |\vec{a}|, |\vec{b}| > 0]$$

$$\Rightarrow \quad 14x^2-8x+x<0 \Rightarrow 7x(2x-1)<0 \Rightarrow x(2x-1)<0 \Rightarrow 0<x<\frac{1}{2}$$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$.

EXAMPLE 37 *Find the values of c for which the vectors $\vec{a} = (c\log_2 x)\hat{i}-6\hat{j}+3\hat{k}$ and $\vec{b} = (\log_2 x)\hat{i}+2\hat{j}+(2c\log_2 x)\hat{k}$ make an obtuse angle for any $x \in (0, \infty)$.*

SOLUTION Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

For θ to be an obtuse angle, we must have

$$\Rightarrow \quad \cos\theta < 0 \quad \text{for all } x \in (0, \infty)$$

$$\Rightarrow \quad \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} < 0 \text{ for all } x \in (0, \infty)$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} < 0 \quad \text{for all } x \in (0, \infty)$$

$$\Rightarrow \quad c(\log_2 x)^2 - 12 + 6c(\log_2 x) < 0 \text{ for all } x \in (0, \infty)$$

$$\Rightarrow \quad cy^2+6cy-12<0 \text{ for all } y \in R, \text{ where } y = \log_2 x \qquad [\because x>0 \Rightarrow y = \log_2 x \in R]$$

$$\Rightarrow \quad c<0 \text{ and } 36c^2+48c<0 \qquad \left[\because ax^2+bx+c<0 \text{ for all } x \Rightarrow a<0 \text{ and Disc} <0\right]$$

$$\Rightarrow \quad c<0 \text{ and } c(3c+4)<0$$

$$\Rightarrow \quad c<0 \text{ and } -\frac{4}{3}<c<0 \Rightarrow c \in (-4/3, 0)$$

EXAMPLE 38 *Find the values of 'a' for which the vector* $\vec{r} = (a^2-4)\,\hat{i} + 2\hat{j} - (a^2-9)\,\hat{k}$ *makes acute angles with the coordinate axes.*

SOLUTION For vector $\vec{r}$ to be inclined with acute angles with the coordinate axes, we must have

$$\vec{r}\cdot\hat{i} > 0,\ \vec{r}\cdot\hat{j} > 0 \text{ and } \vec{r}\cdot\hat{k} > 0$$

$\Rightarrow \quad \vec{r}\cdot\hat{i} > 0$ and $\vec{r}\cdot\hat{k} > 0 \qquad [\because \vec{r}\cdot\hat{j} = 2 > 0]$

$\Rightarrow \quad (a^2-4) > 0$ and $-(a^2-9) > 0 \qquad [\because \vec{r}\cdot\hat{i} = a^2-4 \text{ and } \vec{r}\cdot\hat{k} = -(a^2-9)]$

$\Rightarrow \quad (a-2)(a+2) > 0$ and $(a+3)(a-3) < 0$

$\Rightarrow \quad a < -2$ or $a > 2$ and $-3 < a < 3$

$\Rightarrow \quad a \in (-3, -2) \cup (2, 3).$

EXAMPLE 39 *If* $\vec{a}, \vec{b}$ *are two vectors such that* $|\vec{a}+\vec{b}| = |\vec{a}|$, *then prove that* $2\vec{a}+\vec{b}$ *is perpendicular to* $\vec{b}$.

SOLUTION We have,

$$|\vec{a}+\vec{b}| = |\vec{a}|$$

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{a}|^2$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\cdot\vec{b}) = |\vec{a}|^2$

$\Rightarrow \quad |\vec{b}|^2 + 2(\vec{a}\cdot\vec{b}) = 0 \qquad \text{...(i)}$

Now, $\quad (2\vec{a}+\vec{b})\cdot\vec{b} = 2(\vec{a}\cdot\vec{b}) + (\vec{b}\cdot\vec{b}) = 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2 = 0 \qquad$ [Using (i)]

Hence, $\quad (2\vec{a}+\vec{b})$ is perpendicular to $\vec{b}$.

EXAMPLE 40 *For any three vectors* $\vec{a}, \vec{b}, \vec{c}$, *prove that*

$$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

SOLUTION We know that

$$|\vec{r}|^2 = \vec{r}\cdot\vec{r}$$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = \vec{a}\cdot(\vec{a}+\vec{b}+\vec{c}) + \vec{b}\cdot(\vec{a}+\vec{b}+\vec{c}) + \vec{c}\cdot(\vec{a}+\vec{b}+\vec{c})$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} + \vec{c}\cdot\vec{c}$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c} + 2(\vec{a}\cdot\vec{b}) + 2(\vec{b}\cdot\vec{c}) + 2(\vec{c}\cdot\vec{a})$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$

EXAMPLE 41 *If* $\vec{a}, \vec{b}, \vec{c}$ *are three mutually perpendicular vectors of equal magnitude, prove that* $\vec{a}+\vec{b}+\vec{c}$ *is equally inclined with vectors* $\vec{a}, \vec{b}$ *and* $\vec{c}$. **[NCERT, CBSE 2005]**

SOLUTION Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say).

Since $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors. Therefore,

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0 \qquad \text{...(i)}$$

Now,

$$|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+2\,\vec{a}\cdot\vec{b}+2\,\vec{b}\cdot\vec{c}+2\,\vec{c}\cdot\vec{a}$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2\,|+|\vec{b}|^2+|\vec{c}|^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = 3\lambda^2 \qquad [\because\ |\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda]$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\,\lambda \qquad \text{...(ii)}$$

Suppose $\vec{a}+\vec{b}+\vec{c}$ makes angles $\theta_1, \theta_2, \theta_3$ with $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then,

$$\cos\theta_1 = \frac{\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}|\,|\vec{a}+\vec{b}+\vec{c}|} = \frac{\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}}{|\vec{a}|\,|\vec{a}+\vec{b}+\vec{c}|}$$

$$\Rightarrow \quad \cos\theta_1 = \frac{|\vec{a}|^2}{|\vec{a}|\,|\vec{a}+\vec{b}+\vec{c}|} = \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} = \frac{\lambda}{\sqrt{3}\,\lambda} = \frac{1}{\sqrt{3}} \qquad \text{[Using (ii)]}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Similarly, we have

$$\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \text{ and } \theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

$$\therefore \quad \theta_1 = \theta_2 = \theta_3.$$

Hence, $\vec{a}+\vec{b}+\vec{c}$ is equally inclinded with $\vec{a}, \vec{b}$ and $\vec{c}$.

EXAMPLE 42 *Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors of magnitudes 3, 4 and 5 respectively. If each one is perpendicular to the sum of the other two vectors, prove that $|\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$.*

[NCERT, CBSE 2010]

SOLUTION We have, $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$

It is given that

$$\vec{a}\perp(\vec{b}+\vec{c}),\ \vec{b}\perp(\vec{c}+\vec{a}) \text{ and } \vec{c}\perp(\vec{a}+\vec{b})$$

$$\Rightarrow \quad \vec{a}\cdot(\vec{b}+\vec{c}) = 0,\ \vec{b}\cdot(\vec{c}+\vec{a}) = 0 \text{ and } \vec{c}\cdot(\vec{a}+\vec{b}) = 0$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} = 0,\ \vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{a} = 0 \text{ and } \vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b} = 0$$

Adding all these, we get

$$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0 \Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = 0 \qquad \text{...(i)}$$

Now, $$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = 3^2+4^2+5^2+0 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}.$$

EXAMPLE 43 *If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.*

[NCERT]

SOLUTION We have,

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1 \text{ and } \vec{a}+\vec{b}+\vec{c} = \vec{0}$$

Now, $\quad \vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}| = 0$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = 0$

$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$

$\Rightarrow \quad 3+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$

$\Rightarrow \quad \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = -\frac{3}{2}$

EXAMPLE 44 *Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu = \vec{a}\cdot\vec{b}.\vec{b}+\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}|=1$, $|\vec{b}|=4$ and $|\vec{c}|=2$.* **[NCERT]**

SOLUTION We have,

$|\vec{a}| = 1,\ |\vec{b}| = 4$ and $|\vec{c}| = 2$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}| = 0$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = 0$

$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$

$\Rightarrow \quad 1+16+4+2\mu = 0$

$\Rightarrow \quad 2\mu = -21 \Rightarrow \mu = -\frac{21}{2}$

EXERCISE 24.1

1. Find $\vec{a}\cdot\vec{b}$, when
   (i) $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b}=4\hat{i}-4\hat{j}+7\hat{k}$
   (ii) $\vec{a}=\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{k}$
   (iii) $\vec{a}=\hat{j}-\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}-2\hat{k}$
2. For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other? where:
   (i) $\vec{a}=\lambda\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=4\hat{i}-9\hat{j}+2\hat{k}$
   (ii) $\vec{a}=\lambda\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=5\hat{i}-9\hat{j}+2\hat{k}$
   (iii) $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{b}=3\hat{i}+2\hat{j}-\lambda\hat{k}$
   (iv) $\vec{a}=\lambda\hat{i}+3\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+3\hat{k}$
3. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}|=4$, $|\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=6$. Find the angle between $\vec{a}$ and $\vec{b}$.
4. If $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=-\hat{j}+2\hat{k}$, find $(\vec{a}-2\vec{b})\cdot(\vec{a}+\vec{b})$.
5. Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where :
   (i) $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=\hat{j}+\hat{k}$
   (ii) $\vec{a}=3\hat{i}-2\hat{j}-6\hat{k}$ and $\vec{b}=4\hat{i}-\hat{j}+8\hat{k}$
   (iii) $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=4\hat{i}+4\hat{j}-2\hat{k}$

(iv) $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$

(v) $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

6. Find the angles which the vector $\vec{a} = \hat{i} - \hat{j} + \sqrt{2}\,\hat{k}$ makes with the coordinate axes.
7. Dot product of a vector with $\hat{i} + \hat{j} - 3\hat{k}$, $\hat{i} + 3\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} + 4\hat{k}$ are 0, 5 and 8 respectively. Find the vector. **[CBSE 2003]**
8. If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that

   (i) $\cos\frac{\theta}{2} = \frac{1}{2}\,|\hat{a} + \hat{b}|$ (ii) $\tan\frac{\theta}{2} = \frac{|\hat{a} - \hat{b}|}{|\hat{a} + \hat{b}|}$
9. If the sum of two unit vectors is a unit vector prove that the magnitude of their difference is $\sqrt{3}$.
10. If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular unit vectors, then prove that $|\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}$.
11. If $|\vec{a} + \vec{b}| = 60$, $|\vec{a} - \vec{b}| = 40$ and $|\vec{b}| = 46$, find $|\vec{a}|$
12. Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined with the coordinate axes.
13. Show that the vectors $\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$, $\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$, $\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$ are mutually perpendicular unit vectors.
14. Show that $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0 \Leftrightarrow |\vec{a}| = |\vec{b}|$.
15. If $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, then prove that it is perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.
16. If $|\vec{a}| = a$ and $|\vec{b}| = b$, prove that $\left(\frac{\vec{a}}{a^2} - \frac{\vec{b}}{b^2}\right)^2 = \left(\frac{\vec{a} - \vec{b}}{ab}\right)^2$.
17. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{d} \cdot \vec{a} = \vec{d} \cdot \vec{b} = \vec{d} \cdot \vec{c} = 0$, then show that $\vec{d}$ is the null vector.
18. If a vector $\vec{a}$ is perpendicular to two non-collinear vectors $\vec{b}$ and $\vec{c}$, then $\vec{a}$ is perpendicular to every vector in the plane of $\vec{b}$ and $\vec{c}$.
19. Show that the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$, $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angled triangle. **[CBSE 2005]**
20. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, show that the angle θ between the vectors $\vec{b}$ and $\vec{c}$ is given by

    $$\cos\theta = \frac{|\vec{a}|^2 - |\vec{b}|^2 - |\vec{c}|^2}{2\,|\vec{b}|\,|\vec{c}|}.$$
21. Find the angles of a triangle whose vertices are A (0, –1, –2), B (3, 1, 4) and C (5, 7, 1).
22. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vector such $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. If $|\vec{u}| = 3$, $|\vec{v}| = 4$ and $|\vec{w}| = 5$, then find $\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}$.
23. Show that the points whose position vectors are $\vec{a} = 4\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$, $\vec{c} = \hat{i} - \hat{j}$ form a right triangle.
24. If the vertices A, B, C of ΔABC have position vectors (1, 2, 3), (–1, 0, 0), (0, 1, 2) respectively, what is the magnitude of $\angle ABC$?

25. If A, B, C have position vectors (0, 1, 1,), (3, 1, 5), (0, 3, 3) respectively, show that ΔABC is right angled at C.

26. Let $\vec{a} = x^2\hat{i} + 2\hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = x^2\hat{i} + 5\hat{j} - 4\hat{k}$ be three vectors. Find the values of x for which the angle between $\vec{a}$ and $\vec{b}$ is acute and the angle between $\vec{b}$ and $\vec{c}$ is obtuse.

27. Find the values of x and y if the vectors $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpendicular vectors of equal magnitude.

28. If $\vec{a}$ and $\vec{b}$ are two non-coplinear unit vectors such that $|\vec{a} + \vec{b}| = \sqrt{3}$, find $(2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b})$.

29. Evaluate: $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$ **[NCERT]**

30. If $\vec{a}$ is a unit vector, then find $|\vec{x}|$ in each of the following:

(i) $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$ **[NCERT]**

(ii) $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$ **[NCERT]**

31. Find $|\vec{a}|$ and $|\vec{b}|$, if

(i) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 12$ and $|\vec{a}| = 2|\vec{b}|$

(ii) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$ **[NCERT]**

(iii) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 3$ and $|\vec{a}| = 2|\vec{b}|$

32. Find $|\vec{a} - \vec{b}|$, if

(i) $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $\vec{a} \cdot \vec{b} = 8$

(ii) $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\vec{a} \cdot \vec{b} = 1$

(iii) $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 4$ **[NCERT]**

33. Find the angle between two vectors $\vec{a}$ and $\vec{b}$, if

(i) $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$ **[NCERT]**

(ii) $|\vec{a}| = 3$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 1$

34. Express the vector $\vec{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as the sum of two vectors such that one is parallel to the vector $\vec{b} = 3\hat{i} + \hat{k}$ and other is perpendicular to $\vec{b}$. **[CBSE 2005]**

35. If $\vec{a}$ and $\vec{b}$ are two vectors of the same magnitude inclined at an angle of 30° such that $\vec{a} \cdot \vec{b} = 3$, find $|\vec{a}|$, $|\vec{b}|$.

36. Express $2\hat{i} - \hat{j} + 3\hat{k}$ as the sum of a vector parallel and a vector perpendicular to $2i + 4\hat{j} - 2\hat{k}$.

37. Decompose the vector $6i - 3\hat{j} - 6\hat{k}$ into vectors which are parallel and perpendicular to the vector $\hat{i} + \hat{j} + \hat{k}$.

38. Let $\vec{a} = 5\hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \lambda\hat{k}$. Find λ such that $\vec{a} + \vec{b}$ is orthogonal to $\vec{a} - \vec{b}$.

39. If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, what can you conclude about the vector $\vec{b}$? **[NCERT, CBSE 2004]**

40. A unit vector $\vec{a}$ makes angles $\frac{\pi}{2}$ and $\frac{\pi}{3}$ with $\hat{i}$ and $\hat{j}$ respectively and an acute angle θ with $\hat{k}$. Find the angle θ and components of $\vec{a}$. **[NCERT]**

41. If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal. [CBSE 2004]

42. Find the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. [CBSE 2007]

43. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is 1/2. [NCERT]

44. If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ. [NCERT]

45. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \cdot \vec{b} = 0$. But the converse need not be true. Justify your answer with an example. [NCERT]

ANSWERS

1. (i) 19 (ii) 2 (iii) 5

2. (i) $\lambda = 4$ (ii) $\lambda = \frac{16}{5}$ (iii) 3 (iv) -3

3. $\frac{\pi}{3}$

4. -9

5. (i) $\frac{2\pi}{3}$ (ii) $\cos^{-1}\left(\frac{-34}{63}\right)$ (iii) $\frac{\pi}{2}$ (iv) $\cos^{-1}\left(\frac{-3}{\sqrt{84}}\right)$ (v) $\cos^{-1}\left(-\frac{\sqrt{2}}{3}\right)$

6. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}$

7. $\hat{i} + 2\hat{j} + \hat{k}$

11. 22

21. $\angle A = \frac{\pi}{4}$, $\angle B = \frac{\pi}{2}$, $\angle C = \frac{\pi}{4}$

22. -25

24. $\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$

26. $(-3, -2) \cup (2, 3)$

27. $x = -\frac{31}{12}, y = \frac{41}{12}$

28. $-\frac{11}{2}$

29. $6\,|\vec{a}|^2 + 11\,\vec{a} \cdot \vec{b} - 35\,|\vec{b}|^2$

30. (i) 3 (ii) $\sqrt{13}$

31. (i) $|\vec{a}| = 4, |\vec{b}| = 2$ (ii) $|\vec{a}| = \frac{8\sqrt{8}}{63}, |\vec{b}| = \sqrt{\frac{8}{63}}$ (iii) $|\vec{a}| = 2, |\vec{b}| = 1$

32. (i) $\sqrt{13}$, (ii) $\sqrt{23}$, (iii) $\sqrt{5}$

33. (i) $\frac{\pi}{4}$ (ii) $\cos^{-1}\left(\frac{1}{9}\right)$

34. $6\hat{i} + 2\hat{k}, -\hat{i} - 2\hat{j} + 3\hat{k}$

35. $|\vec{a}| = |\vec{b}| = \sqrt{2\sqrt{3}}$

36. $\left(\frac{-\hat{i}}{2} - \hat{j} + \frac{\hat{k}}{2}\right) + \frac{5}{2}(\hat{i} + \hat{k})$

37. $-\hat{i} - \hat{j} - \hat{k}, 7\hat{i} - 2\hat{j} - 5\hat{k}$

38. $\sqrt{73}$

39. $\vec{b}$ is any vector

40. $\frac{\pi}{3}, \frac{1}{2}\hat{i}, \frac{1}{\sqrt{2}}\hat{j}, \frac{1}{2}\hat{k}$

43. $|\vec{a}| = |\vec{b}| = 1$

44. 8

HINTS TO SELECTED PROBLEMS

6. Find the angles between $\vec{a}$ and $\hat{i}$; between $\vec{a}$ and $\hat{j}$ and between $\vec{a}$ and $\hat{k}$

9. We have, $|\hat{a}| = 1$, $|\hat{b}| = 1$, $|\hat{a} + \hat{b}| = 1$

Now,

$|\hat{a} + \hat{b}|^2 + |\hat{a} - \hat{b}|^2 = 2(|\hat{a}|^2 + |\hat{b}|^2$

$\Rightarrow \quad 1 + |\hat{a} - \hat{b}|^2 = 2(1+1)$

$\Rightarrow |\hat{a} - \hat{b}|^2 = 3 \Rightarrow |\hat{a} - \hat{b}| = \sqrt{3}$

10. Use $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$

11. Use $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$

13. Show that $\vec{a}\cdot\vec{b} = 0 = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a}$ and $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

15. We have,

$\vec{c}\cdot(\vec{a} + \vec{b}) = \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} = 0 + 0 = 0 \qquad [\because \vec{c} \perp \vec{a} \text{ and } \vec{c} \perp \vec{b}]$

$\Rightarrow \vec{c} \perp (\vec{a} + \vec{b})$

Similarly, $\vec{c}$ is perpendicular to $(\vec{a} - \vec{b})$.

18. Let $\vec{r}$ be an arbitrary vector in the plane of $\vec{b}$ and $\vec{c}$. Then,

$\vec{r} = x\vec{b} + y\vec{c}$ for some scalars x, y.

$\Rightarrow \vec{r}\cdot\vec{a} = (x\vec{b} + y\vec{c})\cdot\vec{a} = x(\vec{b}\cdot\vec{a}) + y(\vec{c}\cdot\vec{a}) = x(0) + y(0) = 0 \quad [\because \vec{a} \perp \vec{b} \text{ and } \vec{a} \perp \vec{c}]$

$\Rightarrow \vec{r} \perp \vec{a}$

20. We have,

$\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{b} + \vec{c} = -\vec{a}$

$\Rightarrow (\vec{b} + \vec{c})\cdot(\vec{b} + \vec{c}) = (-\vec{a})\cdot(-\vec{a})$

$\Rightarrow |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos\theta = |\vec{a}|^2 \Rightarrow \cos\theta = \dfrac{|\vec{a}|^2 - |\vec{b}|^2 - |\vec{c}|^2}{2|\vec{b}||\vec{c}|}$

21. Use: $\cos A = \dfrac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}||\vec{AC}|}$, $\cos B = \dfrac{\vec{BC}\cdot\vec{BA}}{|\vec{BC}||\vec{BA}|}$

and, $\cos C = \dfrac{\vec{CB}\cdot\vec{CA}}{|\vec{CB}||\vec{CA}|}$

22. Use, $|\vec{u} + \vec{v} + \vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2(\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u})$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. What is the angle between vectors $\vec{a}$ and $\vec{b}$ with magnitudes 2 and $\sqrt{3}$ respectively? Given $\vec{a}\cdot\vec{b} = \sqrt{3}$. **[CBSE 2008]**
2. If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a}\cdot\vec{b} = 6$, $|\vec{a}| = 3$ and $|\vec{b}| = 4$. Write the projection of $\vec{a}$ on $\vec{b}$.
3. Find the cosine of the angle between the vectors $4\hat{i} - 3\hat{j} + 3\hat{k}$ and $2\hat{i} - \hat{j} - \hat{k}$.
4. If the vectors $3\hat{i} + m\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} - 8\hat{k}$ are orthogonal, find m.
5. If the vectors $3\hat{i} - 2\hat{j} - 4\hat{k}$ and $18\hat{i} - 12\hat{j} - m\hat{k}$ are parallel, find the value of m.
6. If $\vec{a}$ and $\vec{b}$ are vectors of equal magnitude, write the value of $(\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b})$.
7. If $\vec{a}$ and $\vec{b}$ are two vectors such that $(\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b}) = 0$, find the relation between the magnitudes of $\vec{a}$ and $\vec{b}$.
8. For any two vector $\vec{a}$ and $\vec{b}$ write when $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$ holds.

9. For any two vectors $\vec{a}$ and $\vec{b}$ write when $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ holds.
10. If $\vec{a}$ and $\vec{b}$ are two vectors of the same magnitude inclined at an angle of 60° such that $\vec{a}\cdot\vec{b}=8$, write the value of their magnitude.
11. If $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$, what can you conclude about the vector $\vec{b}$?
12. If $\vec{b}$ is a unit vector such that $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$, find $|\vec{a}|$.
13. If $\vec{a}, \vec{b}$ are unit vectors such that $\hat{a}+\hat{b}$ is a unit vector, write the value of $|\hat{a}-\hat{b}|$.
14. If $|\vec{a}|=2$, $|\vec{b}|=5$ and $\vec{a}\cdot\vec{b}=2$, find $|\vec{a}-\vec{b}|$.
15. If $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=-\hat{j}+\hat{k}$, find the projection of $\vec{a}$ on $\vec{b}$.
16. For any two non-zero vectors, write the value of $\dfrac{|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2}{|\vec{a}|^2+|\vec{b}|^2}$.
17. Write the projections of $\vec{r}=3\hat{i}-4\hat{j}+12\hat{k}$ on the coordinate axes.
18. Write the component of $\vec{b}$ along $\vec{a}$.
19. Write the value of $(\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\cdot\hat{k})\hat{k}$.
20. Find the value of $\theta \in (0, \pi/2)$ for which vectors $\vec{a}=(\sin\theta)\hat{i}+(\cos\theta)\hat{j}$ and $\vec{b}=\hat{i}-\sqrt{3}\hat{j}+2\hat{k}$ are perpendicular.
21. Write the projection of $\hat{i}+\hat{j}+\hat{k}$ along the vector $\hat{j}$.
22. Write a vector satisfying $\vec{a}\cdot\hat{i}=\vec{a}\cdot(\hat{i}+\hat{j})=\vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})=1$.
23. If $\vec{a}$ and $\vec{b}$ are unit vectors, find the angle between $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.
24. If $\vec{a}$ and $\vec{b}$ are mutually perpendicular unit vectors, write the value of $|\vec{a}+\vec{b}|$.
25. If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors, write the value of $|\vec{a}+\vec{b}+\vec{c}|$.
26. Find the angle between the vectors $\vec{a}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}-\hat{k}$. **[CBSE 2008]**
27. For what value of λ are the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ perpendicular to each other? **[CBSE 2008]**
28. Find the projection of $\vec{a}$ on $\vec{b}$ if $\vec{a}\cdot\vec{b}=8$ and $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$ **[CBSE 2009]**
29. Write the value of p for which $\vec{a}=3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b}=\hat{i}+p\hat{j}+3\hat{k}$ are parallel vectors. **[CBSE 2009]**
30. Find the value of λ if the vectors $2\hat{i}+\lambda\hat{j}+3\hat{k}$ and $3\hat{i}+2\hat{j}-4\hat{k}$ are perpendicular to each other. **[CBSE 2010]**
31. If $|\vec{a}|=2$, $|\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=3$, find the projection of $\vec{b}$ on $\vec{a}$. **[CBSE 2010]**

ANSWERS

1. $\frac{\pi}{3}$ 2. 3 3. $\frac{4}{\sqrt{51}}$ 4. -2 5. -24

6. 0 7. $|\vec{a}|=|\vec{b}|$ 8. $\vec{a}$ and $\vec{b}$ are parallel

9. $\vec{a}$ and $\vec{b}$ are perpendicular 10. 4 11. $\vec{b}$ is any non-zero vector

12. 3 13. $\sqrt{3}$ 14. 5 15. $\frac{1}{\sqrt{2}}$ 16. 2 17. $3, -4, 12$

18. $\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|^2}\vec{a}$ 19. $\vec{a}$ 20. $\theta=\frac{\pi}{3}$ 21. 1 22. $\hat{i}$ 23. $\frac{\pi}{2}$ 24. $\sqrt{2}$ 25. $\sqrt{3}$

26. $\cos^{-1}\left(\frac{-1}{3}\right)$ 27. $\frac{5}{2}$ 28. $\frac{8}{7}$ 29. $\frac{2}{3}$ 30. 3 31. $3/2$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The vector $\vec{a}$ and $\vec{b}$ satisfy the equation $2\vec{a}+\vec{b}=\vec{p}$ and $\vec{a}+2\vec{b}=\vec{q}$, where $\vec{p}=\hat{i}+\hat{j}$ and $\vec{q}=\hat{i}-\hat{j}$. If θ is the angle between $\vec{a}$ and $\vec{b}$, then
(a) $\cos\theta=\frac{4}{5}$ (b) $\sin\theta=\frac{1}{\sqrt{2}}$ (c) $\cos\theta=-\frac{4}{5}$ (d) $\cos\theta=-\frac{3}{5}$

2. If $\vec{a}\cdot\hat{i}=\vec{a}\cdot(\hat{i}+\hat{j})=\vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})=1$, then $\vec{a}=$
(a) $\vec{0}$ (b) $\hat{i}$ (c) $\hat{j}$ (d) $\hat{i}+\hat{j}+\hat{k}$

3. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, $|\vec{a}|=3$, $|\vec{b}|=5$, $|\vec{c}|=7$, then the angle between $\vec{a}$ and $\vec{b}$ is
(a) $\frac{\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $\frac{5\pi}{3}$ (d) $\frac{\pi}{3}$

4. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and α be the angle between them, then $\vec{a}+\vec{b}$ is a unit vector, if
(a) $\alpha=\frac{\pi}{4}$ (b) $\alpha=\frac{\pi}{3}$ (c) $\alpha=\frac{2\pi}{3}$ (d) $\alpha=\frac{\pi}{2}$

5. The vector $\cos\alpha\cos\beta\hat{i}+\cos\alpha\sin\beta\hat{j}+\sin\alpha\hat{k}$ is a
(a) null vector (b) unit vector
(c) constant vector (d) none of these

6. If the position vectors of P and Q are $\hat{i}+3\hat{j}-7\hat{k}$ and $5\hat{i}-2\hat{j}+4\hat{k}$ then the cosine of the angle between $\vec{PQ}$ and y-axis is
(a) $\frac{5}{\sqrt{162}}$ (b) $\frac{4}{\sqrt{162}}$ (c) $-\frac{5}{\sqrt{162}}$ (d) $\frac{11}{\sqrt{162}}$

7. If $\vec{a}$ and $\vec{b}$ are unit vectors, then which of the following values of $\vec{a}\cdot\vec{b}$ is not possible?
(a) $\sqrt{3}$ (b) $\sqrt{3}/2$ (c) $1/\sqrt{2}$ (d) $-1/2$

8. If the vectors $\hat{i}-2x\hat{j}+3y\hat{k}$ and $\hat{i}+2x\hat{j}-3y\hat{k}$ are perpendicular, then the locus of (x, y) is
(a) a circle (b) an ellipse (c) a hyperbola (d) none of these

9. The vector component of $\vec{b}$ perpendicular to $\vec{a}$ is
(a) $(\vec{b}\cdot\vec{c})\vec{a}$ (b) $\frac{\vec{a}\times(\vec{b}\times\vec{a})}{|\vec{a}|^2}$ (c) $\vec{a}\times(\vec{b}\times\vec{a})$ (d) none of these

10. The length of the longer diagonal of the parallelogram constructed on $5\vec{a}+2\vec{b}$ and $\vec{a}-3\vec{b}$ if it is given that $|\vec{a}|=2\sqrt{2}$, $|\vec{b}|=3$ and angle between $\vec{a}$ and $\vec{b}$ is $\pi/4$, is
(a) 15 (b) $\sqrt{113}$ (c) $\sqrt{593}$ (d) $\sqrt{369}$

11. In the rectangular cartesian axes A is (x_1, y_1) where $x_1=1$ on the curve $y=x^2+x+10$. The tangent at A cuts the x-axis at B. The value of $\vec{OA}\cdot\vec{AB}$ is
(a) $-\frac{520}{3}$ (b) -148 (c) 140 (d) 12

12. The vectors $2\hat{i}-m\hat{j}+3m\hat{k}$ and $(1+m)\hat{i}-2m\hat{j}+\hat{k}$ include an acute angle for
(a) $m=-1/2$ (b) $m\in[-2,-1/2]$
(c) $m\in R$ (d) $m\in(-\infty,-2)\cup(-1/2,\infty)$

13. The values of x for which the angle between $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$, $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse and the angle between $\vec{b}$ and the z-axis is acute and less than $\frac{\pi}{6}$ are

(a) $x > \frac{1}{2}$ or $x < 0$ (b) $0 < x < \frac{1}{2}$ (c) $\frac{1}{2} < x < 15$ (d) ϕ

14. If $\vec{a}, \vec{b}, \vec{c}$ are any three mutually perpendicular vectors of equal magnitude a, then $|\vec{a} + \vec{b} + \vec{c}|$ is equal to

(a) a (b) $\sqrt{2}\,a$ (c) $\sqrt{3}\,a$ (d) $2a$ (e) none of these

15. If the vectors $3\hat{i} + \lambda\hat{j} + \hat{k}$ and $2i - \hat{j} + 8\hat{k}$ are perpendicular, then λ is equal to

(a) -14 (b) 7 (c) 14 (d) $\frac{1}{7}$

16. The projection of the vector $\hat{i} + \hat{j} + \hat{k}$ along the vector of $\hat{j}$ is

(a) 1 (b) 0 (c) 2 (d) -1 (e) -2

17. The vectors $2\hat{i} + 3\hat{j} - 4\hat{k}$ and $a\hat{i} + b\hat{j} + c\hat{k}$ are perpendicular, if

(a) $a = 2, b = 3, c = -4$ (b) $a = 4, b = 4, c = 5$

(c) $a = 4, b = 4, c = -5$ (d) $a = -4, b = 4, c = -5$

18. If $|\vec{a}| = |\vec{b}|$, then $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) =$

(a) positive (b) negative (c) 0 (d) none of these

19. If $\vec{a}$ and $\vec{b}$ are unit vectors inclined at an angle θ, then the value of $|\vec{a} - \vec{b}|$ is

(a) $2\sin\frac{\theta}{2}$ (b) $2\sin\theta$ (c) $2\cos\frac{\theta}{2}$ (d) $2\cos$

20. If $\vec{a}$ and $\vec{b}$ are unit vectors, then the greatest value of $\sqrt{3}\,|\vec{a} + \vec{b}| + |\vec{a} - \vec{b}|$ is

(a) 2 (b) $2\sqrt{2}$ (c) 4 (d) none of these

21. If the angle between the vectors $x\hat{i} + 3\hat{j} - 7\hat{k}$ and $x\hat{i} - x\hat{j} + 4\hat{k}$ is acute, then x lies in the interval

(a) $(-4, 7)$ (b) $[-4, 7]$ (c) $R - [-4, 7]$ (d) $R - (4, 7)$

22. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle θ such that $|\vec{a} + \vec{b}| < 1$, then

(a) $\theta < \frac{\pi}{3}$ (b) $\theta > \frac{2\pi}{3}$ (c) $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$ (d) $\frac{2\pi}{3} < \theta < \pi$

23. Let $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $|\vec{a} + \vec{b} + \vec{c}| = 1$ and $\vec{a}$ is perpendicular to $\vec{b}$. If $\vec{c}$ makes angle α and β with $\vec{a}$ and $\vec{b}$ respectively, then $\cos\alpha + \cos\beta =$

(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$ (c) 1 (d) -1

24. The orthogonal projection of $\vec{a}$ on $\vec{b}$ is

(a) $\frac{(\vec{a} \cdot \vec{b})\vec{a}}{|\vec{a}|^2}$ (b) $\frac{(\vec{a} \cdot \vec{b})\vec{b}}{|\vec{b}|^2}$ (c) $\frac{\vec{a}}{|\vec{a}|}$ (d) $\frac{\vec{b}}{|\vec{b}|}$

25. If θ is an acute angle and the vector $(\sin\theta)\hat{i} + (\cos\theta)\hat{j}$ is perpendicular to the vector $\hat{i} - \sqrt{3}\hat{j}$, then $\theta =$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{5}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

26. If $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector, if $\theta =$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

27. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b} \geq 0$ only when

(a) $0 < \theta < \frac{\pi}{2}$ (b) $0 \leq \theta \leq \frac{\pi}{2}$ (c) $0 < \theta < \pi$ (d) $0 \leq \theta \leq \pi$

28. If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ is a non-zero scalar, then $\lambda\vec{a}$ is a unit vector if

(a) $\lambda = 1$ (b) $\lambda = -1$ (c) $a = |\lambda|$ (d) $a = \frac{1}{|\lambda|}$

ANSWERS

1. (c)	2. (b)	3. (d)	4. (c)	5. (b)	6. (c)	7. (a)	8. (b)
9. (b)	10. (c)	11. (b)	12. (d)	13. (b)	14. (c)	15. (c)	16. (a)
17. (b)	18. (c)	19. (a)	20. (c)	21. (d)	22. (c)	23. (d)	24. (b)
25. (d)	26. (d)	27. (b)	28. (d)				

SUMMARY

1. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors inclined at an angle θ, then

(i) $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

(ii) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$

(iii) Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \vec{b}\cdot\hat{a}$

(iv) Projection vector of $\vec{a}$ on $\vec{b} = \left\{\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right\}\hat{b} = \left\{\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right\}\vec{b}$

(v) Projection vector of $\vec{b}$ on $\vec{a} = \left\{\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right\}\hat{a} = \left\{\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right\}\vec{a}$

(vi) $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}$ is perpendicular to $\vec{b}$

(vii) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(viii) $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

(ix) $m\vec{a}\cdot\vec{b} = m(\vec{a}\cdot\vec{b}) = \vec{a}\cdot m\vec{b}$, for any scalar m

(x) $m\vec{a}\cdot n\vec{b} = mn(\vec{a}\cdot\vec{b}) = mn(\vec{a}\cdot\vec{b}) = \vec{a}\cdot mn\vec{b}$ for scalars m, n

(xi) $|\vec{a}\pm\vec{b}| \leq |\vec{a}| + |\vec{b}|$

(xii) $|\vec{a}-\vec{b}| \geq |\vec{a}| - |\vec{b}|$

(xiii) $|\vec{a}\pm\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \pm 2(\vec{a}\cdot\vec{b})$

(xiv) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

(xv) $\vec{a}\cdot\vec{b} > 0$ iff θ is acute

(xvi) $\vec{a}\cdot\vec{b} < 0$ iff θ is obtuse

2. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, then $\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

3. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

4. If $\vec{a}$ and $\vec{b}$ are two vectors inclined at an angle θ, then $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

5. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors in space and $\vec{r}$ is any vector in space, then

$$\vec{r} = (\vec{r}\cdot\hat{a})\,\hat{a} + (\vec{r}\cdot\hat{b})\,\hat{b} + (\vec{r}\cdot\hat{c})\,\hat{c}$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$ are unit vectors in the directions of $\vec{a}, \vec{b}, \vec{c}$ respectively.

Also, $\vec{r} = \left\{\dfrac{\vec{r}\cdot\vec{a}}{|\vec{a}|^2}\right\}\vec{a} + \left\{\dfrac{\vec{r}\cdot\vec{b}}{|\vec{b}|^2}\right\}\vec{b} + \left\{\dfrac{\vec{r}\cdot\vec{c}}{|\vec{c}|^2}\right\}\vec{c}$

In particular, $\vec{r} = (\vec{r}\cdot\hat{i})\,\hat{i} + (\vec{r}\cdot\hat{j})\,\hat{j} + (\vec{r}\cdot\hat{k})\,\hat{k}$.

VECTOR OR CROSS PRODUCT

25.1 VECTOR (OR CROSS) PRODUCT

DEFINITION *Let $\vec{a}, \vec{b}$ be two non-zero non-parallel vectors. Then the vector product $\vec{a} \times \vec{b}$, in that order, is defined as a vector whose magnitude is $|\vec{a}||\vec{b}| \sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in such a way that $\vec{a}, \vec{b}$ and this direction constitute a right handed system.*

In other words,

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{\eta},$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{\eta}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ form a right handed system.

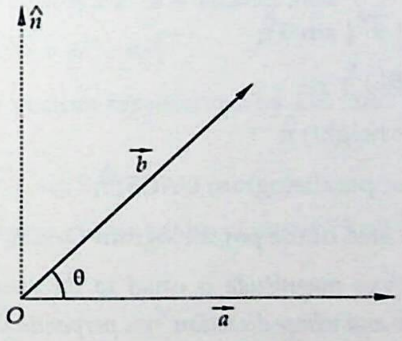


Fig. 25.1

When we say that $\vec{a}, \vec{b}, \hat{\eta}$ form a right handed system it means that if we rotate vector $\vec{a}$ into the vector $\vec{b}$, then $\hat{\eta}$ will point in the direction perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in which a right handed screw will move if it is turned in the same manner.

NOTE 1 *If one of $\vec{a}$ or $\vec{b}$ or both is $\vec{0}$, then θ is not defined as $\vec{0}$ has no direction and so $\hat{\eta}$ is not defined. In this case, we define $\vec{a} \times \vec{b} = \vec{0}$.*

NOTE 2 *If $\vec{a}$ and $\vec{b}$ are collinear i.e. if $\theta = 0$ or π, then the direction of $\hat{\eta}$ is not well defined. So in this case also we define $\vec{a} \times \vec{b} = \vec{0}$.*

NOTE 3 *$\vec{a} \times \vec{b}$ is read as $\vec{a}$ cross $\vec{b}$. Since we are putting cross between $\vec{a}$ and $\vec{b}$ that is why it is called cross-product. As the resulting quantity is a vector so it is also known as the vector product.*

25.1.1 GEOMETRICAL INTERPRETATION OF VECTOR PRODUCT

Let $\vec{a}$, $\vec{b}$ be two non-zero, non-parallel vectors represented by $\vec{OA}$ and $\vec{OB}$ respectively and let θ be the angle between them. Complete the parallelogram $OACB$. Draw $BL \perp OA$.

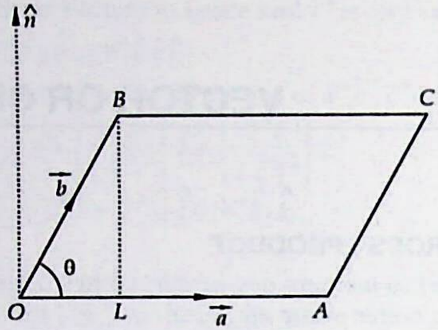


Fig. 25.2

In ΔOBL, we have

$$\sin\theta = \frac{BL}{OB}$$

$\Rightarrow$ $\quad BL = OB \sin\theta = |\vec{b}| \sin\theta$ ...(i)

Now,

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\,\hat{\eta}$$

$\Rightarrow$ $\quad \vec{a} \times \vec{b} = (OA)(BL)\,\hat{\eta}$ [Using (i)]

$\Rightarrow$ $\quad \vec{a} \times \vec{b} = (\text{Base} \times \text{height})\,\hat{\eta}$

$\Rightarrow$ $\quad \vec{a} \times \vec{b} = (\text{Area of parallelogram } OACB)\,\hat{\eta}$

$\Rightarrow$ $\quad \vec{a} \times \vec{b} =$ Vector area of the parallelogram $OACB$.

Thus, $\vec{a} \times \vec{b}$ is a vector whose magnitude is equal to the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as its adjacent sides and whose direction $\hat{\eta}$ is perpendicula to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$, $\hat{\eta}$ form a right handed system.

In other words, $\vec{a} \times \vec{b}$ represents the vector area of the parallelogram having adjacent sides along $\vec{a}$ and $\vec{b}$.

Thus, area of parallelogram $OACB = |\vec{a} \times \vec{b}|$.

Also, $\quad$ Area of $\Delta OAB = \frac{1}{2}$ Area of parallelogram $OACB$

$\Rightarrow$ $\quad$ Area of $\Delta OAB = \frac{1}{2} |\vec{a} \times \vec{b}|$

$\Rightarrow$ $\quad$ Area of $\Delta OAB = \frac{1}{2} |\vec{OA} \times \vec{OB}|$

<u>NOTE</u> *By the term vector area of a plane figure we mean that a vector of magnitude equal to the area of the plane figure and direction normal to the plane of the figure in the sense of right handed rotation.*

25.2 PROPERTIES OF VECTOR PRODUCT

PROPERTY I *Vector product is not commutative i.e. if $\vec{a}$ and $\vec{b}$ are any two vectors, then,* $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$.

PROOF Let $\vec{a}$ and $\vec{b}$ be two non-zero, non-parallel vectors and let θ be the angle between them. Then,

$$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}_1,$$

where $\hat{\eta}_1$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}_1$ form a right-handed system.

and, $\quad \vec{b}\times\vec{a} = |\vec{b}|\,|\vec{a}|\sin\theta\,\hat{\eta}_2,$

where $\hat{\eta}_2$ is a unit vector perpendicular to the plane of $\vec{b}$ and $\vec{a}$ such that $\vec{b}, \vec{a}, \hat{\eta}_2$ form a right-handed system.

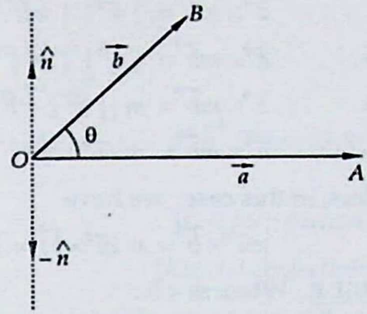


Fig. 25.3

Obviously, $\hat{\eta}_1 = -\hat{\eta}_2$

$$\Rightarrow \quad |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}_1 = -|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}_2$$

$$\Rightarrow \quad \vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$$

Hence, $\quad \vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$

PROPERTY II *If $\vec{a}, \vec{b}$ are two vectors and m is a scalar, then*

$$m\,\vec{a}\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times m\,\vec{b}$$

PROOF Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\vec{OA}$ and $\vec{OB}$ and let θ be the angle between them. Then,

$$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta},$$

where $\hat{\eta}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ form a right handed system.

We have the following cases:

CASE I When $m > 0$:
In this case, $m\vec{a} = \vec{OA_1}$ and $\vec{a} = \vec{OA}$ are in the same direction. Also, $\vec{b} = \vec{OB}$ and $m\vec{b} = \vec{OB_1}$ are in the same direction.

$$\therefore \quad m\vec{a}\times\vec{b} = |m\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}$$

$$\Rightarrow \quad m\vec{a}\times\vec{b} = |m|\,|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}$$

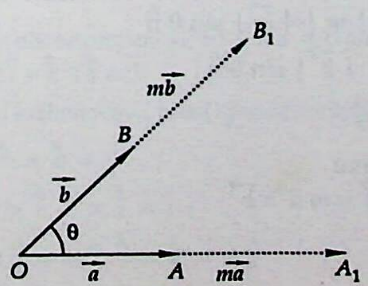


Fig. 25.4

$\Rightarrow \quad m\vec{a} \times \vec{b} = m\,|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}$ $\quad [\because\ m > 0\ \therefore\ |m| = m]$

$\Rightarrow \quad m\vec{a} \times \vec{b} = m\,(|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta})$

$\Rightarrow \quad m\vec{a} \times \vec{b} = m\,(\vec{a} \times \vec{b}).$

and,

$\vec{a} \times m\vec{b} = |\vec{a}|\,|m\vec{b}|\sin\theta\,\hat{\eta}$

$\Rightarrow \quad \vec{a} \times m\vec{b} = |\vec{a}|\,|m|\,|\vec{b}|\sin\theta\,\hat{\eta}$

$\Rightarrow \quad \vec{a} \times m\vec{b} = m\{|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}\}$

$\Rightarrow \quad \vec{a} \times m\vec{b} = m\,(\vec{a} \times \vec{b})$

Thus, in this case, we have

$$m\vec{a} \times \vec{b} = m\,(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}.$$

CASE II When $m < 0$:

In this case, the angle between $m\vec{a} = \vec{OA_1}$ and $\vec{b} = \vec{OB}$ is $(\pi + \theta)$.

$\therefore \quad m\vec{a} \times \vec{b} = |m\vec{a}|\,|\vec{b}|\sin(\pi+\theta)\,\hat{\eta}$

$\Rightarrow \quad m\vec{a} \times \vec{b} = -\,|m|\,|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}$

$\Rightarrow \quad m\vec{a} \times \vec{b} = m\{|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}\}$ $\quad [\because\ m < 0\ \therefore\ |m| = -m]$

$\Rightarrow \quad m\vec{a} \times \vec{b} = m\,(\vec{a} \times \vec{b})$

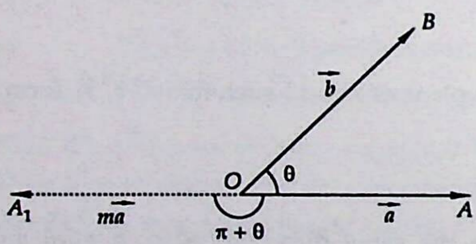


Fig. 25.5

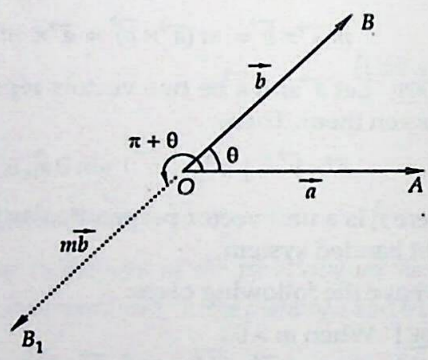


Fig. 25.6

The angle between $\vec{a}$ and $m\vec{b}$ is $(\pi + \theta)$.

$\therefore \quad \vec{a} \times m\vec{b} = |\vec{a}|\,|m\vec{b}|\sin(\pi+\theta)\,\hat{\eta}$

$\Rightarrow \quad \vec{a} \times m\vec{b} = -\,|\vec{a}|\,|m|\,|\vec{b}|\sin\theta\,\hat{\eta}$

$\Rightarrow \quad \vec{a} \times m\vec{b} = m\{|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}\}$ $\quad [\because\ |m| = -m]$

$\Rightarrow \quad \vec{a} \times m\vec{b} = m\,(\vec{a} \times \vec{b})$

Thus, in this case also, we have

$$\vec{a} \times m\vec{b} = m\,(\vec{a} \times \vec{b}) = m\,\vec{a} \times \vec{b}$$

CASE III. When $m = 0$:

In this case, we have

$m\vec{a} \times \vec{b} = \vec{0} \times \vec{b} = \vec{0}$ $\quad$ [See Note 1 in section 25.1]

$\Rightarrow \quad \vec{a} \times m\vec{b} = \vec{a} \times \vec{0} = \vec{0}$ [See Note 1 in section 25.1]

and, $\quad m(\vec{a} \times \vec{b}) = 0\,(\vec{a} \times \vec{b}) = \vec{0}$

So, in this case also, we have

$$m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}.$$

Hence, $\quad m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$ for all values of m.

PROPERTY III *If $\vec{a}, \vec{b}$ are two vectors and m, n are scalars, then*

$$m\vec{a} \times n\vec{b} = mn(\vec{a} \times \vec{b}) = m(\vec{a} \times n\vec{b}) = n(m\vec{a} \times \vec{b})$$

PROPERTY IV *(Distributivity of vector product over vector addition). Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors. Then,*

(i) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ *[Left distributivity]*

(ii) $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$ *[Right distributivity]*

PROOF To prove this property we need the concept of scalar triple product. So, it will be proved in scalar triple product.

PROPERTY V *For any three vectors $\vec{a}, \vec{b}, \vec{c}$, we have*

$$\vec{a} \times (\vec{b} - \vec{c}) = \vec{a} \times \vec{b} - \vec{a} \times \vec{c}$$

PROOF We have,

$$\vec{a} \times (\vec{b} - \vec{c}) = \vec{a} \times \{\vec{b} + (-\vec{c})\}$$

$\Rightarrow \quad \vec{a} \times (\vec{b} - \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times (-\vec{c})$ [By Prop. (IV)]

$\Rightarrow \quad \vec{a} \times (\vec{b} - \vec{c}) = \vec{a} \times \vec{b} - \vec{a} \times \vec{c}$ $[\because \vec{a} \times (-\vec{c}) = -(\vec{a} \times \vec{c})]$

PROPERTY VI *The vector product of two non-zero vectors is zero vector iff they are parallel (collinear).*

i.e. $\quad \vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \,||\, \vec{b}$, *$\vec{a}, \vec{b}$ are non-zero vectors.*

PROOF Let $\vec{a}, \vec{b}$ be two non-zero vectors. Then,

$$\vec{a} \times \vec{b} = \vec{0}$$

$\Leftrightarrow \quad |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{\eta} = \vec{0}$

$\Leftrightarrow \quad \sin\theta = 0$ $[\because |\vec{a}| \neq 0, |\vec{b}| \neq 0]$

$\Leftrightarrow \quad \theta = 0$ or, π

$\Leftrightarrow \quad \vec{a}, \vec{b}$ are parallel vectors

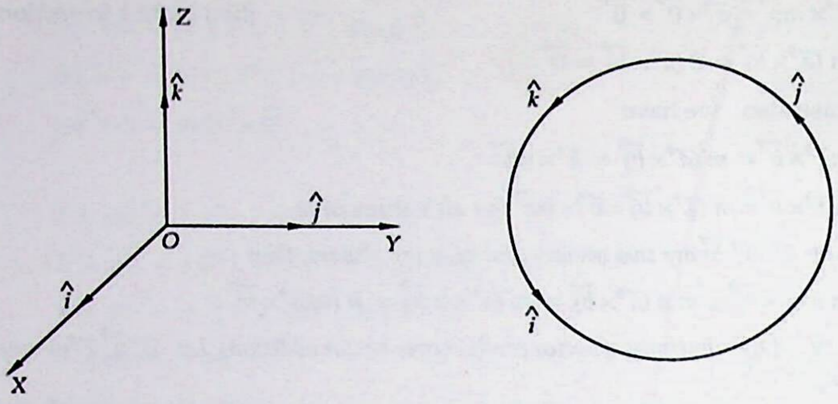
REMARK 1 *It follows from the above property that $\vec{a} \times \vec{a} = \vec{0}$ for every non-zero vector $\vec{a}$ which in turn implies that $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$.*

REMARK 2 *Vector product of orthonormal triad of unit vectors $\hat{i}, \hat{j}, \hat{k}$ is given by*

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j}$$



(i) (ii)

Fig. 25.7

25.3 VECTOR PRODUCT IN TERMS OF COMPONENTS

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be two vectors. Then,

$$\vec{a}\times\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$\Rightarrow \quad \vec{a}\times\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times b_1\hat{i} + (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times b_2\hat{j} + (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times b_3\hat{k}$$

[By left distributivity]

$$\Rightarrow \quad \vec{a}\times\vec{b} = a_1 b_1 (\hat{i}\times\hat{i}) + (a_2 b_1)(\hat{j}\times\hat{i}) + (a_3 b_1)(\hat{k}\times\hat{i}) + a_1 b_2 (\hat{i}\times\hat{j}) + a_2 b_2 (\hat{j}\times\hat{j})$$
$$+ a_3 b_2 (\hat{k}\times\hat{j}) + a_1 b_3 (\hat{i}\times\hat{k}) + a_2 b_3 (\hat{j}\times\hat{k}) + a_3 b_3 (\hat{k}\times\hat{k})$$

$$\Rightarrow \quad \vec{a}\times\vec{b} = (a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

[Using Remark 2 in section 25.2]

$$\Rightarrow \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

25.4 VECTORS NORMAL TO THE PLANE OF TWO GIVEN VECTORS

Let $\vec{a}, \vec{b}$ be two non-zero, non-parallel vectors and let θ be the angle between them.

$$\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta},$$

where $\hat{\eta}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ from a right-handed system

$$\therefore \quad (\vec{a}\times\vec{b}) = |\vec{a}\times\vec{b}|\,\hat{\eta}$$

$$\Rightarrow \quad \hat{\eta} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

Thus, $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$

Note that $-\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is also a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$

Vectors of magnitude 'λ' normal to the plane of $\vec{a}$ and $\vec{b}$ are given by

$$\pm\frac{\lambda(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$$

25.5 SOME IMPORTANT RESULTS

RESULT I *The area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.*

PROOF See section 25.1.1

RESULT II *The area of a triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$.*

PROOF See section 25.1.1

RESULT III *The area of a triangle ABC is $\frac{1}{2}|\vec{AB}\times\vec{AC}|$ or, $\frac{1}{2}|\vec{BC}\times\vec{BA}|$ or, $\frac{1}{2}|\vec{CB}\times\vec{CA}|$.*

PROOF See section 25.1.1

RESULT IV *The area of a parallelogram with diagonals $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$.*

PROOF Let $ABCD$ be a parallelogram. With A as the origin, let the position vectors of B and D be $\vec{p}$ and $\vec{q}$ respectively. Then,

$$\vec{AB}=\vec{p} \text{ and } \vec{AD}=\vec{q}$$

But, $\vec{BC}=\vec{AD}$. Therefore, $\vec{BC}=\vec{q}$

By triangle law of addition of vectors, we have

$$\vec{AB}+\vec{BC}=\vec{AC}$$

$$\Rightarrow \quad \vec{AB}+\vec{AD}=\vec{AC}$$

$$\Rightarrow \quad \vec{p}+\vec{q}=\vec{AC} \qquad \text{...(i)}$$

Let $\vec{AC}=\vec{a}$ and $\vec{BD}=\vec{b}$ be the diagonals of the parallelogram $ABCD$. Then, from (i), we have

$$\vec{p}+\vec{q}=\vec{a} \qquad \text{...(ii)}$$

and, $\vec{BD}$ = Position vector of D – Position vector of B

$$\Rightarrow \quad \vec{b}=\vec{q}-\vec{p} \qquad \text{...(iii)}$$

Adding (i) and (ii), we obtain

$$2\vec{q}=\vec{a}+\vec{b} \Rightarrow \vec{q}=\frac{1}{2}(\vec{a}+\vec{b})$$

Subtracting (iii) from (ii), we obtain

$$2\vec{p}=\vec{a}-\vec{b} \Rightarrow \vec{p}=\frac{1}{2}(\vec{a}-\vec{b})$$

Now,

$$\vec{p}\times\vec{q}=\frac{1}{2}(\vec{a}-\vec{b})\times\frac{1}{2}(\vec{a}+\vec{b})$$

$$\Rightarrow \quad \vec{p}\times\vec{q}=\frac{1}{4}[(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})]$$

$$\Rightarrow \quad \vec{p}\times\vec{q}=\frac{1}{4}[\vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{b}\times\vec{b}]$$

$$\Rightarrow \quad \vec{p}\times\vec{q}=\frac{1}{4}[\vec{a}\times\vec{b}+\vec{a}\times\vec{b}]$$

$$\Rightarrow \quad \vec{p}\times\vec{q}=\frac{1}{2}(\vec{a}\times\vec{b})$$

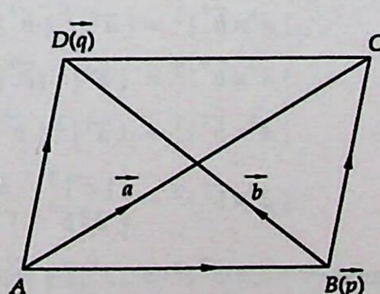


Fig. 25.8

So, area of parallelogram $ABCD = |\vec{p}\times\vec{q}| = \left|\frac{1}{2}(\vec{a}\times\vec{b})\right| = \frac{1}{2}|\vec{a}\times\vec{b}|$.

RESULT V *The area of a plane quadrilateral ABCD is*

$\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$, *where AC and BD are its diagonals.*

PROOF Let the diagonals AC and BD intersect at O. Then,

Vector area of quadrilateral $ABCD$

= (Vector area of triangle ABC) + (Vector area of triangle ADC)

$= \frac{1}{2}(\overrightarrow{AB}\times\overrightarrow{AC}) + \frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{AD})$

$= -\frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{AB}) + \frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{AD})$

$= \frac{1}{2}[\overrightarrow{AC}\times(\overrightarrow{AD} - \overrightarrow{AB})$

$= \frac{1}{2}[\overrightarrow{AC}\times(\overrightarrow{BA} + \overrightarrow{AD})]$ $\quad [\because -\overrightarrow{AB} = \overrightarrow{BA}]$

$= \frac{1}{2}[\overrightarrow{AC}\times\overrightarrow{BD}]$ $\quad [\because \overrightarrow{BA} + \overrightarrow{AD} = \overrightarrow{BD}]$

So, area of quadrilateral $ABCD = \frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$

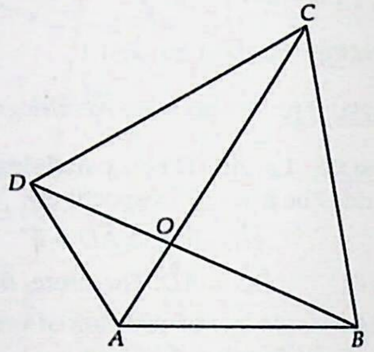


Fig. 25.9

25.6 LAGRANGE'S IDENTITY

THEOREM *If $\vec{a}, \vec{b}$ are any two vectors, then*

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = \begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$$

or, $\quad |\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2\,|\vec{b}|^2$ **[CBSE 2004]**

PROOF We have,

$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$

$\therefore \quad |\vec{a}\times\vec{b}|^2$

$\Rightarrow \quad |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2\sin^2\theta$

$\Rightarrow \quad |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2(1-\cos^2\theta)$

$\Rightarrow \quad |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2 - |\vec{a}|^2\,|\vec{b}|^2\cos^2\theta$

$\Rightarrow \quad |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2 - (|\vec{a}|\,|\vec{b}|\cos\theta)^2$

$\Rightarrow \quad |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$\Rightarrow \quad |\vec{a}\times\vec{b}|^2 = \begin{vmatrix}|\vec{a}|^2 & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & |\vec{b}|^2\end{vmatrix} = \begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$

Hence, $\quad |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$\Rightarrow \quad |\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2\,|\vec{b}|^2$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find $\vec{a}\times\vec{b}$, if $\vec{a} = 2\hat{i}+\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}+\hat{k}$.*

SOLUTION We have,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \vec{a}\times\vec{b} = (0-1)\,\hat{i}-(2-1)\,\hat{j}+(2-0)\,\hat{k} = -\hat{i}-\hat{j}+2\,\hat{k}$$

EXAMPLE 2 *Find the magnitude of $\vec{a}$ given by $\vec{a} = (\hat{i}+3\,\hat{j}-2\,\hat{k})\times(-\hat{i}+3\,\hat{k})$*

SOLUTION We have,

$$\vec{a} = (\hat{i}+3\,\hat{j}-2\,\hat{k})\times(-\hat{i}+0\,\hat{j}+3\,\hat{k})$$

$$\Rightarrow \quad \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix} = (9-0)\,\hat{i}-(3-2)\,\hat{j}+(0+3)\,\hat{k} \quad = 9\,\hat{i}-\hat{j}+3\,\hat{k}$$

$$\therefore \quad |\vec{a}| = \sqrt{9^2+(-1)^2+3^2} = \sqrt{91}.$$

EXAMPLE 3 *Find a unit vector perpendicular to both the vectors $\hat{i}-2\,\hat{j}+3\,\hat{k}$ and $\hat{i}+2\,\hat{j}-\hat{k}$.*

SOLUTION Let $\vec{a} = \hat{i}-2\,\hat{j}+3\,\hat{k}$ and $\vec{b} = -2\,\hat{i}+\hat{j}-2\,\hat{k}$. Then,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & 2 & -1 \end{vmatrix} = (2-6)\,\hat{i}-(-1-3)\,\hat{j}+(2+2)\,\hat{k} = -4\,\hat{i}+4\,\hat{j}+4\,\hat{k}.$$

$$\Rightarrow \quad |\vec{a}\times\vec{b}| = \sqrt{(-4)^2+4^2+4^2} = 4\sqrt{3}.$$

So, a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{(-4\,\hat{i}+4\,\hat{j}+4\,\hat{k})}{4\sqrt{3}} = \frac{1}{\sqrt{3}}(-\hat{i}+\hat{j}+\hat{k})$$

EXAMPLE 4 *Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$.* **[NCERT]**

SOLUTION We have,

$$\vec{a}+\vec{b} = 2\hat{i}+3\hat{j}+4\hat{k} \text{ and } \vec{a}-\vec{b} = 0\,\hat{i}-\hat{j}-2\hat{k}$$

A vector $\vec{c}$ perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is given by

$$\vec{c} = (\vec{a}+\vec{b})\times(\vec{a}-\vec{b})$$

$$\text{or} \quad \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i}+4\,\hat{j}-2\,\hat{k}$$

$$\Rightarrow \quad |\vec{c}| = \sqrt{4+16+4} = 2\sqrt{6}$$

$$\therefore \quad \text{Required unit vector} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{2\sqrt{6}}(-2\,\hat{i}+4\,\hat{j}-2\,\hat{k}) = -\frac{1}{\sqrt{6}}\hat{i}+\frac{2}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}$$

EXAMPLE 5 *Find a vector of magnitude 9, which is perpendicular to both the vectors* $4\hat{i}-\hat{j}+3\hat{k}$ *and* $-2\hat{i}+\hat{j}-2\hat{k}$.

SOLUTION Let $\vec{a}=4\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$. Then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 3\\ -2 & 1 & -2\end{vmatrix}=(2-3)\hat{i}-(-8+6)\hat{j}+(4-2)\hat{k}=-\hat{i}+2\hat{j}+2\hat{k}$$

$\Rightarrow$ $$|\vec{a}\times\vec{b}|=\sqrt{(-1)^2+2^2+2^2}=3$$

$\therefore$ $$\text{Required vector}=9\left\{\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right\}=\frac{9}{3}(-\hat{i}+2\hat{j}+2\hat{k})=-3\hat{i}+6\hat{j}+6\hat{k}.$$

EXAMPLE 6 *Find a unit vector perpendicular to the plane ABC where A, B, C are the points* $(3,-1,2),(1,-1,-3),(4,-3,1)$ *respectively.*

SOLUTION The vector $\vec{AB}\times\vec{AC}$ is perpendicular to the vectors $\vec{AB}$ and $\vec{AC}$.

$\therefore$ $$\text{Required vector}=\frac{\vec{AB}\times\vec{AC}}{|\vec{AB}\times\vec{AC}|}$$

Now, $\vec{AB}=$ P.V. of $B-$ P.V. of $A=(\hat{i}-\hat{j}-3\hat{k})-(3\hat{i}-\hat{j}+2\hat{k})=-2\hat{i}+0\hat{j}-5\hat{k}$

and, $\vec{AC}=$ P.V. of $C-$ P.V. of $A=(4\hat{i}-3\hat{j}+\hat{k})-(3\hat{i}-\hat{j}+2\hat{k})=\hat{i}-2\hat{j}-\hat{k}$

$\therefore$ $$\vec{AB}\times\vec{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & -5\\ 1 & -2 & -1\end{vmatrix}=(0-10)\hat{i}-(2+5)\hat{j}+(4-0)\hat{k}=-10\hat{i}-7\hat{j}+4\hat{k}$$

$\Rightarrow$ $$|\vec{AB}\times\vec{AC}|=\sqrt{(-10)^2+(-7)^2+4^2}=\sqrt{165}$$

Hence, required vector $$=\frac{\vec{AB}\times\vec{AC}}{|\vec{AB}\times\vec{AC}|}=\frac{1}{\sqrt{165}}(-10\hat{i}-7\hat{j}+4\hat{k})$$

EXAMPLE 7 *Find the area of the parallelogram determined by the vectors* $\hat{i}+2\hat{j}+3\hat{k}$ *and* $3\hat{i}-2\hat{j}+\hat{k}$.

SOLUTION Let $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$. The vector area of the parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\times\vec{b}$.

Now, $$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 3 & -2 & 1\end{vmatrix}=(2+6)\hat{i}-(1-9)\hat{j}+(-2-6)\hat{k}=8\hat{i}+8\hat{j}-8\hat{k}$$

So, area of the parallelogram $=|\vec{a}\times\vec{b}|=\sqrt{8^2+8^2+(-8)^2}=8\sqrt{3}$ square units

EXAMPLE 8 *Show that the area of a parallelogram having diagonals* $3\hat{i}+\hat{j}-2\hat{k}$ *and* $\hat{i}-3\hat{j}+4\hat{k}$ *is* $5\sqrt{3}$. **[CBSE 2008]**

SOLUTION Let $\vec{a}=3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}-3\hat{j}+4\hat{k}$. Then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -2\\ 1 & -3 & 4\end{vmatrix}=(4-6)\hat{i}-(12+2)\hat{j}+(-9-1)\hat{k}=-2\hat{i}-14\hat{j}-10\hat{k}$$

$\Rightarrow$ $$|\vec{a}\times\vec{b}|=\sqrt{(-2)^2+(-14)^2+(-10)^2}=\sqrt{300}$$

$\therefore$ $$\text{Area of the parallelogram}=\frac{1}{2}|\vec{a}\times\vec{b}|=\frac{1}{2}\sqrt{300}=5\sqrt{3}\text{ sq. units.}$$

EXAMPLE 9 *Given* $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 12$, *find* $|\vec{a}\times\vec{b}|$.

SOLUTION We have,

$$(\vec{a}\cdot\vec{b})^2 + |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2$$

$$\therefore\quad 12^2 + |\vec{a}\times\vec{b}|^2 = (10)^2\times(2)^2$$

$$\Rightarrow\quad |\vec{a}\times\vec{b}|^2 = 400 - 144$$

$$\Rightarrow\quad |\vec{a}\times\vec{b}|^2 = 256$$

$$\Rightarrow\quad |\vec{a}\times\vec{b}| = 16$$

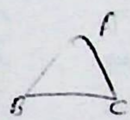
EXAMPLE 10 *Find the area of the triangle whose vertices are A* (3, − 1, 2), *B* (1, − 1, − 3) *and* C (4, − 3, 1). **[CBSE 2010]**

SOLUTION Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of points A, B and C respectively. Then,

$$\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k},\ \vec{b} = \hat{i} - \hat{j} - 3\hat{k} \text{ and } \vec{c} = 4\hat{i} - 3\hat{j} + \hat{k}$$

We have,

$$\text{Area of } \Delta ABC = \frac{1}{2}\,|\vec{AB}\times\vec{AC}|$$



Now, $\vec{AB}$ = Position vector of B – Position vector of A

$$\Rightarrow\quad \vec{AB} = \vec{b} - \vec{a}$$

$$\Rightarrow\quad \vec{AB} = (\hat{i} - \hat{j} - 3\hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k}) = -2\hat{i} + 0\hat{j} - 5\hat{k},$$

$\vec{AC}$ = Position vector of C – Position vector of A

$$\Rightarrow\quad \vec{AC} = \vec{c} - \vec{a}$$

$$\Rightarrow\quad \vec{AC} = (4\hat{i} - 3\hat{j} + \hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k}) = \hat{i} - 2\hat{j} - \hat{k},$$

$$\therefore\quad \vec{AB}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix} = (0-10)\,\hat{i} - (2+5)\,\hat{j} + (4-0)\,\hat{k}$$

$$\Rightarrow\quad \vec{AB}\times\vec{AC} = -10\hat{i} - 7\hat{j} + 4\hat{k}$$

$$\Rightarrow\quad |\vec{AB}\times\vec{AC}| = \sqrt{(-10)^2 + (-7)^2 + 4^2} = \sqrt{165}$$

So, area of $\Delta ABC = \frac{1}{2}\,|\vec{AB}\times\vec{AC}| = \frac{1}{2}\sqrt{165}.$

EXAMPLE 11 *Show that* $(\vec{a}\times\vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$

SOLUTION We have,

$$(\vec{a}\times\vec{b})^2 = |\vec{a}\times\vec{b}|^2$$

$$\Rightarrow\quad (\vec{a}\times\vec{b})^2 = \left\{|\vec{a}|\,|\vec{b}|\sin\theta\right\}^2$$

$$\Rightarrow\quad (\vec{a}\times\vec{b})^2 = |\vec{a}|^2\,|\vec{b}|^2\sin^2\theta$$

$$\Rightarrow\quad (\vec{a}\times\vec{b})^2 = \left\{|\vec{a}|^2\,|\vec{b}|^2\right\}(1-\cos^2\theta)$$

$\Rightarrow \quad (\vec{a}\times\vec{b})^2 = |\vec{a}|^2\,|\vec{b}|^2 - |\vec{a}|^2\,|\vec{b}|^2\cos^2\theta$

$\Rightarrow \quad (\vec{a}\times\vec{b})^2 = (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{b}) \qquad [\because \vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta]$

$\Rightarrow \quad (\vec{a}\times\vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$

EXAMPLE 12 *If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C of a triangle ABC, show that the area of triangle ABC is $\frac{1}{2}\,|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$.*

Deduce the condition for points $\vec{a}, \vec{b}, \vec{c}$ to be collinear.

SOLUTION We have,

$$\text{Area of } \Delta ABC = \frac{1}{2}\,|\vec{AB}\times\vec{AC}|$$

Now, $\vec{AB} = \vec{b}-\vec{a}$, and $\vec{AC} = \vec{c}-\vec{a}$

$\therefore \quad \vec{AB}\times\vec{AC} = (\vec{b}-\vec{a})\times(\vec{c}-\vec{a})$

$\Rightarrow \quad \vec{AB}\times\vec{AC} = \vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a} \qquad$ [By distributivity]

$\Rightarrow \quad \vec{AB}\times\vec{AC} = \vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+\vec{0} \qquad [\because \vec{a}\times\vec{a} = \vec{0}]$

$\Rightarrow \quad \vec{AB}\times\vec{AC} = \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$

$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\,|\vec{AB}\times\vec{AC}| = \frac{1}{2}\,|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

If the points, A, B, C are collinear, then

$$\text{Area of } \Delta ABC = 0$$

$\Rightarrow \quad \frac{1}{2}\,|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}| = 0$

$\Rightarrow \quad |\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}| = 0$

$\Rightarrow \quad \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a} = \vec{0}$

Thus, $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}$ is the required condition of collinearity of three points $\vec{a}, \vec{b}, \vec{c}$.

EXAMPLE 13 *Prove that the points A, B and C with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively are collinear if and only if*

$$\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a} = \vec{0}$$

SOLUTION The points A, B and C are collinear

$\Leftrightarrow \quad \vec{AB}$ and $\vec{BC}$ are parallel vectors.

$\Leftrightarrow \quad \vec{AB}\times\vec{BC} = \vec{0}$

$\Leftrightarrow \quad (\vec{b}-\vec{a})\times(\vec{c}-\vec{b}) = \vec{0}$

$\Leftrightarrow \quad (\vec{b}-\vec{a})\times\vec{c}-(\vec{b}-\vec{a})\times\vec{b} = \vec{0}$

$\Leftrightarrow \quad (\vec{b}\times\vec{c}-\vec{a}\times\vec{c})-(\vec{b}\times\vec{b}-\vec{a}\times\vec{b}) = \vec{0}$

$\Leftrightarrow \quad (\vec{b}\times\vec{c}+\vec{c}\times\vec{a})-(\vec{0}-\vec{a}\times\vec{b}) = \vec{0} \qquad [\because (\vec{a}\times\vec{c}) = \vec{c}\times\vec{a}]$

$\Leftrightarrow \quad \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a} = \vec{0}.$

EXAMPLE 14 *Show that distance of the point $\vec{c}$ from the line joining $\vec{a}$ and $\vec{b}$ is*

$$\frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|}{|\vec{b}-\vec{a}|}$$

SOLUTION Let ABC be a triangle and let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of its vertices A, B, C respectively. Let CM be the perpendicular from C on AB. Then,

$$\text{Area of } \Delta ABC = \frac{1}{2}(AB)\cdot CM = \frac{1}{2}|\vec{AB}|(CM)$$

Also, $$\text{Area of } \Delta ABC = \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}| \qquad \text{[See Example 11]}$$

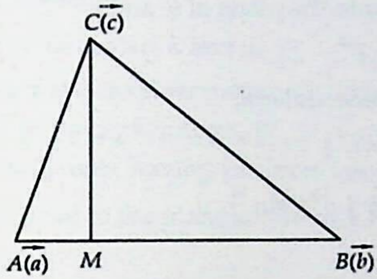


Fig. 25.10

$$\therefore \quad \frac{1}{2}|\vec{AB}|(CM) = \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$$

$$\Rightarrow \quad CM = \frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{AB}|}$$

$$\Rightarrow \quad CM = \frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{b}-\vec{a}|}$$

EXAMPLE 15 *For any three vectors $\vec{a}, \vec{b}, \vec{c}$. Show that*

$$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b}) = \vec{0}$$

SOLUTION We have,

$$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$$
$$= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b} \qquad \text{[Using distributive law]}$$
$$= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c}$$
$$= \vec{0}. \qquad [\because \vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \text{ etc.}]$$

EXAMPLE 16 *If $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$, show that $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$.* **[CBSE 2001, 2009]**

SOLUTION Recall that two non-zero vectors are parallel iff their cross-product is zero vector.

We have,

$$(\vec{a}-\vec{d})\times(\vec{b}-\vec{c}) = \vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{d}\times\vec{b}+\vec{d}\times\vec{c} \qquad \text{[Using distributive law]}$$

$$\Rightarrow \quad (\vec{a}-\vec{d})\times(\vec{b}-\vec{c}) = \vec{c}\times\vec{d}-\vec{b}\times\vec{d}+\vec{b}\times\vec{d}-\vec{c}\times\vec{d}$$

$$\left[\because \vec{a}\times\vec{b}=\vec{c}\times\vec{d}, \vec{a}\times\vec{c}=\vec{b}\times\vec{d}, -\vec{d}\times\vec{b}=\vec{b}\times\vec{d} \text{ and } \vec{d}\times\vec{c}=-\vec{b}\times\vec{d}\right]$$

$\Rightarrow \quad (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0}$

So, $\quad (\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$

EXAMPLE 17 *Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0$ and the angle between* $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, prove that $\vec{a} = \pm 2 (\vec{b} \times \vec{c})$.

SOLUTION We have ,

$$\vec{a} \cdot \vec{b} = 0 \text{ and } \vec{a} \cdot \vec{c} = 0$$

$\Rightarrow \quad \vec{a} \perp \vec{b}$ and $\vec{a} \perp \vec{c}$

$\Rightarrow \quad \vec{a}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

$\Rightarrow \quad \vec{a}$ is parallel to $\vec{b} \times \vec{c}$

$\Rightarrow \quad \vec{a} = \lambda (\vec{b} \times \vec{c})$ for some scalar λ.

$\Rightarrow \quad |\vec{a}| = |\lambda| \, |\vec{b} \times \vec{c}|$

$\Rightarrow \quad |\vec{a}| = |\lambda| \, |\vec{b}| \, |\vec{c}| \sin \frac{\pi}{6}$

$\Rightarrow \quad 1 = |\lambda|/2 \qquad [\because \ |\vec{a}| = |\vec{b}| = |\vec{c}| = 1]$

$\Rightarrow \quad |\lambda| = 2$

$\Rightarrow \quad \lambda = \pm 2$

$\therefore \quad \vec{a} = \lambda (\vec{b} \times \vec{c})$

$\Rightarrow \quad \vec{a} = \pm 2 (\vec{b} \times \vec{c})$.

EXAMPLE 18 *If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then prove that*

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

[CBSE 2001, 2004]

SOLUTION We have,

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$\Rightarrow \quad \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0} \qquad$ [Taking cross-product with $\vec{a}$]

$\Rightarrow \quad \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0} \qquad$ [Using distributive law]

$\Rightarrow \quad \vec{a} \times \vec{b} - \vec{c} \times \vec{a} = \vec{0} \qquad [\because \vec{a} \times \vec{a} = \vec{0} \text{ and } \vec{a} \times \vec{c} = -\vec{c} \times \vec{a}]$

$\Rightarrow \quad \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \qquad$ ...(i)

Again, $\quad \vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \quad \vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times \vec{0} \qquad$ [Taking cross-product with $\vec{b}$]

$\Rightarrow \quad \vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}$

$\Rightarrow \quad -\vec{a} \times \vec{b} + \vec{0} + \vec{b} \times \vec{c} = \vec{0} \qquad [\because \vec{b} \times \vec{b} = \vec{0} \text{ and } \vec{b} \times \vec{a} = -\vec{a} \times \vec{b}]$

$\Rightarrow \quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} \qquad$ ...(ii)

Form (i) and (ii), we obtain

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$$

EXAMPLE 19 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors such that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ and $\vec{b}$ and $\vec{c}$ are not parallel vectors, prove that $\vec{a} = \lambda \vec{b} + \mu \vec{c}$, where λ and μ are scalars.*

SOLUTION We have,

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$$

$\Rightarrow$ $\vec{a}=\vec{0}$ or, $\vec{b}\times\vec{c}=\vec{0}$ or, $\vec{a}\perp(\vec{b}\times\vec{c})$

$\Rightarrow$ $\vec{a}=\vec{0}$ or, $\vec{b}\parallel\vec{c}$ or, $\vec{a}\perp(\vec{b}\times\vec{c})$

But, $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

$\therefore$ $\vec{a}\perp(\vec{b}\times\vec{c})$

$\Rightarrow$ $\vec{a}$ lies in the plane of $\vec{b}$ and $\vec{c}$

$\Rightarrow$ $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors

$\Rightarrow$ $\vec{a}=\lambda\vec{b}+\mu\vec{c}$ for some scalars λ and μ.

EXAMPLE 20 *Prove that the normal to the plane containing three points whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ lies in the direction $\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}$.* **[CBSE 2001C]**

SOLUTION Let A, B, C be the points having position vectors $\vec{a}$ $\vec{b}$ and $\vec{c}$ respectively Then, $\vec{AB}\times\vec{AC}$ is a vector normal to the plane containing the points A, B and C.

Now,

$$\vec{AB}\times\vec{AC} = (\vec{b}-\vec{a})\times(\vec{c}-\vec{a})$$

$\Rightarrow$ $\vec{AB}\times\vec{AC} = \vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}$

$\Rightarrow$ $\vec{AB}\times\vec{AC} = \vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+\vec{0}$ $\quad[\because \vec{a}\times\vec{a}=\vec{0}]$

$\Rightarrow$ $\vec{AB}\times\vec{AC} = \vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}$

Hence, $\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}$ is a vector normal to the plane containing points $\vec{a}, \vec{b}, \vec{c}$.

EXAMPLE 21 *If $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, $\vec{a}\neq\vec{0}$ and $\vec{b}\neq\vec{c}$, show that $\vec{b}=\vec{c}+t\vec{a}$ for some scalar t.*

SOLUTION We have,

$$\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$$

$\Rightarrow$ $\vec{a}\times\vec{b}-\vec{a}\times\vec{c}=\vec{0}$

$\Rightarrow$ $\vec{a}\times(\vec{b}-\vec{c})=\vec{0}$

$\Rightarrow$ $\vec{a}=\vec{0}$ or, $\vec{b}-\vec{c}=\vec{0}$ or, $\vec{a}\parallel(\vec{b}-\vec{c})$

$\Rightarrow$ $\vec{a}=\vec{0}$ or, $\vec{b}=\vec{c}$ or, $\vec{a}\parallel(\vec{b}-\vec{c})$

$\Rightarrow$ $\vec{a}\parallel(\vec{b}-\vec{c})$ $\quad[\because \vec{a}\neq\vec{0}$ and $\vec{b}\neq\vec{c}]$

$\Rightarrow$ $\vec{b}-\vec{c}=t\vec{a}$ for some scalar t

$\Rightarrow$ $\vec{b}=\vec{c}+t\vec{a}$

EXAMPLE 22 *If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$, $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, $\vec{a}\neq\vec{0}$, then show that $\vec{b}=\vec{c}$.*

SOLUTION We have,

$$\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c} \text{ and } \vec{a}\neq\vec{0}$$

$\Rightarrow$ $\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=0$ and $\vec{a}\neq\vec{0}$

$\Rightarrow$ $\vec{a}\cdot(\vec{b}-\vec{c})=0$ and $\vec{a}\neq\vec{0}$

$\Rightarrow \quad \vec{b} - \vec{c} = \vec{0}$ or, $\vec{a} \perp (\vec{b} - \vec{c}) \qquad [\because \vec{a} \neq \vec{0}]$

$\Rightarrow \quad \vec{b} = \vec{c}$ or, $\vec{a} \perp (\vec{b} - \vec{c}) \qquad$ ...(i)

Again, $\quad \vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow \quad \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow \quad \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow \quad \vec{b} - \vec{c} = \vec{0}$ or, $\vec{a} \parallel (\vec{b} - \vec{c}) \qquad [\because \vec{a} \neq \vec{0}]$

$\Rightarrow \quad \vec{b} = \vec{c}$ or, $\vec{a} \parallel (\vec{b} - \vec{c}) \qquad$ ...(ii)

From (i) and (ii), it follows that $\vec{b} = \vec{c}$, because $\vec{a}$ cannot be both parallel and perpendicular to $(\vec{b} - \vec{c})$.

EXAMPLE 23 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$, prove that $\vec{a}, \vec{b}, \vec{c}$ are mutually at right angles and* $|\vec{b}| = 1$ *and* $|\vec{c}| = |\vec{a}|$.

SOLUTION We have,

$$\vec{a} \times \vec{b} = \vec{c} \text{ and } \vec{a} = \vec{b} \times \vec{c}$$

$\Rightarrow \quad \vec{c} \perp \vec{a}, \vec{c} \perp \vec{b}$ and $\vec{a} \perp \vec{b}, \vec{a} \perp \vec{c}$

$\Rightarrow \quad \vec{a} \perp \vec{b}, \vec{b} \perp \vec{c}$ and $\vec{c} \perp \vec{a}$

$\Rightarrow \quad \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors.

Again, $\quad \vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$

$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{c}|$ and $|\vec{b} \times \vec{c}| = |\vec{a}|$

$\Rightarrow \quad |\vec{a}|\,|\vec{b}| \sin\frac{\pi}{2} = |\vec{c}|$ and $|\vec{b}|\,|\vec{c}| \sin\frac{\pi}{2} = |\vec{a}| \qquad [\because \vec{a} \perp \vec{b}$ and $\vec{b} \perp \vec{c}]$

$\Rightarrow \quad |\vec{a}|\,|\vec{b}| = |\vec{c}|$ and $|\vec{b}|\,|\vec{c}| = |\vec{a}|$

$\Rightarrow \quad |b|^2\,|\vec{c}| = |\vec{c}| \qquad \left[\text{Putting } |\vec{a}| = |\vec{b}|\,|\vec{c}| \text{ in } |\vec{a}|\,|\vec{b}| = |\vec{c}|\right]$

$\Rightarrow \quad |\vec{b}|^2 = 1 \qquad [\because |\vec{c}| \neq 0]$

$\Rightarrow \quad |\vec{b}| = 1$

Putting $|\vec{b}| = 1$ in $|\vec{a}|\,|\vec{b}| = |\vec{c}|$, we obtain $|\vec{a}| = |\vec{c}|$

EXAMPLE 24 *Prove that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$ and interpret it geometrically.* [NCERT]

SOLUTION We have,

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$$

$\Rightarrow \quad (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{b} \quad [\because \vec{a} \times \vec{a} = \vec{0} = \vec{b} \times \vec{b}$ and $-\vec{b} \times \vec{a} = \vec{a} \times \vec{b}]$

$\Rightarrow \quad (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$

Geometrical Interpretation: Let *ABCD* be a parallelogram. Taking *A* as the origin, let the position vectors of *B* and *D* be $\vec{a}$ and $\vec{b}$ respectively.

Then, $\vec{AB} = \vec{a}$ and $\vec{AD} = \vec{b}$.

By triangle law of addition of vectors, we have

$$\vec{AB} + \vec{BC} = \vec{AC} \Rightarrow \vec{AC} = \vec{a} + \vec{b}$$

In ΔABD, we have

$$\vec{AD} + \vec{DB} = \vec{AB}$$

$$\Rightarrow \quad \vec{DB} = \vec{AB} - \vec{AD} = \vec{a} - \vec{b}.$$

$$\therefore \quad (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2 (\vec{a} \times \vec{b}).$$

$$\Rightarrow \quad \vec{DB} \times \vec{AC} = 2 (\vec{AB} \times \vec{AD})$$

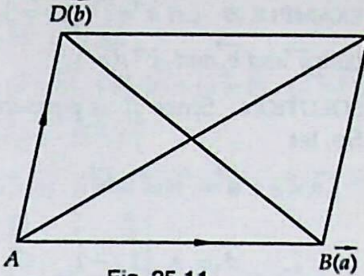


Fig. 25.11

Thus, area of a parallelogram whose adjacent sides are the diagonals of a given parallelogram is twice the area of the given parallelogram.

EXAMPLE 25 *For any vector $\vec{a}$, prove that* $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2 |\vec{a}|^2$

SOLUTION Let $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$. Then,

$$\vec{a} \times \hat{i} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times \hat{i} = a_1 (\hat{i} \times \hat{i}) + a_2 (\hat{j} \times \hat{i}) + a_3 (\hat{k} \times \hat{i}) = -a_2 \hat{k} + a_3 \hat{j}$$

$$\Rightarrow \quad |\vec{a} \times \hat{i}|^2 = a_2^2 + a_3^2$$

$$\vec{a} \times \hat{j} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times \hat{j} = a_1 (\hat{i} \times \hat{j}) + a_2 (\hat{j} \times \hat{j}) + a_3 (\hat{k} \times \hat{j}) = a_1 \hat{k} - a_3 \hat{i}$$

$$\Rightarrow \quad |\vec{a} \times \hat{j}|^2 = a_1^2 + a_3^2$$

and, $$\vec{a} \times \hat{k} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times \hat{k} = a_1 (\hat{i} \times \hat{k}) + a_2 (\hat{j} \times \hat{k}) + a_3 (\hat{k} \times \hat{k}) = -a_1 \hat{j} + a_2 \hat{i}$$

$$\Rightarrow \quad |\vec{a} \times \hat{k}|^2 = a_1^2 + a_2^2$$

$$\therefore \quad |\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = a_2^2 + a_3^2 + a_1^2 + a_3^2 + a_1^2 + a_2^2$$

$$\Rightarrow \quad |\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2 (a_1^2 + a_2^2 + a_3^2) = 2 |\vec{a}|^2$$

EXAMPLE 26 *Let $\vec{OA} = \vec{a}$, $\vec{OB} = 10 \vec{a} + 2\vec{b}$, and $\vec{OC} = \vec{b}$ where O is origin. Let p denote the area of the quadrilateral OABC and q denote the area of the parallelogram with OA and OC as adjacent sides. Prove that $p = 6q$.*

SOLUTION We have,

$$p = \text{Area of the quadrilateral } OABC$$

$$\Rightarrow \quad p = \frac{1}{2} |\vec{OB} \times \vec{AC}|$$

$$\Rightarrow \quad p = \frac{1}{2} | \vec{OB} \times (\vec{OC} - \vec{OA}) | \qquad [\because \vec{AC} = \vec{OC} - \vec{OA}]$$

$$\Rightarrow \quad p = \frac{1}{2} | (10\vec{a} + 2\vec{b}) \times (\vec{b} - \vec{a}) | \qquad [\because \vec{OB} = 10\vec{a} + 2\vec{b}, \vec{OC} = \vec{b} \text{ and } \vec{OA} = \vec{a}]$$

$$\Rightarrow \quad p = \frac{1}{2} |10 (\vec{a} \times \vec{b}) - 10 (\vec{a} \times \vec{a}) + 2 (\vec{b} \times \vec{b}) - 2 (\vec{b} \times \vec{a})|$$

$$\Rightarrow \quad p = \frac{1}{2} | 10 (\vec{a} \times \vec{b}) - 0 + 0 + 2 (\vec{a} \times \vec{b})|$$

$$\Rightarrow \quad p = \frac{1}{2} | 12 (\vec{a} \times \vec{b}) | = 6 |\vec{a} \times \vec{b}| \qquad \text{...(i)}$$

and, $$q = \text{Area of the parallelogram with } OA \text{ and } OC \text{ as adjacent sides}$$

$$\Rightarrow \quad q = |\vec{OA} \times \vec{OC}| = |\vec{a} \times \vec{b}| \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$p = 6q$$

EXAMPLE 27 *Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = 3\hat{j} - \hat{k}$ and $\vec{c} = 7\hat{i} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c} \cdot \vec{d} = 1$.* **[NCERT]**

SOLUTION Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Therefore, it is parallel to $\vec{a} \times \vec{b}$. So, let

$$\vec{d} = \lambda (\vec{a} \times \vec{b})$$

$$\Rightarrow \quad \vec{d} = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 3 & -1 \end{vmatrix}$$

$$\Rightarrow \quad \vec{d} = \lambda (\hat{i} + \hat{j} + 3\hat{k}) \quad \text{...(i)}$$

Now, $\vec{c} \cdot \vec{d} = 1$

$$\Rightarrow \quad (7\hat{i} - \hat{k}) \cdot \lambda (\hat{i} + \hat{j} + 3\hat{k}) = 1$$

$$\Rightarrow \quad 7\lambda - 3\lambda = 1$$

$$\Rightarrow \quad \lambda = \frac{1}{4}$$

Putting $\lambda = \frac{1}{4}$ in (i), we get $\vec{d} = \frac{1}{4}(\hat{i} + \hat{j} + 3\hat{k})$.

EXAMPLE 28 *Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.* **[NCERT]**

SOLUTION We have,

$$(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$$

$$\Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = \vec{0}$$

$$\Rightarrow \quad (6\mu - 27\lambda)\hat{i} - (2\mu - 27)\hat{j} + (2\lambda - 6)\hat{k} = \vec{0}$$

$$\Rightarrow \quad 6\mu - 27\lambda = 0, 2\mu - 27 = 0 \text{ and } 2\lambda - 6 = 0$$

$$\Rightarrow \quad \lambda = 3 \text{ and } \mu = \frac{27}{2}$$

ALITER We have,

$$(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$$

$$\Rightarrow \quad 2\hat{i} + 6\hat{j} + 27\hat{k} \text{ is parallel to } \hat{i} + \lambda\hat{j} + \mu\hat{k}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{\lambda}{6} = \frac{\mu}{27} \quad \left[\begin{array}{l} \because \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \\ \text{are parallel, iff } \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \end{array}\right]$$

$$\Rightarrow \quad \lambda = 3 \text{ and } \mu = \frac{27}{2}$$

EXAMPLE 29 *For any two vectors $\vec{a}$ and $\vec{b}$, show that:*

$$\left(1 + |\vec{a}|^2\right)\left(1 + |\vec{b}|^2\right) = \{(1 - \vec{a} \cdot \vec{b})\}^2 + |\vec{a} + \vec{b} + (\vec{a} \times \vec{b})|^2 \quad \textbf{[CBSE 2002]}$$

SOLUTION We have,

$$(1 - \vec{a} \cdot \vec{b})^2 + |\vec{a} + \vec{b} + (\vec{a} \times \vec{b})|^2$$

$$= \{1 - 2(\vec{a}\cdot\vec{b}) + (\vec{a}\cdot\vec{b})^2\} + \{(\vec{a}+\vec{b}+\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b}+\vec{a}\times\vec{b})\}$$

$$= \{1 - 2(\vec{a}\cdot\vec{b}) + (\vec{a}\cdot\vec{b})^2\} + \{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) + (\vec{a}+\vec{b})\cdot(\vec{a}\times\vec{b}) + (\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b}) + (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})\}$$

$$= \{1 - 2(\vec{a}\cdot\vec{b}) + (\vec{a}\cdot\vec{b})^2\} + \{|\vec{a}+\vec{b}|^2 + \vec{a}\cdot(\vec{a}\times\vec{b}) + \vec{b}\cdot(\vec{a}\times\vec{b}) + (\vec{a}\times\vec{b})\cdot\vec{a} + (\vec{a}\times\vec{b})\cdot\vec{b} + |\vec{a}\times\vec{b}|^2\}$$

$$= \{1 - 2(\vec{a}\cdot\vec{b}) + (\vec{a}\cdot\vec{b})^2\} + \{|\vec{a}+\vec{b}|^2 + |\vec{a}\times\vec{b}|^2\}$$

$$[\because \vec{a}\perp(\vec{a}\times\vec{b}), \vec{b}\perp(\vec{a}\times\vec{b}) \therefore \vec{a}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{a}\times\vec{b}) = 0]$$

$$= 1 - 2(\vec{a}\cdot\vec{b}) + (\vec{a}\cdot\vec{b})^2 + |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{a}\times\vec{b}|^2$$

$$= 1 + |\vec{a}|^2 + |\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 + |\vec{a}\times\vec{b}|^2$$

$$= 1 + |\vec{a}|^2 + |\vec{b}|^2 + |\vec{a}|^2\,|\vec{b}^2| \qquad [\because (\vec{a}\cdot\vec{b})^2 + |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2]$$

$$= (1 + |\vec{a}|^2)(1 + |\vec{b}|^2)$$

Hence, $(1 + |\vec{a}|^2)(1 + |\vec{b}|^2) = 1 - (\vec{a}\cdot\vec{b})^2 + |\vec{a}+\vec{b}+\vec{a}\times\vec{b}|^2$

EXAMPLE 30 *ABCD is a quadrilateral such that $\vec{AB} = \vec{b}, \vec{AD} = \vec{d}, \vec{AC} = m\vec{b} + p\vec{d}$. Show that the area of the quadrilateral ABCD is*

$$\frac{1}{2}|m+p|\,|\vec{b}\times\vec{d}|$$

SOLUTION We have,

$\vec{AB} = \vec{b}, \vec{AD} = \vec{d}$ and $\vec{AC} = m\vec{b} + p\vec{d}$.

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Now, $\vec{AB} + \vec{BD} = \vec{AD}$

$\Rightarrow \quad \vec{BD} = \vec{AD} - \vec{AB}$

$\Rightarrow \quad \vec{BD} = \vec{d} - \vec{b}$

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$\therefore \quad \Delta =$ Area of quadrilateral $ABCD$

$\Rightarrow \quad \Delta = \frac{1}{2}|\vec{AC}\times\vec{BD}|$

$\Rightarrow \quad \Delta = \frac{1}{2}|(m\vec{b} + p\vec{d})\times(\vec{d} - \vec{b})|$

$\Rightarrow \quad \Delta = \frac{1}{2}|m(\vec{b}\times\vec{d}) - p(\vec{d}\times\vec{b})|$

$\Rightarrow \quad \Delta = \frac{1}{2}|m(\vec{b}\times\vec{d}) + p(\vec{b}\times\vec{d})|$

$\Rightarrow \quad \Delta = \frac{1}{2}|(m+p)(\vec{b}\times\vec{d})|$

$\Rightarrow \quad \Delta = \frac{1}{2}|m+p|\,|\vec{b}\times\vec{d}|$.

EXAMPLE 31 *If A, B, C, D be any four points in space, prove that*

$$|\vec{AB}\times\vec{CD} + \vec{BC}\times\vec{AD} + \vec{CA}\times\vec{BD}| = 4\,(\textit{Area of triangle ABC}).$$

SOLUTION Taking A as the origin, let the position vectors of B, C and D be $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively. Then,

$\overrightarrow{AB} = \vec{b}, \overrightarrow{CD} = \vec{d} - \vec{c}, \quad \overrightarrow{BC} = \vec{c} - \vec{b}, \overrightarrow{AD} = \vec{d}, \quad \overrightarrow{CA} = -\vec{c}$ and, $\overrightarrow{BD} = \vec{d} - \vec{b}$

$$\therefore \quad |\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}|$$

$$= |\vec{b} \times (\vec{d} - \vec{c}) + (\vec{c} - \vec{b}) \times \vec{d} + (-\vec{c}) \times (\vec{d} - \vec{b})|$$

$$= |\vec{b} \times \vec{d} - \vec{b} \times \vec{c} + \vec{c} \times \vec{d} - \vec{b} \times \vec{d} - \vec{c} \times \vec{d} + \vec{c} \times \vec{b}|$$

$$= |-\vec{b} \times \vec{c} + \vec{c} \times \vec{b}|$$

$$= |-2(\vec{b} \times \vec{c})|$$

$$= 2\,|\vec{b} \times \vec{c}|$$

$$= 2\,|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= 4 \text{ (Area of } \Delta ABC) \qquad \left[\because \text{ Area of } \Delta ABC = \frac{1}{2}\,|\overrightarrow{AB} \times \overrightarrow{AC}|\right]$$

EXAMPLE 32 *Let $\vec{a} = 2\hat{i} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ be three vectors. Find a vector $\vec{r}$ which satisfies $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0$.*

SOLUTION We have,

$$\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$$

$$\Rightarrow \quad \vec{r} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0}$$

$$\Rightarrow \quad (\vec{r} - \vec{c}) \times \vec{b} = \vec{0}$$

$\Rightarrow \quad \vec{r} - \vec{c}$ is parallel to $\vec{b}$

$\Rightarrow \quad \vec{r} - \vec{c} = \lambda \vec{b}$ for some scalar λ.

$$\Rightarrow \quad \vec{r} = \vec{c} + \lambda \vec{b}.$$

Now, $\quad \vec{r} \cdot \vec{a} = 0$

$$\Rightarrow \quad (\vec{c} + \lambda \vec{b}) \cdot \vec{a} = 0$$

$$\Rightarrow \quad (\vec{c} \cdot \vec{a}) + \lambda (\vec{b} \cdot \vec{a}) = 0$$

$$\Rightarrow \quad \lambda = -\left(\frac{\vec{c} \cdot \vec{a}}{\vec{b} \cdot \vec{a}}\right)$$

$$\Rightarrow \quad \lambda = -\left(\frac{8+7}{2+0+1}\right) = -\frac{15}{3} = -5$$

$$\therefore \quad \vec{r} = \vec{c} - 5\vec{b} = (4\hat{i} - 3\hat{j} + 7\hat{k}) - 5(\hat{i} + \hat{j} + \hat{k}) = -\hat{i} - 8\hat{j} + 2\hat{k}$$

EXAMPLE 33 *If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{j} - \hat{k}$ are given vectors, then find a vector $\vec{b}$ satisfying the equations $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a} \cdot \vec{b} = 3$.* **[CBSE 2008]**

SOLUTION Let $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$. Then,

$$\vec{a} \times \vec{b} = \vec{c}$$

$$\Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = (\hat{j} - \hat{k})$$

$$\Rightarrow \quad (z - y)\hat{i} - (z - x)\hat{j} + (y - x)\hat{k} = \hat{j} - \hat{k}$$

$$\Rightarrow \quad z - y = 0, -(z - x) = 1, y - x = -1$$

$$\Rightarrow \quad y = z, x - z = 1, x - y = 1$$

$$\Rightarrow \quad x - z = 1, x - y = 1 \qquad [\because \text{ These two equations are equivalent to } y = z]$$

$\Rightarrow \quad z = x - 1$ and $y = x - 1.$ ...(i)

Now, $\quad \vec{a}.\vec{b} = 3$

$\Rightarrow \quad (\hat{i}+\hat{j}+\hat{k}).(x\hat{i}+y\hat{j}+z\hat{k}) = 3$

$\Rightarrow \quad x+y+z = 3$

$\Rightarrow \quad x+x-1+x-1 = 3$ [Using (i)]

$\Rightarrow \quad 3x = 5 \Rightarrow x = \frac{5}{3}$

$\therefore \quad y = x-1 \Rightarrow y = \frac{5}{3}-1 = \frac{2}{3}.$

Also, $\quad y = z \Rightarrow z = \frac{2}{3}.$

Hence, $\quad \vec{b} = \frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}.$

EXERCISE 25.1

1. If $\vec{a} = \hat{i}+3\hat{j}-2\hat{k}$ and $\vec{b} = -\hat{i}+3\hat{k}$, find $|\vec{a}\times\vec{b}|$
2. (i) If $\vec{a} = 3\hat{i}+4\hat{j}$ and $\vec{b} = \hat{i}+\hat{j}+\hat{k}$, find the value of $|\vec{a}\times\vec{b}|$.
   (ii) If $\vec{a} = 2\hat{i}+\hat{k}, \vec{b} = \hat{i}+\hat{j}+\hat{k}$, find the magnitude of $\vec{a}\times\vec{b}$.
3. (i) Find a unit vector perpendicular to both the vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$
   (ii) Find a unit vector perpendicular to the plane containing the vectors $\vec{a} = 2\hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+\hat{k}$
4. Find the magnitude of $\vec{a} = (3\hat{k}+4\hat{j})\times(\hat{i}+\hat{j}-\hat{k})$
5. If $\vec{a} = 4\hat{i}+3\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}-2\hat{k}$, then *find* $|2\hat{b}\times\vec{a}|$.
6. If $\vec{a} = 3\hat{i}-\hat{j}-2\hat{k}$ and $\vec{b} = 2\hat{i}+3\hat{j}+\hat{k}$, find $(\vec{a}+2\vec{b})\times(2\vec{a}-\vec{b})$.
7. (i) Find a vector of magnitude 49, which is perpendicular to both the vectors $2\hat{i}+3\hat{j}+6\hat{k}$ and $3\hat{i}-6\hat{j}+2\hat{k}$.
   (ii) Find a vector whose length is 3 and which is perpendicular to the vectors $\vec{a} = 3\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b} = 6\hat{i}+5\hat{j}-2\hat{k}$.
8. Find the area of the parallelogram determined by the vectors :
   (i) $2\hat{i}$ and $3\hat{j}$
   (ii) $2\hat{i}+\hat{j}+3\hat{k}$ and $\hat{i}-\hat{j}$
   (iii) $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$
   (iv) $\hat{i}-3\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$. [CBSE 2002C]
9. Find the area of the parallelogram whose diagonals are :
   (i) $4\hat{i}-\hat{j}-3\hat{k}$ and $-2\hat{j}+\hat{j}-2\hat{k}$
   (ii) $2\hat{i}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$
   (iii) $3\hat{i}+4\hat{j}$ and $\hat{i}+\hat{j}+\hat{k}$
   (iv) $2\hat{i}+3\hat{j}+6\hat{k}$ and $3\hat{i}-6\hat{j}+2\hat{k}$

10. If $\vec{a} = 2\hat{i}+5\hat{j}-7\hat{k}, \vec{b} = -3\hat{i}+4\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}-2\hat{j}-3\hat{k}$, compute $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$ and verify that these are not equal.
11. If $|\vec{a}| = 2, |\vec{b}| = 5$ and $|\vec{a}\times\vec{b}| = 8$, find $\vec{a}\cdot\vec{b}$.
12. Given $\vec{a} = \frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}), \vec{b} = \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}), \vec{c} = \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$, $\hat{i}, \hat{j}, \hat{k}$ being a right handed orthogonal system of unit vectors in space, show that $\vec{a}, \vec{b}, \vec{c}$ is also another system.
13. If $|\vec{a}| = 13, |\vec{b}| = 5$ and $\vec{a}\cdot\vec{b} = 60$, then find $|\vec{a}\times\vec{b}|$.
14. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ if $|\vec{a}\times\vec{b}| = \vec{a}\cdot\vec{b}$.
15. If $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} \neq 0$, then show that $\vec{a}+\vec{c} = m\vec{b}$, where m is any scalar.
16. If $|\vec{a}| = 2, |\vec{b}| = 7$ and $\vec{a}\times\vec{b} = 3\hat{i}+2\hat{j}+6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$.
17. What inference can you draw if $\vec{a}\times\vec{b} = \vec{0}$ and $\vec{a}\cdot\vec{b} = 0$.
18. If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that $\vec{a}\times\vec{b} = \vec{c}, \vec{b}\times\vec{c} = \vec{a}, \vec{c}\times\vec{a} = \vec{b}$. Show that $\vec{a}, \vec{b}, \vec{c}$ form an orthonormal right handed triad of unit vectors.
19. Find a unit vector perpendicular to the plane ABC, where the coordinates of A, B and C are $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$.
20. If a, b, c are the lengths of sides, BC, CA and AB of a triangle ABC, prove that
$\vec{BC} + \vec{CA} + \vec{AB} = \vec{0}$ and deduce that
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
21. If $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$, and $\vec{b} = 2\hat{i}+3\hat{j}-5\hat{k}$, then find $\vec{a}\times\vec{b}$. Verify that $\vec{a}$ and $\vec{a}\times\vec{b}$ are perpendicular to each other.
22. If $\vec{p}$ and $\vec{q}$ are unit vectors forming an angle of 30°; find the area of the parallelogram having $\vec{a} = \vec{p}+2\vec{q}$ and $\vec{b} = 2\vec{p}+\vec{q}$ as its diagonals.
23. For any two vectors $\vec{a}$ and $\vec{b}$, prove that
$$|\vec{a}\times\vec{b}|^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} \end{vmatrix}$$
24. Define $\vec{a}\times\vec{b}$ and prove that $|\vec{a}\times\vec{b}| = (\vec{a}\cdot\vec{b})\tan\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.
25. If $|\vec{a}| = \sqrt{26}, |\vec{b}| = 7$ and $|\vec{a}\times\vec{b}| = 35$, find $\vec{a}\cdot\vec{b}$. **[CBSE 2002]**
26. Find the area of the triangle formed by O, A, B when $\vec{OA} = \hat{i}+2\hat{j}+3\hat{k}$, $\vec{OB} = -3\hat{i}-2\hat{j}+\hat{k}$.
27. Let $\vec{a} = \hat{i}+4\hat{j}+2\hat{k}$, $\vec{b} = 3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}\cdot\vec{d} = 15$. **[CBSE 2010]**
28. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$.
29. Using vectors find the area of the triangle with vertices, $A(2, 3, 5)$, $B(3, 5, 8)$ and $C(2, 7, 8)$. **[CBSE 2010]**
30. Let $\vec{a} = \hat{i}+4\hat{j}+2\hat{k}, \vec{b} = 3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}\cdot\vec{d} = 15$. **[NCERT]**

31. Given that $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$? **[NCERT]**

32. If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{0}$. Is the converse true? Justify your answer with an example. **[NCERT]**

33. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then verify that $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$. **[NCERT]**

ANSWERS

1. $\sqrt{91}$ 2. (i) $\sqrt{26}$ (ii) $\sqrt{6}$
3. (i) $\frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})$ (ii) $\pm\frac{1}{\sqrt{11}}(\hat{i}+\hat{j}-3\hat{k})$
4. $\sqrt{74}$ 5. $\sqrt{504}$ 6. $-25\hat{i}+35\hat{j}-55\hat{k}$
7. $42\hat{i}+14\hat{j}-21\hat{k}$
8. (i) 6 sq. units (ii) $3\sqrt{3}$ sq. units (iii) $\frac{5\sqrt{3}}{2}$ sq. units (iv) $4\sqrt{2}$ sq. units
9. (i) $\frac{15}{2}$ sq. units (ii) $\frac{\sqrt{6}}{2}$ sq. units (iii) $\frac{\sqrt{74}}{2}$ sq. units (iv) $\frac{7\sqrt{19}}{2}$ sq. units
11. 6 13. 25 14. $\frac{\pi}{4}$ 16. $\frac{\pi}{6}$
17. either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ 19. $\frac{1}{\sqrt{165}}(10\hat{i}+7\hat{j}-4\hat{k})$
21. $\hat{i}+11\hat{j}+7\hat{k}$ 22. $\frac{3}{4}$ sq. units 25. 7 26. $3\sqrt{5}$ sq. units
27. $\frac{1}{3}(160\hat{i}-5\hat{j}-70\hat{k})$ 28. $\frac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})$ 30. $\frac{1}{3}(160\hat{i}-5\hat{j}+70\hat{k})$
31. Either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ 32. No, Take any two collinear vectors

HINTS TO SELECTED PROBLEMS

11. Use : $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2=|\vec{a}|^2\,|\vec{b}|^2$

12. Show that $\vec{a},\vec{b},\vec{c}$ satisfy $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$,
$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$ and $\vec{a}\times\vec{b}=\vec{c}$, $\vec{b}\times\vec{c}=\vec{a}$, and $\vec{c}\times\vec{a}=\vec{b}$

13. Use : $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2=|\vec{a}|^2\,|\vec{b}|^2$

14. We have,
$$|\vec{a}\times\vec{b}|=\vec{a}\cdot\vec{b}$$
$$\Rightarrow |\vec{a}|\,|\vec{b}|\sin\theta=|\vec{a}|\,|\vec{b}|\cos\theta \Rightarrow \sin\theta=\cos\theta \Rightarrow \tan\theta=1 \Rightarrow \theta=\frac{\pi}{4}$$

15. We have,
$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}$
$\Rightarrow \vec{a}\times\vec{b}=-\vec{c}\times\vec{b}$
$\Rightarrow \vec{a}\times\vec{b}+\vec{c}\times\vec{b}=\vec{0}$
$\Rightarrow (\vec{a}+\vec{c})\times\vec{b}=\vec{0}$
$\Rightarrow \vec{a}+\vec{c}\parallel\vec{b} \Rightarrow \vec{a}+\vec{c}=m\vec{b}$ for some scalar m.

16. We have,

$|\vec{a}\times\vec{b}| = |3\hat{i}+2\hat{j}+6\hat{k}| = \sqrt{9+4+36} = 7.$

Now,

$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$

$\Rightarrow 7 = 2\times7\times\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

17. We have,

$\vec{a}\times\vec{b}=\vec{0} \Rightarrow \vec{a}=\vec{0}$ or, $\vec{b}=\vec{0}$ or, $\vec{a}\,||\,\vec{b}$ and, $\vec{a}\cdot\vec{b}=0 \Rightarrow \vec{a}=\vec{0}$ or, $\vec{b}=\vec{0}$ or, $\vec{a}\perp\vec{b}.$

Since that $\vec{a}$ cannot be both $\perp$ and $||$ to $\vec{b}.$ Therefore, $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$

19. Required vector $= \dfrac{\vec{AB}\times\vec{AC}}{|\vec{AB}\times\vec{AC}|}$.

29. Required area $= \frac{1}{2}|\vec{AB}\times\vec{AC}|$ or, $\frac{1}{2}|\vec{BC}\times\vec{BA}|$ or, $\frac{1}{2}|\vec{CB}\times\vec{CA}|$.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Define vector product of two vectors.
2. Write the value $(\hat{i}\times\hat{j}).\hat{k}+\hat{i}.\hat{j}$.
3. Write the value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{k}\times\hat{i})+\hat{k}.(\hat{j}\times\hat{i})$.
4. Write the value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{k}\times\hat{i})+\hat{k}.(\hat{i}\times\hat{j})$.
5. Write the value of $\hat{i}\times(\hat{j}+\hat{k})+\hat{j}\times(\hat{k}+\hat{i})+\hat{k}\times(\hat{i}+\hat{j})$.
6. Write the expression for the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as its diagonals.
7. For any two vectors $\vec{a}$ and $\vec{b}$ write the value of $(\vec{a}.\vec{b})^2+|\vec{a}\times\vec{b}|^2$ in terms of their magnitudes.
8. If $\vec{a}$ and $\vec{b}$ are two vectors of magnitudes 3 and $\frac{\sqrt{2}}{3}$ respectively such that $\vec{a}\times\vec{b}$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}.$
9. If $|\vec{a}|=10$, $|\vec{b}|=2$ and $|\vec{a}\times\vec{b}|=16$, find $\vec{a}.\vec{b}.$
10. For any two vectors $\vec{a}$ and $\vec{b}$, find $\vec{a}.(\vec{b}\times\vec{a})$.
11. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}\times\vec{b}|=\sqrt{3}$ and $\vec{a}.\vec{b}=1$, find the angle between.
12. For any three vectors write the value of $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$.
13. For any two vectors $\vec{a}$ and $\vec{b}$, find $(\vec{a}\times\vec{b}).\vec{b}.$
14. Write the value of $\hat{i}\times(\hat{j}\times\hat{k})$.
15. If $\vec{a}=3\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-\hat{k}$, then find $(\vec{a}\times\vec{b}).\vec{a}.$
16. Write a unit vector perpendicular to $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$.
17. If $|\vec{a}\times\vec{b}|^2+(\vec{a}.\vec{b})^2=144$ and $|\vec{a}|=4$, find $|\vec{b}|$.
18. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then write the value of $|\vec{r}\times\hat{i}|^2$.
19. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $\vec{a}\times\vec{b}$ is also a unit vector, find the angle between $\vec{a}$ and $\vec{b}.$

20. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$, write the angle between $\vec{a}$ and $\vec{b}$.
21. If $\vec{a}$ and $\vec{b}$ are unit vectors, then write the value of $|\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2$.
22. If $\vec{a}$ is a unit vector such that $\vec{a}\times\hat{i} = \hat{j}$, find $\vec{a}\cdot\hat{i}$.
23. If $\vec{c}$ is a unit vector perpendicular to the vectors $\vec{a}$ and $\vec{b}$, write another unit vector perpendicular to $\vec{a}$ and $\vec{b}$.
24. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $|\vec{a}\times\vec{b}| = \sqrt{3}$. [CBSE 2009]
25. Vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = \frac{2}{3}$ and $(\vec{a}\times\vec{b})$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$. [CBSE 2010]
26. Find λ, if $(2\hat{i} + 6\hat{j} + 14\hat{k}) \times (\hat{i} - \lambda\hat{j} + 7\hat{k}) = \vec{0}$. [CBSE 2010]

ANSWERS

2. 1 3. 1 4. 3 5. 0 6. $\frac{1}{2}\,|\vec{a}\times\vec{b}|$

7. $|\vec{a}|^2\,|\vec{b}|^2$ 8. 45°, 135° 9. ± 12 10. 0 11. 60°

12. $\vec{0}$ 13. $\vec{0}$ 14. $\vec{0}$ 15. Not meaningful 16. $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} + \hat{k})$ 17. 3

18. $y^2 + z^2$ 19. $\frac{\pi}{2}$ 20. $\frac{\pi}{4}$ 21. 1 22. 0 23. $-\vec{c}$ 24. $\pi/3$ 25. $\pi/3$ 26. -3

HINTS TO SELECTED PROBLEM

24. We have, $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $|\vec{a}\times\vec{b}| = \sqrt{3}$

Now, $|\vec{a}\times\vec{b}| = \sqrt{3} \Rightarrow |\vec{a}|\,|\vec{b}|\sin\theta = \sqrt{3} \Rightarrow 2\sin\theta = \sqrt{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $\vec{a}$ is any vector, then $(\vec{a}\times\hat{i})^2 + (\vec{a}\times\hat{j})^2 + (\vec{a}\times\hat{k})^2 =$
   (a) $\vec{a}^2$ (b) $2\vec{a}^2$ (c) $3\vec{a}^2$ (d) $4\vec{a}^2$
2. If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, then
   (a) either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{c}$ (b) $\vec{a} \parallel (\vec{b} - \vec{c})$ (c) $\vec{a} \perp (\vec{b} - \vec{c})$ (d) none of these

   or

   If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, $\vec{a} \neq 0$, then
   (a) $\vec{b} = \vec{c}$ (b) $\vec{b} = \vec{0}$ (c) $\vec{b} + \vec{c} = \vec{0}$ (d) none of these
3. The vector $\vec{b} = 3\hat{i} + 4\hat{k}$ is to be written as the sum of a vector $\vec{\alpha}$ parallel to $\vec{a} = \hat{i} + \hat{j}$ and a vector $\vec{\beta}$ perpendicular to $\vec{a}$. Then $\vec{\alpha} =$
   (a) $\frac{3}{2}(\hat{i} + \hat{j})$ (b) $\frac{2}{3}(\hat{i} + \hat{j})$ (c) $\frac{1}{2}(\hat{i} + \hat{j})$ (d) $\frac{1}{3}(\hat{i} + \hat{j})$
4. The unit vector perpendicular to the plane passing through points $P(\hat{i} - \hat{j} + 2\hat{k})$, $Q(2\hat{i} - \hat{k})$ and $R(2\hat{j} + \hat{k})$ is
   (a) $2\hat{i} + \hat{j} + \hat{k}$ (b) $\sqrt{6}(2\hat{i} + \hat{j} + \hat{k})$ (c) $\frac{1}{\sqrt{6}}(2\hat{i} + \hat{j} + \hat{k})$ (d) $\frac{1}{6}(2\hat{i} + \hat{j} + \hat{k})$
5. If $\vec{a}, \vec{b}$ represent the diagonals of a rhombus, then
   (a) $\vec{a}\times\vec{b} = \vec{0}$ (b) $\vec{a}\cdot\vec{b} = 0$ (c) $\vec{a}\cdot\vec{b} = 1$ (d) $\vec{a}\times\vec{b} = \vec{a}$

6. Vectors $\vec{a}$ and $\vec{b}$ are inclined at angle $\theta = 120°$.
If $|\vec{a}| = 1$, $|\vec{b}| = 2$, then $[(\vec{a} + 3\vec{b}) \times (3\vec{a} - \vec{b})]^2$ is equal to
(a) 300 (b) 325 (c) 275 (d) 225

7. If $\vec{a} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$, then a unit vector normal to the vectors $\vec{a} + \vec{b}$ and $\vec{b} - \vec{c}$ is
(a) $\hat{i}$ (b) $\hat{j}$ (c) $\hat{k}$ (d) none of these

8. A unit vector perpendicular to both $\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$ is
(a) $\hat{i} - \hat{j} + \hat{k}$ (b) $\hat{i} + \hat{j} + \hat{k}$ (c) $\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$ (d) $\frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}$

9. If $\vec{a} = 2\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 4\hat{j} - 2\hat{k}$, then $\vec{a} \times \vec{b}$ is
(a) $10\hat{i} + 2\hat{j} + 11\hat{k}$ (b) $10\hat{i} + 3\hat{j} + 11\hat{k}$
(c) $10\hat{i} - 3\hat{j} + 11\hat{k}$ (d) $10\hat{i} - 3\hat{j} - 10\hat{k}$

10. If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors, then
(a) $\hat{i} \cdot \hat{j} = 1$ (b) $\hat{i} \cdot \hat{i} = 1$ (c) $\hat{i} \times \hat{j} = 1$ (d) $\hat{i} \times (\hat{j} \times \hat{k}) = 1$

11. If θ is the angle between the vectors $2\hat{i} - 2\hat{j} + 4\hat{k}$ and $3\hat{i} + \hat{j} + 2\hat{k}$, then $\sin\theta =$
(a) $\frac{2}{3}$ (b) $\frac{2}{\sqrt{7}}$ (c) $\frac{\sqrt{2}}{7}$ (d) $\sqrt{\frac{2}{7}}$

12. If $|\vec{a} \times \vec{b}| = 4$, $|\vec{a} \cdot \vec{b}| = 2$, then $|\vec{a}|^2 \, |\vec{b}|^2 =$
(a) 6 (b) 2 (c) 20 (d) 8

13. $(\vec{a} \times \vec{b})^2 = ?$
(a) $|\vec{a}|^2 + |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$ (b) $|\vec{a}|^2 \, |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$
(c) $|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a} \cdot \vec{b})$ (d) $|\vec{a}|^2 + |\vec{b}|^2 - \vec{a} \cdot \vec{b}$

14. The value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$, is
(a) 0 (b) -1 (c) 1 (d) 3

15. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to
(a) 0 (b) $\pi/4$ (c) $\pi/2$ (d) π

ANSWERS

1. (b) 2. (a) 3. (a) 4. (c) 5. (b) 6. (a) 7. (a) 8. (c)
9. (b) 10. (b) 11. (b) 12. (c) 13. (b) 14. (c) 15. (b)

SUMMARY

1. If $\vec{a}, \vec{b}$ are two vectors inclined at an angle θ, then $\vec{a} \times \vec{b} = |\vec{a}| \, |\vec{b}| \sin\theta \, \hat{n}$, where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{n}$ form a right handed system.
2. $\vec{a} \times \vec{b} = \vec{0}$ iff $\vec{a}$ and $\vec{b}$ are parallel.
3. $\vec{a} \times \vec{b} =$ Vector area of the parallelogram having two adjacent sides as $\vec{a}$ and $\vec{b}$.

4. Unit vectors perpendicular to the plane of $\vec{a}$ and $\vec{b}$ are $\pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

5. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

6. (i) $m(\vec{a} \times \vec{b}) = m\vec{a} \times \vec{b} = \vec{a} \times m\vec{b}$, for any scalar m

   (ii) $m\vec{a} \times n\vec{b} = mn(\vec{a} \times \vec{b}) = n\vec{a} \times m\vec{b} = mn\vec{a} \times \vec{b} = \vec{a} \times mn\vec{b}$ for scalars m, n.

7. $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$, where $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

8. For any two vectors $\vec{a}$ and $\vec{b}$, we have $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

9. For any vector $\vec{a}$, we have $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2|\vec{a}|^2$

10. (i) Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}| = \frac{1}{2}|\overrightarrow{CB} \times \overrightarrow{CA}|$

    (ii) Area of a plane convex quadrilateral $ABCD$

    $= \frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$, where AC and BD are diagonal.

11. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C of ΔABC, then

    Area of $\Delta ABC = \frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$

    Length of the perpendicular from C on AB $= \dfrac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}{|\vec{a} - \vec{b}|}$

    Length of the perpendicular from A on BC $= \dfrac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}{|\vec{b} - \vec{c}|}$

    Length of the perpendicular from B on AC $= \dfrac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}{|\vec{c} - \vec{a}|}$

26

DIRECTION COSINES AND DIRECTION RATIOS

26.1 INTRODUCTION

In class XI, we have had a brief introduction of three dimensional geometry in which we used cartesian methods only. In previous chapters, we have studied some basic concepts of vectors. In this chapter and two more chapters to follow, we will use vector algebra to three dimensional geometry.

26.2 RECAPITULATION

26.2.1 COORDINATES OF A POINT IN SPACE

In this section, we will recapitulate various concepts learnt in class XI. We have leant in class XI that three mutually perpendicular lines in space define three mutually perpendicular planes which in turn divide the space into eight parts known as *octants* and the lines are known as the coordinate axes.

Let $X'OX$, $Y'OY$ and $Z'OZ$ be three mutually perpendicular lines intersecting at O such that two of them viz. $Y'OY$ and $Z'OZ$ lie in the plane of the paper and the third $X'OX$ is perpendicular to the plane of the paper and is projecting out from the plane of the paper (see Fig. 26.1). Let O be the origin and the lines $X'OX$, $Y'OY$ and $Z'OZ$ be x-axis, y-axis and z-axis. respectively. These three lines are also called the *rectangular axes of coordinates*. The planes containing the lines $X'OX$, $Y'OY$ and $Z'OZ$ in pairs, determine three mutually perpendicular planes XOY, YOX and ZOZ or simply XY, YZ and ZX which are called *rectangular coordinate planes*.

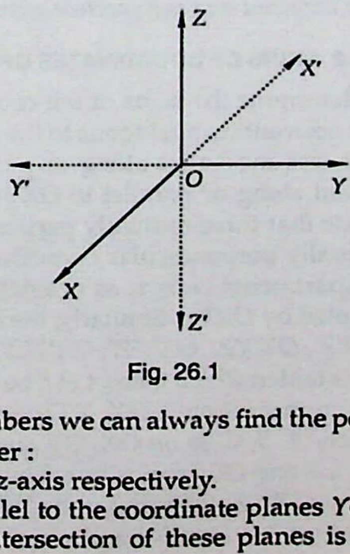

Fig. 26.1

Let P be a point in space (Fig. 26.2). Through P draw three planes parallel to the coordinate planes to meet the axes in A, B and C respectively. Let $OA = x$, $OB = y$ and $OC = z$. These three real numbers taken in this order determined by the point P are called the coordinates of the point P, written as (x, y, z), x, y, z are positive or negative according as they are measured along positive or negative directions of the coordinate axes.

Conversely, given an ordered triad (x, y, z) of real numbers we can always find the point whose coordinates are (x, y, z) in the following manner :

(i) Measure OA, OB, OC along x-axis, y-axis and z-axis respectively.

(ii) Through the points A, B, C draw planes parallel to the coordinate planes YOZ, ZOX and XOY respectively. The point of intersection of these planes is the required point P.

To give another explanation about the coordinates of a point P we draw three planes through P parallel to the coordinate planes. These three planes determine a rectangular parallelopiped which has three pairs of rectangular faces, viz. $PB'AC'$, $OCA'B$; $PA'BC'$, $OAB'C$; $PA'CB'$, $OAC'B$ as shown in Fig. 26.2. Then, we have

$$x = OA = CB' = PA' = \text{perpendicular distance from } P \text{ on the } YOZ \text{ plane;}$$

$$y = OB = A'C = PB' = \text{perpendicular distance from } P \text{ on the } ZOX \text{ plane;}$$

$$z = OC = A'B = PC' = \text{perpendicular distance from } P \text{ on the } XOY \text{ plane.}$$

Thus, *the coordinates of the point P are the perpendicular distances from P on the three mutually rectangular coordinate planes YOZ, ZOX and XOY respectively.*

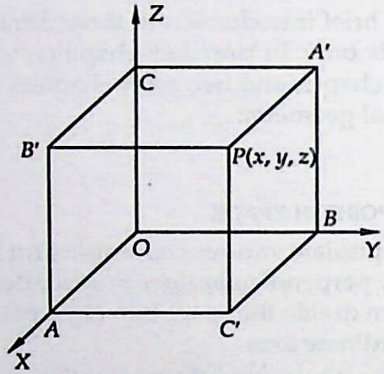

Fig. 26.2

Further, since the line PA lies in the plane $PB'AC'$ which is perpendicular to the line OA, we have PA perpendicular to OA. Similarly, PB perpendicular to OB and PC perpendicular to OC.

Thus, *the coordinates of a point are the distances from the origin of the feet of the perpendiculars from the point on the respective coordinate axes.*

26.2.2 SIGNS OF COORDINATES OF A POINT

To determine the signs of the coordinates of a point in three dimension we follow the sign convention analogous to the sign convention in two dimensional geometry that all distances measured along or parallel to OX, OY, OZ will be positive and distances moved along or parallel to OX', OY', OZ' will be negative. As discussed in previous article that three mutually perpendicular lines $X'OX$, $Y'OY$ and $Z'OZ$ determine three mutually perpendicular coordinate planes which in turn divide the space into eight compartments known as octants. The octant having OX, OY and OZ as its edges is denoted by $OXYZ$. Similarly, the other octants are denoted by $OX'YZ$, $OXY'Z$, $OX'Y'Z$, $OXYZ'$, $OX'YZ'$, $OXY'Z'$, $OX'Y'Z'$. The signs of the coordinates of a point depend upon the octant in which it lies. Let P be a point and let A, B, C be the feet of the perpendiculars drawn from P on $X'OX$, $Y'OY$ and $Z'OZ$ respectively. If P lies in octant $OXYZ$, then clearly A, B, C lie on OX, OY and OZ respectively. Therefore, by our sign convention OA, OB and OC are positive. Thus all the three coordinates of P are positive. If P lies in octant $OX'YZ$, then A, B and C lie on OX', OY and OZ respectively. Therefore, x-coordinate of P is negative and y and z coordinates are positive.

The following table shows the signs of coordinates of points in various octants :

Coordinate \ Octant	OXYZ	OX′YZ	OXY′Z	OX′Y′Z	OXYZ′	OX′YZ′	OXY′Z′	OX′Y′Z′
x	+	−	+	−	+	−	+	−
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

REMARK 1 *If a point P lies in x y-plane, then by the definition of coordinates of a point, z-coordinate of P is zero. Therefore, the coordinates of a point on xy-plane are of the form (x, y, 0) and we may take the equation of xy-plane as z = 0. Similarly the coordinates of any point in yz and zx-planes are of the forms (0, y, z) and (x, 0, z) respectively and their equations may be taken as x = 0 and y = 0 respectively.*

REMARK 2 *If a point lies on the x-axis, then its y and z-coordinates are both zero. Therefore, the coordinates of a point on x-axis are of the form (x, 0, 0) and we may take the equation of x-axis as y = 0, z = 0. Similarly, the coordinates of a point on y and z-axes are of the form (0, y, 0) and (0, 0, z) respectively and their equations may be taken as x = 0, z = 0 and x = 0, y = 0 respectively.*

26.2.3 DISTANCE FORMULA

THEOREM *Prove that the distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by*

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

PROOF Let O be the origin and let $P(x_1, y_1\, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points.

Then,

$$\vec{OP} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k},\ \vec{OQ} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}$$

Now,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$\Rightarrow \vec{PQ} = (x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}) - (x_1 \hat{i} + y_1 \hat{j} + z_1) \hat{k}$$

$$\Rightarrow \vec{PQ} = (x_2 - x_1) \hat{i} + (y_2 - y_1) \hat{j} + (z_2 - z_1) \hat{k}$$

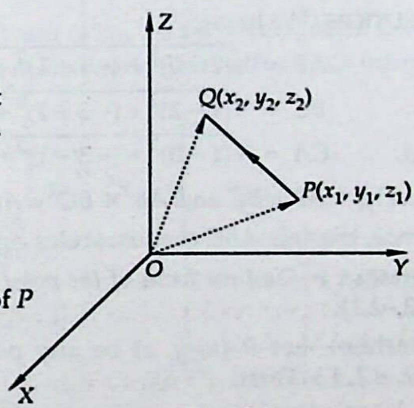

Fig. 26.3

$$\therefore\ PQ = |\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Hence, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the distance between the points $P(-2, 4, 1)$ and $Q(1, 2, -5)$.*

SOLUTION We have,

$$PQ = \sqrt{(1-(-2))^2 + (2-4)^2 + (-5-1)^2} = \sqrt{9+4+36} = 7 \text{ units}$$

EXAMPLE 2 *Prove by using distance formula that the points $P(1, 2, 3)$, $Q(-1, -1, -1)$ and $R(3, 5, 7)$ are collinear.*

SOLUTION We have,

$$PQ = \sqrt{(-1-1)^2 + (-1-2)^2 + (-1-3)^2} = \sqrt{4+9+16} = \sqrt{29},$$

$$QR = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2} = \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$$

and, $PR = \sqrt{(3-1)^2 + (5-2)^2 + (7-3)^2} = \sqrt{4+9+16} = \sqrt{29}$

Since $QR = PQ + PR$. Therefore, the given points are collinear.

EXAMPLE 3 *Determine the point in XY-plane which is equidistant from three points A (2, 0, 3), B (0, 3, 2) and C (0, 0, 1).*

SOLUTION We know that z-coordinate of every point on xy-plane is zero. So, let $P(x, y, 0)$ be a point on xy-plane such that $PA = PB = PC$.

Now, $PA = PB$

$\Rightarrow \quad PA^2 = PB^2$

$\Rightarrow \quad (x-2)^2 + (y-0)^2 + (0-3)^2 = (x-0)^2 + (y-3)^2 + (0-2)^2$

$\Rightarrow \quad 4x - 6y = 0 \Rightarrow 2x - 3y = 0$ ...(i)

$PB = PC$

$\Rightarrow \quad PB^2 = PC^2$

$\Rightarrow \quad (x-0)^2 + (y-3)^2 + (0-2)^2 = (x-0)^2 + (y-0)^2 + (0-1)^2$

$\Rightarrow \quad -6y + 12 = 0 \Rightarrow y = 2$ ...(ii)

Putting $y = 2$ in (i), we obtain $x = 3$.

Hence, the required point is (3, 2, 0).

EXAMPLE 4 *Show that the points A (0, 1, 2), B (2, – 1, 3) and C (1, – 3, 1) are vertices of an isosceles right-angled triangle.*

SOLUTION We have,

$$AB = \sqrt{(2-0)^2 + (-1-1)^2 + (3-2)^2} = \sqrt{4+4+1} = 3,$$

$$BC = \sqrt{(1-2)^2 + (-3+1)^2 + (1-3)^2} = \sqrt{1+4+4} = 3$$

and, $$CA = \sqrt{(1-0)^2 + (-3-1)^2 + (1-2)^2} = \sqrt{1+16+1} = 3\sqrt{2}$$

Clearly, $AB = BC$ and $AB^2 + BC^2 = AC^2$.

Hence, triangle ABC is an isosceles right-angled triangle.

EXAMPLE 5 *Find the locus of the point which is equidistant from the points A (0, 2, 3) and B (2,–2,1).*

SOLUTION Let $P(x, y, z)$ be any point which is equidistant from $A(0, 2, 3)$ and $B(2, -2, 1)$ .Then,

$PA = PB$

$\Rightarrow \quad PA^2 = PB^2$

$\Rightarrow \quad \sqrt{(x-0)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-2)^2 + (y+2)^2 + (z-1)^2}$

$\Rightarrow \quad 4x - 8y - 4z + 4 = 0 \Rightarrow x - 2y - z + 1 = 0$

Hence, the required locus is $x - 2y - z + 1 = 0$.

EXAMPLE 6 *Find the coordinates of a point equidistant from the four points O (0, 0, 0), A (a, 0, 0), B (0, b, 0) and C (0, 0, c).*

SOLUTION Let $P(x, y, z)$ be the required point. Then, $OP = PA = PB = PC$.

Now, $OP = PA$

$\Rightarrow \quad OP^2 = PA^2$

$\Rightarrow \quad x^2 + y^2 + z^2 = (x-a)^2 + (y-0)^2 + (z-0)^2$

$\Rightarrow \quad 0 = -2ax + a^2 \Rightarrow x = a/2$

Similarly, $OP = PB \Rightarrow y = b/2$ and $OP = PC \Rightarrow z = c/2$.

Hence, the coordinates of the required point are $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

EXAMPLE 7 *Using vector method : prove that the points* $A(3,-2,4)$, $B(1,1,1)$ *and* $C(-1,4,-2)$ *are collinear.*

SOLUTION We have,

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$\Rightarrow \quad \overrightarrow{AB} = (\hat{i}+\hat{j}+\hat{k}) - (3\hat{i}-2\hat{j}+4\hat{k}) = -2\hat{i}+3\hat{j}-3\hat{k}$

and, $\quad \overrightarrow{BC} = (-\hat{i}+4\hat{j}-2\hat{k}) - (\hat{i}+\hat{j}+\hat{k}) = -2\hat{i}+3\hat{j}-3\hat{k}$

Clearly, $\overrightarrow{AB} = \overrightarrow{BC}$

This shows that AB is parallel to BC. But, B is common to AB and BC.

Hence, A, B, C are collinear.

EXAMPLE 8 *Find the distance between the points A and B with position vectors* $\hat{i}-\hat{j}$ *and* $2\hat{i}+\hat{j}+2\hat{k}$.

SOLUTION We have,

$\overrightarrow{AB}$ = Position vector of B – Position vector of A

$\Rightarrow \quad \overrightarrow{AB} = (2\hat{i}+\hat{j}+2\hat{k}) - (\hat{i}-\hat{j}+0\hat{k}) = \hat{i}+2\hat{j}+2\hat{k}$

$\therefore \quad AB = |\overrightarrow{AB}| = \sqrt{1+4+4} = 3$

26.2.4 SECTION FORMULAS

THEOREM 1 *(Internal Division) Let* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *be two points. Let R be a point on the line segment joining P and Q such that it divides the join of P and Q internally in the ratio* $m_1 : m_2$. *Then, the coordinates of R are*

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1+m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1+m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1+m_2}\right)$$

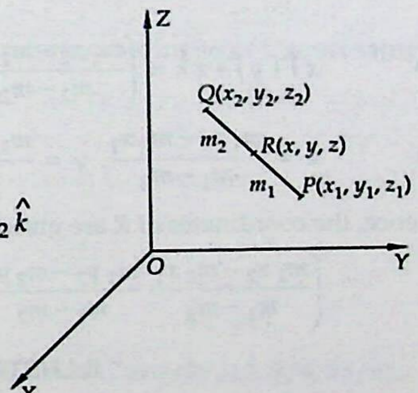

Fig. 26.4

PROOF Let the coordinates of R be (x, y, z). Let $\vec{r_1}, \vec{r_2}$ and $\vec{r}$ be the position vectors of P, Q and R respectively. Then,

$$\vec{r_1} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k}, \quad \vec{r_2} = x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$$

and, $\quad \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

Since R divides PQ internally in the ratio $m_1 : m_2$.

$\therefore$ Position vector $\vec{r}$ of point R is given by

$$\vec{r} = \frac{m_1\vec{r_2}+m_2\vec{r_1}}{m_1+m_2}$$

$$\Rightarrow \quad x\hat{i}+y\hat{j}+z\hat{k} = \frac{m_1(x_2\hat{i}+y_2\hat{j}+z_2\hat{k}) + m_2(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})}{m_1+m_2}$$

$$\Rightarrow \quad x\hat{i}+y\hat{j}+z\hat{k} = \left(\frac{m_1 x_2 + m_2 x_1}{m_1+m_2}\right)\hat{i}+\left(\frac{m_1 y_2 + m_2 y_1}{m_1+m_2}\right)\hat{j}+\left(\frac{m_1 z_2 + m_2 z_1}{m_1+m_2}\right)\hat{k}$$

$$\Rightarrow \quad x = \frac{m_1 x_2 + m_2 x_1}{m_1+m_2}, \quad y = \frac{m_1 y_2 + m_2 y_1}{m_1+m_2}, \quad z = \frac{m_1 z_2 + m_2 z_1}{m_1+m_2}$$

Hence, the coordinates of R are

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1+m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1+m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1+m_2}\right)$$

Q.E.D.

<u>COROLLARY</u> *If R is the mid-point of the segment joining P (x_1, y_1, z_1) and Q (x_2, y_2, z_2), then $m_1 = m_2 = 1$ and the coordinates of R are given by*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

THEOREM 2 *(External Division) Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be two points, and let R be a point on PQ produced dividing it externally in the ratio $m_1 : m_2$ $(m_1 \neq m_2)$. Then, the coordinates of R are*

$$\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}\right)$$

<u>PROOF</u> Let the coordinates of R be (x, y, z). Let $\vec{r_1}, \vec{r_2}$ and $\vec{r}$ be the position vectors of P, Q and R respectively. Then ,

$$\vec{r_1} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}, \vec{r_2} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k} \text{ and } \vec{r} = x \hat{i} + y \hat{j} + z \hat{k}.$$

Since R divides PQ externally in the ratio $m_1 : m_2$. Therefore, position vector $\vec{r}$ of point R is given by

$$\vec{r} = \frac{m_1 \vec{r_2} - m_2 \vec{r_1}}{m_1 - m_2}$$

$$\Rightarrow \quad x \hat{i} + y \hat{j} + z \hat{k} = \frac{m_1 (x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}) - m_2 (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k})}{m_1 - m_2}$$

$$\Rightarrow \quad x \hat{i} + y \hat{j} + z \hat{k} = \left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}\right)\hat{i} + \left(\frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}\right)\hat{j} + \left(\frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}\right)\hat{k}$$

$$\Rightarrow \quad x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, z = \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$$

Hence, the coordinates of R are given by

$$\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}\right)$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the coordinates of the point which divides the join of P $(2, -1, 4)$ and Q $(4, 3, 2)$ in the ratio $2 : 3$ (i) internally (ii) externally.*

SOLUTION Let R (x, y, z) be the required point. Then,

(i) $$x = \frac{2 \times 4 + 3 \times 2}{2 + 3}, y = \frac{2 \times 3 + 3 \times -1}{2 + 3}, z = \frac{2 \times 2 + 3 \times 4}{2 + 3}$$

$$\Rightarrow \quad x = \frac{14}{5}, y = \frac{3}{5}, z = \frac{16}{5}$$

So, coordinates of the required point are $R\left(\frac{14}{5}, \frac{3}{5}, \frac{16}{5}\right)$

(ii) $$x = \frac{2 \times 4 - 3 \times 2}{2 - 3}, y = \frac{2 \times 3 - 3 \times -1}{2 - 3}, z = \frac{2 \times 2 - 3 \times 4}{2 - 3}$$

$$\Rightarrow \quad x = -2, y = -9, z = 8$$

So, coordinates of the required point are $R(-2, -9, 8)$.

EXAMPLE 2 *Find the ratio in which the line joining the points (1, 2, 3) and (– 3, 4, – 5) is divided by the xy-plane.. Also find the coordinates of the point of division.* **[HSB 1993]**

SOLUTION Suppose the line joining the points $P(1, 2, 3)$ and $Q(-3, 4-5)$ is divided by the xy-plane at a point R in the ratio $\lambda : 1$. Then, the coordinates of R are

$$\left(\frac{-3\lambda+1}{\lambda+1}, \frac{4\lambda+2}{\lambda+1}, \frac{-5\lambda+3}{\lambda+1}\right) \quad \text{...(i)}$$

Since R lies on xy-plane i.e. $z = 0$. Therefore,

$$\frac{-5\lambda+3}{\lambda+1} = 0 \Rightarrow \lambda = \frac{3}{5}$$

So, the required ratio is $\frac{3}{5} : 1$ or, $3 : 5$

Putting $\lambda = \frac{3}{5}$ in (i), we obtain the coordinates of R as $(-1/2, 11/4, 0)$.

EXAMPLE 3 *Find the ratio in which the join the A (2, 1, 5) and B (3, 4, 3) is divided by the plane* $2x + 2y - 2z = 1$. *Also, find the coordinates of the point of division.*

SOLUTION Suppose the plane $2x + 2y - 2z = 1$ divides the line joining the points $A(2, 1, 5)$ and $B(3, 4, 3)$ at a point C in the ratio $\lambda : 1$. Then, the coordinates of C are

$$\left(\frac{3\lambda+2}{\lambda+1}, \frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+5}{\lambda+1}\right) \quad \text{...(i)}$$

Since point C lies on the plane $2x + 2y - 2z = 1$. Therefore, coordinates of C must satisfy the equation of the plane

i.e. $$2\left(\frac{3\lambda+2}{\lambda+1}\right) + 2\left(\frac{4\lambda+1}{\lambda+1}\right) - 2\left(\frac{3\lambda+5}{\lambda+1}\right) = 1$$

$$\Rightarrow \quad 8\lambda - 4 = \lambda + 1 \Rightarrow \lambda = \frac{5}{7}$$

So, the required ratio is $\frac{5}{7} : 1$ or, $5 : 7$.

Putting $\lambda = \frac{5}{7}$ in (i), the coordinates of the point of division C are $(29/12, 9/4, 25/6)$.

EXAMPLE 4 *Using section formula, prove that the three points A (– 2, 3, 5), B (1, 2, 3) and C (7, 0, – 1) are collinear.*

SOLUTION Suppose the given points are collinear and C divides AB in the ratio $\lambda : 1$. Then, coordinates of C are

$$\left(\frac{\lambda-2}{\lambda+1}, \frac{2\lambda+3}{\lambda+1}, \frac{3\lambda+5}{\lambda+1}\right)$$

But, coordinates of C are $(7, 0, -1)$.

$$\therefore \quad \frac{\lambda-2}{\lambda+1} = 7, \frac{2\lambda+3}{\lambda+1} = 0 \text{ and } \frac{3\lambda+5}{\lambda+1} = -1$$

From each of these equations, we get $\lambda = -\frac{3}{2}$.

Since each of these equations give the same value of λ. Therefore, the given points are collinear and C divides AB externally in the ratio $3 : 2$.

EXAMPLE 5 *The mid-points of the sides of a triangle are* $(1, 5, -1)$, $(0, 4, -2)$ *and* $(2, 3, 4)$. *Find its vertices.* **[PSB 1990]**

SOLUTION Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of the given triangle, and let $D(1, 5, -1)$, $E(0, 4, -2)$ and $F(2, 3, 4)$ be the mid-points of the sides BC, CA and AB respectively.

Now, D is the mid-point of BC

$$\therefore \quad \frac{x_2 + x_3}{2} = 1, \ \frac{y_2 + y_3}{2} = 5, \ \frac{z_2 + z_3}{2} = -1$$

$$\Rightarrow \quad x_2 + x_3 = 2, \ y_2 + y_3 = 10, \ z_2 + z_3 = -2 \qquad \text{...(i)}$$

E is the mid-point of CA

$$\therefore \quad \frac{x_1 + x_3}{2} = 0, \ \frac{y_1 + y_3}{2} = 4, \ \frac{z_1 + z_3}{2} = -2$$

$$\Rightarrow \quad x_1 + x_3 = 0, \ y_1 + y_3 = 8, \ z_1 + z_3 = -4 \qquad \text{...(ii)}$$

F is the mid-point of AB

$$\therefore \quad \frac{x_1 + x_2}{2} = 2, \ \frac{y_1 + y_2}{2} = 3, \ \frac{z_1 + z_2}{2} = 4$$

$$\Rightarrow \quad x_1 + x_2 = 4, \ y_1 + y_2 = 6, \ z_1 + z_2 = 8 \qquad \text{...(iii)}$$

Adding first three equations in (i), (ii) and (iii), we obtain

$$2(x_1 + x_2 + x_3) = 2 + 0 + 4 \Rightarrow x_1 + x_2 + x_3 = 3.$$

Solving first three equations in (i), (ii) and (iii) with $x_1 + x_2 + x_3 = 3$, we obtain $x_1 = 1, x_2 = 3, x_3 = -1$.

Adding next three equations in (i), (ii) and (iii), we obtain

$$2(y_1 + y_2 + y_3) = 10 + 8 + 6 \Rightarrow y_1 + y_2 + y_3 = 12$$

Solving next three equations in (i), (ii) and (iii) with $y_1 + y_2 + y_3 = 12$, we obtain $y_1 = 2$, $y_2 = 4$, $y_3 = 6$.

Adding last three equations in (i), (ii) and (iii), we obtain

$$2(z_1 + z_2 + z_3) = -2 - 4 + 8 \Rightarrow z_1 + z_2 + z_3 = 1.$$

Solving last three equations in (i), (ii) and (iii) with $z_1 + z_2 + z_3 = 1$, we obtain

$$z_1 = 3, \ z_2 = 5, \ z_3 = -7.$$

Thus, the vertices of the triangle are $A(1, 2, 3)$, $B(3, 4, 5)$ and $C(-1, 6, -7)$.

EXAMPLE 6 *Given that* $P(3, 2, -4)$, $Q(5, 4, -6)$ *and* $R(9, 8, -10)$ *are collinear. Find the ratio in which* Q *divides* PR.

SOLUTION Suppose Q divides PR in the ratio $\lambda : 1$. Then, coordinates of Q are

$$\left(\frac{9\lambda + 3}{\lambda + 1}, \frac{8\lambda + 2}{\lambda + 1}, \frac{-10\lambda - 4}{\lambda + 1}\right)$$

But, coordinates of Q are $(5, 4, -6)$.

$$\therefore \quad \frac{9\lambda + 3}{\lambda + 1} = 5, \ \frac{8\lambda + 2}{\lambda + 1} = 4, \ \frac{-10\lambda - 4}{\lambda + 1} = 6.$$

These three equations give $\lambda = \frac{1}{2}$. So, Q divides PR in the ratio $\frac{1}{2} : 1$ or, $1 : 2$.

EXAMPLE 7 *Find the coordinates of the points which trisect the line segment AB, given that A (2, 1, – 3) and B (5, – 8, 3).*

SOLUTION Let P and Q be the points which trisect AB. Then, $AP = PQ = QB$. Therefore, P divides AB in the ratio 1 : 2 and Q divides it in the ratio 2 : 1.

As P divides AB in the ratio 1 : 2. So coordinates of P are

$$\left(\frac{1\times 5+2\times 2}{1+2}, \frac{1\times -8+2\times 1}{1+2}, \frac{1\times 3+2\times -3}{1+2}\right) \text{ or, } (3, -2, -1)$$

Since Q divides AB in the ratio 2 : 1. So coordinates of Q are

$$\left(\frac{2\times 5+1\times 2}{2+1}, \frac{2\times -8+1\times 1}{2+1}, \frac{2\times 3+1\times -3}{1+2}\right) \text{ or, } (4, -5, 1)$$

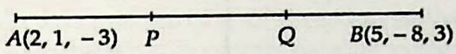

Fig. 26.5

EXAMPLE 8 *Show that the centroid of the triangle with vertices A (x_1, y_1, z_1), B (x_2, y_2, z_2) and C (x_3, y_3, z_3) is*

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

SOLUTION Let D be the mid-point of AC. Then, coordinates of D are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$.

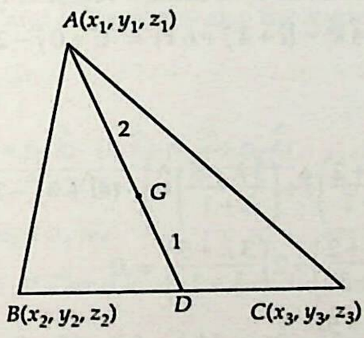

Fig. 26.6

Let G be the centroid of ΔABC. Then G divides AD in the ratio 2 : 1. So, coordinates of D are

$$\left(\frac{1\cdot x_1+2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1\cdot y_1+2\left(\frac{y_2+y_3}{2}\right)}{1+2}, \frac{1\cdot z_1+2\left(\frac{z_2+z_3}{2}\right)}{1+2}\right)$$

i.e. $$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

EXAMPLE 9 *Find the coordinates of the foot of the perpendicular drawn from the point A (1, 2, 1) to the line joining B (1, 4, 6) and C (5, 4, 4).* **[CBSE 1995]**

SOLUTION Let D be the foot of the perpendicular drawn from A on BC, and let D divide BC in the ratio $\lambda : 1$. Then, coordinates of D are

$$\left(\frac{5\lambda+1}{\lambda+1}, \frac{4\lambda+4}{\lambda+1}, \frac{4\lambda+6}{\lambda+1}\right) \quad \text{...(i)}$$

Now, $\vec{AD}$ = Position vector of D – Position vector of A

$$\Rightarrow \quad \vec{AD} = \left(\frac{5\lambda+1}{\lambda+1} - 1\right)\hat{i} + \left(\frac{4\lambda+4}{\lambda+1} - 2\right)\hat{j} + \left(\frac{4\lambda+6}{\lambda+1} - 1\right)\hat{k}$$

$$\Rightarrow \quad \vec{AD} = \left(\frac{4\lambda}{\lambda+1}\right)\hat{i} + \left(\frac{2\lambda+2}{\lambda+1}\right)\hat{j} + \left(\frac{3\lambda+5}{\lambda+1}\right)\hat{k}$$

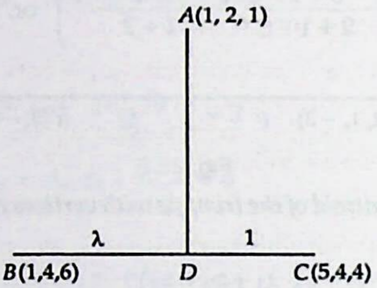

Fig. 26.7

and,

$\vec{BC}$ = Position vector of C – Position vector of B

$$\Rightarrow \quad \vec{BC} = (5\hat{i} + 4\hat{j} + 4\hat{k}) - (\hat{i} + 4\hat{j} + 6\hat{k}) = 4\hat{i} + 0\hat{j} - 2\hat{k}.$$

Since $\vec{AD} \perp \vec{BC}$.

$$\therefore \quad \vec{AD} \cdot \vec{BC} = 0$$

$$\Rightarrow \quad \left\{\left(\frac{4\lambda}{\lambda+1}\right)\hat{i} + \left(\frac{2\lambda+2}{\lambda+1}\right)\hat{j} + \left(\frac{3\lambda+5}{\lambda+1}\right)\hat{k}\right\} \cdot (4\hat{i} + 0\hat{j} - 2\hat{k}) = 0$$

$$\Rightarrow \quad 4\left(\frac{4\lambda}{\lambda+1}\right) + 0\left(\frac{2\lambda+2}{\lambda+1}\right) - 2\left(\frac{3\lambda+5}{\lambda+1}\right) = 0$$

$$\Rightarrow \quad \frac{16\lambda}{\lambda+1} + 0 - 2\frac{(3\lambda+5)}{\lambda+1} = 0 \Rightarrow 16\lambda - 6\lambda - 10 = 0 \Rightarrow \lambda = 1$$

Putting $\lambda = 1$ in (i), we obtain that the coordinates of D are $(3, 4, 5)$.

26.3 DIRECTION COSINES AND DIRECTION RATIOS OF A LINE

In chapter 23, we have learnt about the direction cosines and direction ratios of a vector. In this section, we will introduce the notion of direction cosines and direction ratios of a line.

DEFINITION *The direction cosines of a line are defined as the direction cosines of any vector whose support is the given line.*

It follows from the above definition if A and B are two points on a given line L, then direction cosines of vectors $\vec{AB}$ or, $\vec{BA}$ are the direction cosines of line L. Thus, if α, β, γ are the angles which the line L makes with positive directions of x-axis, y-axis and z-axis respectively, then its direction cosines are either, $\cos\alpha, \cos\beta \cos\gamma$ *or,* $-\cos\alpha, -\cos\beta, -\cos\gamma$.

Therefore, if l, m, n are direction cosines of a line, then $-l, -m, -n$ are also its direction cosines and we always have

$$l^2 + m^2 + n^2 = 1$$

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points on a line L, then its direction cosines are

$$\frac{x_2 - x_1}{AB}, \frac{y_2 - y_1}{AB}, \frac{z_2 - z_1}{AB} \text{ or } \frac{x_1 - x_2}{AB}, \frac{y_1 - y_2}{AB}, \frac{z_1 - z_2}{AB}.$$

DEFINITION *The direction ratios of a line are proportional to the direction ratios of any vector whose support is the given line*

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points on a line, then its direction ratios are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

26.4 ANGLE BETWEEN TWO VECTORS

In this section, we will find the formula for the angle between two vectors in terms of their direction cosines and also interms of their direction ratios. The angle between two lines is defined as the angle between two vectors parallel to them. So, the results derived for vectors will also be applicable to lines.

26.4.1 ANGLE BETWEEN TWO VECTORS IN TERMS OF THEIR DIRECTION COSINES

Let $\vec{a}$ and $\vec{b}$ be two vectors with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Then,

$$\hat{a} = \text{Unit vector along } \vec{a} = l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}$$

and, $$\hat{b} = \text{Unit vector along } \vec{b} = l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k}$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then, θ is also the angle between $\hat{a}$ and $\hat{b}$.

$$\therefore \quad \cos\theta = \frac{\hat{a} \cdot \hat{b}}{|\hat{a}| \cdot |\hat{b}|}$$

$$\Rightarrow \quad \cos\theta = \frac{(l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}) \cdot (l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k})}{(1)(1)} \quad [\because |\hat{a}| = |\hat{b}| = 1]$$

$$\Rightarrow \quad \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

Condition for perpendicularity : If $\vec{a}$ and $\vec{b}$ are perpendicular , then

$$\vec{a} \perp \vec{b}$$

$$\Leftrightarrow \quad \hat{a} \perp \hat{b}$$

$$\Leftrightarrow \quad \hat{a} \cdot \hat{b} = 0$$

$$\Leftrightarrow \quad (l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}) \cdot (l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k}) = 0$$

$$\Leftrightarrow \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0.$$

Condition for parallelism: If $\vec{a}$ and $\vec{b}$ are parallel, then

$$\Leftrightarrow \quad \hat{a} \text{ and } \hat{b} \text{ are parallel}$$

$$\Leftrightarrow \quad \hat{a} = \lambda \hat{b} \text{ for some scalar } \lambda$$

$$\Leftrightarrow \quad (l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}) = \lambda (l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k})$$

$$\Leftrightarrow \quad l_1 = \lambda l_2, m_1 = \lambda m_2, n_1 = \lambda n_2$$

$$\Leftrightarrow \quad \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

26.4.2 ANGLE BETWEEN TWO VECTORS IN TERMS OF THEIR DIRECTION RATIOS

Let $\vec{a}$ and $\vec{b}$ be two vectors with direction ratios proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively. Then,

$$\vec{A} = A \text{ vector along } \vec{a} = a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}$$

and, $$\vec{B} = A \text{ vector along } \vec{b} = a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}.$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then, θ is also the angle between $\vec{A}$ and $\vec{B}$.

$$\therefore \quad \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$

$$\Rightarrow \quad \cos\theta = \frac{(a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k})\cdot(a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k})}{|a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}|\,|a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}|}$$

$$\Rightarrow \quad \cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Condition for perpendicularity : We have,

$\vec{a}$ and $\vec{b}$ are perpendicular

$\Leftrightarrow \quad \vec{A} \perp \vec{B}$

$\Leftrightarrow \quad \vec{A}\cdot\vec{B} = 0$

$\Leftrightarrow \quad a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

Condition for parallelism : We have,

$\vec{a}$ and $\vec{b}$ are parallel

$\Leftrightarrow \quad \vec{A}$ and $\vec{B}$ are parallel

$\Leftrightarrow \quad \vec{A} = \lambda \vec{B}$ for some scalar λ

$\Leftrightarrow \quad (a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}) = \lambda\,(a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k})$

$\Leftrightarrow \quad a_1 = \lambda a_2,\ b_1 = \lambda b_2,\ c_1 = \lambda c_2$

$$\Leftrightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

In order to find the angle between two vectors when their direction ratios or cosines are given, we may use the following algorithm.

ALGORITHM

<u>STEP 1</u> *Obtain direction ratios or direction cosines of two vectors. Let the direction ratios of two vectors be proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively.*

<u>STEP II</u> *Write vectors parallel to the given vectors.*
Let $\vec{a}$ = A vector parallel to the vector having direction ratios a_1, b_1, c_1

$\Rightarrow \quad \vec{a} = a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}$

$\vec{b}$ = *A vector parallel to the vector having direction ratios a_2, b_2, c_2*

$\Rightarrow \quad \vec{b} = a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}$

<u>STEP III</u> *Use the formula :* $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the angle between the vectors with direction ratios proportional to* 4, – 3, 5 *and* 3, 4, 5.

SOLUTION Let $\vec{a}$ = A vector parallel to the vector having direction ratios 4, – 3, 5

Then, $\vec{a} = 4\hat{i} - 3\hat{j} + 5\hat{k}$

and, $\vec{b}$ = A vector parallel to the vector having direction ratios 3, 4, 5.

$\Rightarrow$ $\vec{b} = 3\hat{i} + 4\hat{j} + 5\hat{k}$.

Let θ be the angle between the given vectors. Then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{12-12+25}{\sqrt{16+9+25}\,\sqrt{9+16+25}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Thus, the angle between the vectors with direction ratios proportional to 4, – 3, 5 and 3, 4, 5 is of 60°.

EXAMPLE 2 *Find the angle between the lines whose direction ratios are proportional to* 4, – 3, 5 *and* 3, 4, 5.

SOLUTION Let θ be the angle between the given lines.
We have,

$$a_1 = 4, b_1 = -3, c_1 = 5 \text{ and } a_2 = 3, b_2 = 4, c_2 = 5$$

$$\therefore \quad \cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\Rightarrow \quad \cos\theta = \frac{12-12+25}{\sqrt{16+9+25}\,\sqrt{9+16+25}} = \frac{25}{50} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \pi/3$$

Thus, the angle between the lines with direction ratios proportional to 4, – 3, 5 and 3, 4, 5 is of 60°.

EXAMPLE 3 $P(6, 3, 2)$, $Q(5, 1, 4)$ and $R(3, 3, 5)$ *are the vertices of a triangle PQR. Find* $\angle PQR$.

SOLUTION We know that the direction ratios of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Therefore,

Direction ratios of QP are proportional to $6-5, 3-1, 2-4$ *i.e.* $1, 2, -2$

Direction ratios of QR are proportional to $3-5, 3-1, 5-4$ i.e. $-2, 2, 1$

Let $\angle PQR = \theta$. Then,

$$\cos\theta = \frac{1\times -2 + 2\times 2 + -2\times 1}{\sqrt{1^2+2^2+(-2)^2}\,\sqrt{(-2)^2+2^2+(-1)^2}} = \frac{-2+4-2}{\sqrt{9}\,\sqrt{9}} = 0$$

$$\therefore \quad \theta = \frac{\pi}{2}$$

EXAMPLE 4 *Find the coordinates of the foot of the perpendicular drawn from the point* $A(1, 2, 1)$ *to the line joining* $B(1, 4, 6)$ *and* $C(5, 4, 4)$.

SOLUTION Let D be the foot of the perpendicular drawn from A on BC, and let D divide BC in the ratio $\lambda : 1$. Then, the coordinates of D are

$$\left(\frac{5\lambda+1}{\lambda+1}, \frac{4\lambda+4}{\lambda+1}, \frac{4\lambda+6}{\lambda+1}\right) \quad \text{...(i)}$$

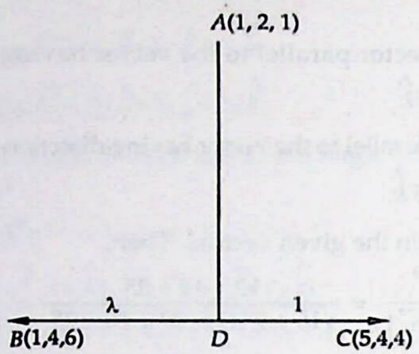

Fig. 26.8

Now,

Direction ratios of BC are proportional to

$5-1, 4-4, 4-6$ *i.e.* $4, 0, -2$

and, Direction ratios of AD are proportional to

$$\frac{5\lambda+1}{\lambda+1}-1, \frac{4\lambda+4}{\lambda+1}-2, \frac{4\lambda+6}{\lambda+1}-1, \text{ i.e. } \frac{4\lambda}{\lambda+1}, \frac{2\lambda+2}{\lambda+1}, \frac{3\lambda+5}{\lambda+1}$$

Since $AD \perp BC$. Therefore,

$$4\times\left(\frac{4\lambda}{\lambda+1}\right)+0\times\left(\frac{2\lambda+2}{\lambda+1}\right)+(-2)\left(\frac{3\lambda+5}{\lambda+1}\right)=0 \quad [\text{Using} : a_1 a_2 + b_1 b_2 + c_1 c_2 = 0]$$

$\Rightarrow \quad 16\lambda - 6\lambda - 10 = 0$

$\Rightarrow \quad \lambda = 1$

Putting $\lambda = 1$ in (i) the coordinates of D are $(3, 4, 5)$.

EXAMPLE 5 *Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to $1, -2, -2$ and $0, 2, 1$.*

SOLUTION Let l, m, n be the direction cosines of the required line. Since it is perpendicular to the lines whose direction cosines are proportional to $1, -2, -2$ and $0, 2, 1$ respectively.

$\therefore \quad l - 2m - 2n = 0$ ...(i)

and, $\quad 0l + 2m + n = 0$ ...(ii)

On solving (i) and (ii) by cross-multiplication, we get

$$\frac{l}{-2+4} = \frac{m}{0-1} = \frac{n}{2}$$

$$\Rightarrow \quad \frac{l}{2} = \frac{m}{-1} = \frac{n}{2}$$

Thus, the direction ratios of the required line are proportional to $2, -1, 2$.

Hence, its direction cosines are

$$\frac{2}{\sqrt{2^2+(-1)^2+2^2}}, \frac{-1}{\sqrt{2^2+(-1)^2+2^2}}, \frac{2}{\sqrt{2^2+(-1)^2+2^2}}$$

i.e. $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$.

EXAMPLE 6 *If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two lines, show that the direction cosines of the line perpendicular to both of them are proportional to $(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, $(l_1 m_2 - l_2 m_1)$.*

SOLUTION Let l, m, n be the direction cosines of the line perpendicular to each of the given lines. Then,

$$ll_1 + mm_1 + nn_1 = 0 \quad \text{...(i)}$$

and, $$ll_2 + mm_2 + nn_2 = 0 \quad \text{...(ii)}$$

On sloving (i) and (ii) by cross-multiplication, we get

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

Hence, the direction cosines of the line perpendicular to the given lines are proportional to $(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, $(l_1m_2 - l_2m_1)$.

EXAMPLE 7 *If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of them are*

$$(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1)$$ **[NCERT]**

SOLUTION Let l, m, n be the direction cosines of the line perpendicular to each of the given lines. Then, proceeding as in Example 5, we get

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

Thus, the direction cosines of the given line are proportional to $(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, $(l_1m_2 - l_2m_1)$

So, its direction cosines are

$$\frac{m_1 n_2 - m_2 n_1}{\lambda}, \frac{n_1 l_2 - n_2 l_1}{\lambda}, \frac{l_1 m_2 - l_2 m_1}{\lambda}, \text{ where}$$

$$\lambda = \sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}$$

We know that,

$$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1m_2 + n_1n_2)^2$$

$$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 \quad \text{...(i)}$$

It is given that the given lines are perpendicular to each other. Therefore,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

Also, we have

$$l_1^2 + m_1^2 + n_1^2 = 1$$

and, $$l_2^2 + m_2^2 + n_2^2 = 1$$

Putting these values in (i), we get

$$(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 1 \Rightarrow \lambda = 1$$

Hence, the direction cosines of the given line are

$$(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1)$$

EXAMPLE 8 *Find the direction cosines of the sides of the triangle whose vertices are (3, 5, – 4), (– 1, 1, 2) and (– 5, – 5, – 2) and also find the angles of the triangle. What types of triangle it is?*

SOLUTION Let ABC be the triangle the coordinates of whose vertices are A (3, 5, – 4), B (– 1, 1, 2) and C (– 5, – 5, – 2).

The direction ratios of AB are proportional to

$$-1-3, 1-5, 2+4 \text{ or, } -4, -4, 6 \text{ or, } -2, -2, 3$$

$\therefore$ Direction cosines of AB are

$$\frac{-2}{\sqrt{(-2)^2+(-2)^2+3^2}}, \frac{-2}{\sqrt{(-2)^2+(-2)^2+3^2}}, \frac{3}{\sqrt{(-2)^2+(-2)^2+3^2}}$$

or, $\dfrac{-2}{\sqrt{17}}, \dfrac{-2}{\sqrt{17}}, \dfrac{3}{\sqrt{17}}$

Let $l_1 = \dfrac{-2}{\sqrt{17}}, m_1 = \dfrac{-2}{\sqrt{17}}, n_1 = \dfrac{3}{\sqrt{17}}$

The direction ratios of AC are proportional to

$$-5-3, -5-5, -2-(-4) \text{ or, } -8, -10, 2 \text{ or, } -4, -5, 1$$

$\therefore$ Direction cosines of AC are

$$\frac{-4}{\sqrt{(-4)^2+(-5)^2+1^2}}, \frac{-5}{\sqrt{(-4)^2+(-5)^2+1^2}}, \frac{1}{\sqrt{(-4)^2+(-5)^2+1^2}}$$

or, $\dfrac{-4}{\sqrt{42}}, -\dfrac{5}{\sqrt{42}}, \dfrac{1}{\sqrt{42}}$

Let $l_2 = \dfrac{-4}{\sqrt{42}}, m_2 = \dfrac{-5}{\sqrt{42}}, n_2 = \dfrac{1}{\sqrt{42}}$

The direction ratios of BC are proportional to

$$-5-(-1), -5-1, -2-2 \text{ or, } -4, -6, -4 \text{ or, } -2, -3, -2$$

$\therefore$ Direction cosines of BC are

$$\frac{-2}{\sqrt{(-2)^2+(-3)^2+(-2)^2}}, \frac{-3}{\sqrt{(-2)^2+(-3)^2+(-2)^2}}, \frac{-2}{\sqrt{(-2)^2+(-3)^2+(-2)^2}}$$

or, $\dfrac{-2}{\sqrt{17}}, \dfrac{-3}{\sqrt{17}}, \dfrac{-2}{\sqrt{17}}$

Let $l_3 = \dfrac{-2}{\sqrt{17}}, m_3 = \dfrac{-3}{\sqrt{17}}, n_3 = \dfrac{-2}{\sqrt{17}}$

Since A is the angle between sides AB and AC.

$$\therefore \quad \cos A = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$\Rightarrow \quad \cos A = \frac{8}{\sqrt{17}\sqrt{42}} + \frac{10}{\sqrt{17}\sqrt{42}} + \frac{3}{\sqrt{17}\sqrt{42}} = \frac{21}{\sqrt{17}\sqrt{42}} = \sqrt{\frac{21}{34}}$$

$$\Rightarrow \quad A = \cos^{-1}\left(\sqrt{\frac{21}{34}}\right)$$

Since B is the angle between sides BA and BC.

$$\therefore \quad \cos B = (-l_1)\, l_3 + (-m_1)\, m_3 + (-n_1)\, n_3 \qquad \left[\because \text{Direction cosines of } BA \text{ are } -l_1, -m_1, -n_1\right]$$

$\Rightarrow$ $\cos B = \frac{2}{\sqrt{17}} \times \frac{-2}{\sqrt{17}} + \frac{2}{17} \times \frac{-3}{\sqrt{17}} + \frac{-3}{\sqrt{17}} \times \frac{-2}{\sqrt{17}}$

$\Rightarrow$ $\cos B = \frac{-4-6+6}{17} = -\frac{4}{17}$

$\Rightarrow$ $B = \cos^{-1}\left(-\frac{4}{17}\right)$

Since C is the angle between sides CB and CA and direction cosines of CB and CA are $-l_3, -m_3, -n_3$ and $-l_2, -m_2, -n_2$ respectively

$\therefore$ $\cos C = (-l_3)(-l_2) + (-m_3)(-m_2) + (-n_3)(-n_2)$

$\Rightarrow$ $\cos C = l_2 l_3 + m_2 m_3 + n_2 n_3$

$\Rightarrow$ $\cos C = \frac{-4}{\sqrt{42}} \times \frac{-2}{\sqrt{17}} + \frac{-5}{\sqrt{42}} \times \frac{-3}{\sqrt{17}} + \frac{1}{\sqrt{42}} \times \frac{-2}{\sqrt{17}}$

$\Rightarrow$ $\cos C = \frac{8+15-2}{\sqrt{42}\sqrt{17}} = \frac{21}{\sqrt{42}\sqrt{17}} = \sqrt{\frac{21}{34}}$

$\Rightarrow$ $C = \cos^{-1}\sqrt{\frac{21}{34}}$

Since $\angle A = \angle C$ and $\cos B$ is negative. Therefore, ΔABC is isosceles obtuse angled triangle.

EXAMPLE 9 *Find the angle between the lines whose direction cosines are given by the equations*

$$3l + m + 5n = 0,\ 6mn - 2nl + 5lm = 0$$

SOLUTION The given equations are

$$3l + m + 5n = 0 \quad ...(i)$$

and, $$6mn - 2nl + 5\,lm = 0 \quad ...(ii)$$

From (i), we have $m = -3l - 5n$.

Putting $m = -3l - 5n$ in (ii), we get

$$6(-3l - 5n)\,n - 2nl + 5l(-3l - 5n) = 0$$

$\Rightarrow$ $30n^2 + 45ln + 15l^2 = 0$

$\Rightarrow$ $2n^2 + 3ln + l^2 = 0$

$\Rightarrow$ $2n^2 + 2nl + nl + l^2 = 0$

$\Rightarrow$ $2n(n + l) + l(n + l) = 0$

$\Rightarrow$ $(n + l)(2n + l) = 0$

$\Rightarrow$ either $l = -n$ or, $l = -2n$.

If $l = -n$, then putting $l = -n$ in (i), we obtain $m = -2n$.

If $l = -2n$, then putting $l = -2n$ in (i), we obtain $m = n$.

Thus, the direction ratios of two lines are proportional to

$$-n, -2n, n \text{ and } -2n, n, n$$

i.e. $1, 2, -1$ and $-2, 1, 1$.

So, vectors parallel these lines are $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} + \hat{k}$ respectively.

If θ is the angle between the lines, then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{-2+2-1}{\sqrt{1+4+1}\,\sqrt{4+1+1}} = -\frac{1}{6} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{6}\right)$$

EXAMPLE 10 *Find the direction cosines of the two lines which are connected by the relations.*

$$l - 5m + 3n = 0 \text{ and } 7l^2 + 5m^2 - 3n^2 = 0$$

SOLUTION The given equations are

$$l - 5m + 3n = 0 \quad \text{...(i)}$$

$$7l^2 + 5m^2 - 3n^2 = 0 \quad \text{...(ii)}$$

From (i), we have

$$l = 5m - 3n$$

Putting $l = 5m - 3n$ in (ii), we get

$$7(5m-3n)^2 + 5m^2 - 3n^2 = 0$$

$$\Rightarrow \quad 6m^2 - 7mn + 2n^2 = 0$$

$$\Rightarrow \quad 6m^2 - 3mn - 4mn + 2n^2 = 0$$

$$\Rightarrow \quad (3m-2n)(2m-n) = 0$$

$$\Rightarrow \quad m = \frac{2}{3}n \text{ or, } m = \frac{n}{2}$$

If $m = \frac{2}{3}n$, then from (i), we obtain $l = \frac{1}{3}n$

If $m = \frac{n}{2}$, then from (i), we obtain $l = -\frac{n}{2}$

Thus, direction ratios of two lines are proportional to

$$\frac{n}{3}, \frac{2}{3}n, n \text{ and } \frac{-n}{2}, \frac{n}{2}, n \text{ i.e. } 1, 2, 3 \text{ and } -1, 1, 2$$

Hence, their direction cosines are

$$\pm\frac{1}{\sqrt{14}}, \pm\frac{2}{\sqrt{14}}, \pm\frac{3}{\sqrt{14}} \text{ and } \pm\frac{-1}{\sqrt{6}}, \pm\frac{1}{6}, \pm\frac{2}{\sqrt{6}}$$

EXAMPLE 11 *If a variable line in two adjacent positions has direction cosines l, m, n and $l+\delta l$, $m+\delta m$, $n+\delta n$, show that the small angle $\delta\theta$ between two positions is given by*

$$(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

SOLUTION Since l, m, n and $l+\delta l$, $m+\delta m$, $n+\delta n$ are direction cosines of a variable line in two different positions. Therefore,

$$l^2 + m^2 + n^2 = 1 \quad \text{...(i)}$$

and, $$(l+\delta l)^2 + (m+\delta m)^2 + (n+\delta n)^2 = 1 \quad \text{...(ii)}$$

Now, $$(l+\delta l)^2 + (m+\delta m)^2 + (n+\delta n)^2 = 1$$

$$\Rightarrow \quad (l^2+m^2+n^2) + 2(l\,\delta l + n\,\delta m + n\,\delta n) + (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = 1$$

$$\Rightarrow \quad 1 + 2(l\,\delta l + m\,\delta m + n\,\delta n) + (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = 1$$

$$\Rightarrow \quad 2(l\,\delta l + m\,\delta m + n\,\delta n) = -((\delta l)^2 + (\delta m)^2 + (\delta n)^2)$$

$$\Rightarrow \quad l\,\delta l + m\,\delta m + n\,\delta n = -\frac{1}{2}[(\delta l)^2 + (\delta m)^2 + (\delta n)^2] \quad ...(iii)$$

Now, $\hat{a}$ = Unit vector along a line with direction cosines l, m, n

$$\Rightarrow \quad \hat{a} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$$

and, $\hat{b}$ = Unit vector along a line with direction cosines $l + \delta l, m + \delta m, n + \delta n$

$$\Rightarrow \quad \hat{b} = (l + \delta l)\,\hat{i} + (m + \delta m)\,\hat{j} + (n + \delta n)\,\hat{k}$$

$$\therefore \quad \cos\delta\theta = \frac{\hat{a}\cdot\hat{b}}{|\hat{a}|\,|\hat{b}|} = \hat{a}\cdot\hat{b} \qquad [\because |\hat{a}| = |\hat{b}| = 1]$$

$$\Rightarrow \quad \cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$$

$$\Rightarrow \quad \cos\delta\theta = (l^2 + m^2 + n^2) + (l\,\delta l + m\,\delta m + n\,\delta n)$$

$$\Rightarrow \quad \cos\delta\theta = 1 - \frac{1}{2}[(\delta l)^2 + (\delta m)^2 + (\delta n)^2] \qquad \text{[Using (i) and (iii)]}$$

$$\Rightarrow \quad 2(1 - \cos\delta\theta) = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

$$\Rightarrow \quad 2\times 2\sin^2\frac{\delta\theta}{2} = (\delta l)^2 + (\delta m)^2 + (\delta n)^2 \qquad \left[\because 1 - \cos\delta\theta = 2\sin^2\frac{\delta\theta}{2}\right]$$

$$\Rightarrow \quad 4\left(\frac{\delta\theta}{2}\right)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2 \qquad \left[\because \frac{\delta\theta}{2} \text{ is small, } \therefore \sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2}\right]$$

$$\Rightarrow \quad (\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

EXAMPLE 12 *Prove that the straight lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $f\,mn + g\,nl + h\,lm = 0$ are perpendicular, if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$ and parallel, if $a^2f^2 + b^2g^2 + c^2h^2 - 2abfg - 2bcgh - 2achf = 0$.*

SOLUTION The given relations are

$$al + bm + cn = 0 \quad ...(i)$$

and,
$$fmn + fnl + hlm = 0 \quad ...(ii)$$

From (i), we obtain $n = -\left(\frac{al + bm}{c}\right)$

Putting this value of n in (ii), we get

$$-fm\left(\frac{al + bm}{c}\right) - gl\left(\frac{al + bm}{c}\right) + hlm = 0$$

$$\Rightarrow \quad agl^2 + (af + bg - ch)\,lm + bfm^2 = 0$$

$$\Rightarrow \quad ag\left(\frac{l}{m}\right)^2 + (af + bg - ch)\frac{l}{m} + bf = 0 \quad ...(iii)$$

This is a quadratic equation in $\frac{l}{m}$. So, it will have two roots, say $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

$$\therefore \quad \frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \frac{bf}{ag} \Rightarrow \frac{l_1l_2}{bf} = \frac{m_1m_2}{ag} \Rightarrow \frac{l_1l_2}{f/a} = \frac{m_1m_2}{g/b} \quad ...(iv)$$

Similarly, by making a quadratic in $\frac{m}{n}$, by using (i) and (ii), we get

$$\frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c} \quad ...(v)$$

From (iv) and (v), we obtain

$$\frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c} = \lambda \quad \text{(say)}$$

$$\Rightarrow \quad l_1 l_2 = \lambda\left(\frac{f}{a}\right), m_1 m_2 = \lambda\left(\frac{g}{b}\right), n_1 n_2 = \lambda\left(\frac{h}{c}\right)$$

The given lines will be perpendicular, if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow \quad \lambda\left(\frac{f}{a} + \frac{g}{b} + \frac{h}{c}\right) = 0 \quad \Rightarrow \quad \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$$

The given lines will be parallel, if their direction cosines are same. This is possible only when the roots of (iii) are equal. The condition for equal roots is

$$\text{Disc. of equations (iii)} = 0$$

$$\Rightarrow \quad (af + bg - ch)^2 - 4agbf = 0$$

$$\Rightarrow \quad a^2 f^2 + b^2 g^2 + c^2 h^2 - 2abfg - 2bcgh - 2achf = 0$$

EXAMPLE 13 *Show that the straight lines whose direction cosines are given by the equations $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ are perpendicular, if*

$$a^2 (v + w) + b^2 (u + w) + c^2 (u + v) = 0 \text{ and, parallel, if } \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$$

SOLUTION The given equations are

$$al + bm + cn = 0 \quad ...(i)$$

and, $$ul^2 + vm^2 + wn^2 = 0 \quad ...(ii)$$

From (i), we obtain

$$n = -\left(\frac{al + bm}{c}\right)$$

Putting $n = -\left(\frac{al + bm}{c}\right)$ in (ii), we get

$$ul^2 + vm^2 + w\frac{(al + bm)^2}{c^2} = 0$$

$$\Rightarrow \quad (c^2 u + a^2 w)\, l^2 + 2abwlm + (c^2 v + b^2 w)\, m^2 = 0$$

$$\Rightarrow \quad (a^2 w + c^2 u)\left(\frac{l}{m}\right)^2 + 2abw\left(\frac{l}{m}\right) + (b^2 w + c^2 v) = 0 \quad ...(iii)$$

This is a quadratic equation in $\frac{l}{m}$. So, it gives two values of $\frac{l}{m}$. Let the two values be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

$$\therefore \quad \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b^2 w + c^2 v}{a^2 w + c^2 u} \Rightarrow \frac{l_1 l_2}{b^2 w + c^2 v} = \frac{m_1 m_2}{a^2 w + c^2 u} \quad ...(iv)$$

Similarly, by making a quadratic equation in $\frac{m}{n}$, we obtain

$$\frac{m_1 m_2}{a^2 w + c^2 u} = \frac{n_1 n_2}{a^2 v + b^2 u} \qquad ...(v)$$

From (iv) and (v), we get

$$\frac{l_1 l_2}{b^2 w + c^2 v} = \frac{m_1 m_2}{a^2 w + c^2 u} = \frac{n_1 n_2}{a^2 v + b^2 u} = \lambda \qquad \text{(say).}$$

$$\Rightarrow \quad l_1 l_2 = \lambda (b^2 w + c^2 v), m_1 m_2 = \lambda (a^2 w + c^2 u), n_1 n_2 = \lambda (a^2 v + b^2 u).$$

For the given lines to be perpendicular, we must have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow \quad \lambda (b^2 w + c^2 v) + \lambda (a^2 w + c^2 u) + \lambda (a^2 v + b^2 u) = 0$$

$$\Rightarrow \quad a^2 (v + w) + b^2 (u + w) + c^2 (u + v) = 0$$

For the given lines to be parallel, the direction cosines must be equal and so the roots of the equation (iii) must be equal.

$\therefore$ Disc. of (iii) = 0

$$\Rightarrow \quad 4 a^2 b^2 w^2 - 4 (a^2 w + c^2 u)(b^2 w + c^2 v) = 0$$

$$\Rightarrow \quad a^2 c^2 vw + b^2 c^2 uw + c^4 uv = 0$$

$$\Rightarrow \quad a^2 vw + b^2 uw + c^2 uv = 0$$

$$\Rightarrow \quad \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0 \qquad \text{[Dividing throughout by } uvw]$$

EXAMPLE 14 *If the edges of a rectangular parallelopiped are a, b, c ; prove that the angles between the four diagonals are given by*

$$\cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right)$$

SOLUTION Let O be the origin and OX, OY, OZ be the coordinate axes. Let OA, OB, OC be the coterminus edges of the parallelopiped such that $OA = a$, $OB = b$ and $OC = c$. Then, the coordinates of the vertices of the parallelopiped are : $O\,(0, 0, 0)$, $A\,(a, 0, 0)$, $B\,(0, b, 0)$, $C\,(0, 0, c)$, $P\,(a, b, c)$ $Q\,(a, b, 0)$, $R\,(0, b, c)$, $S\,(a, 0, c)$

Clearly, OP, AR, CQ and BS are four diagonals of the rectangular parallelopiped.

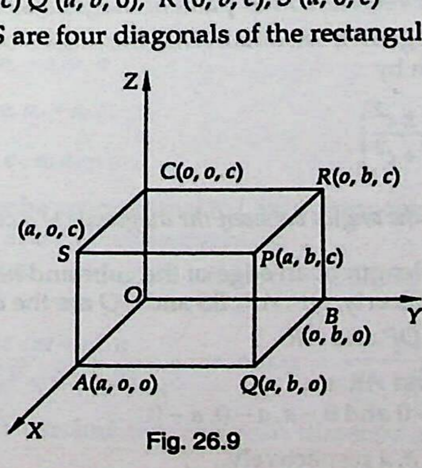

Fig. 26.9

Since the direction ratios of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Therefore, direction ratios of the diagonals OP, AR, BS and CQ are proportional

$$a, b, c\ ; -a, b, c\ ; a, -b, c\ ; a, b, -c$$

Let θ_1 be the angle between OP and AR. Then,

$$\cos\theta_1 = \frac{a \times -a + b \times b + c \times c}{\sqrt{a^2+b^2+c^2}\sqrt{(-a)^2+b^2+c^2}}$$

$$\Rightarrow \qquad \cos\theta_1 = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

$$\Rightarrow \qquad \theta_1 = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right)$$

Again, let θ_2 be the angle between OP and BS. Then,

$$\cos\theta_2 = \frac{a \times a + b \times -b + c \times c}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+(-b)^2+c^2)^2}}$$

$$\Rightarrow \qquad \cos\theta_2 = \frac{a^2-b^2+c^2}{a^2+b^2+c^2}$$

$$\Rightarrow \qquad \theta_2 = \cos^{-1}\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right)$$

Let θ_3 be the angle between diagonals CQ and BS. Then,

$$\cos\theta_3 = \frac{a \times a + (-b) \times b + c \times -c}{\sqrt{a^2+(-b)^2+c^2}\sqrt{a^2+b^2+(-c^2)}}$$

$$\Rightarrow \qquad \cos\theta_3 = \frac{a^2-b^2-c^2}{a^2+b^2+c^2}$$

$$\Rightarrow \qquad \theta_3 = \cos^{-1}\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right)$$

Similarly, the angles between the other pairs of diagonals can be obtained.

Putting all the results together, we obtain that the angles between the diagonals of the parallelopiped are given by

$$\cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$$

EXAMPLE 15 *Show that the angles between the diagonals of a cube is* $\cos^{-1}\left(\frac{1}{3}\right)$.

SOLUTION Let a be the length of an edge of the cube and let one corner be at the origin as shown in Fig. 26.10. Clearly, OP, AR, BS and CQ are the diagonals of the cube.

Consider the diagonals OP and AR.

Direction ratios of OP and AR are

$$a-0, a-0, a-0 \text{ and } 0-a, a-0, a-0$$

i.e. $\qquad a, a, a$ and $-a, a, a$ respectively.

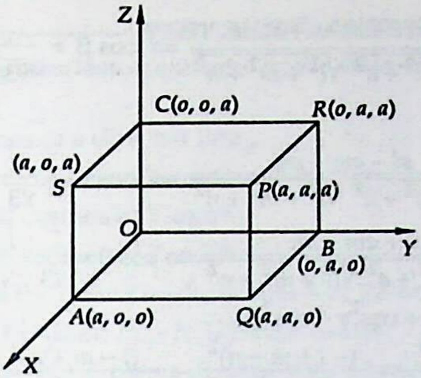

Fig. 26.10

Let θ be the angle between OP and AR. Then,

$$\cos\theta = \frac{a\times -a + a\times a + a\times a}{\sqrt{a^2+a^2+a^2}\,\sqrt{(-a)^2+a^2+a^2}}$$

$$\Rightarrow \quad \cos\theta = \frac{-a^2+a^2+a^2}{\sqrt{3a^2}\,\sqrt{3a^2}}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Similarly, the angles between the other pairs of diagonals are each equal to $\cos^{-1}\left(\frac{1}{3}\right)$.

Hence, the angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 16 *A line makes angles α, β, γ, δ with the four diagonals of a cube, prove that*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

SOLUTION Let a be the length of an edge of the cube and let one corner be at the origin as shown in Fig. .... Clearly, OP, AR, BS and CQ are the diagonals of the cube. The direction ratios of OP, AR, BS and CQ are

$a-0, a-0, a-0$ i.e. a, a, a

$0-a, a-0, a-0$ i.e. $-a, a, a$

$a-0, 0-a, a-0$ i.e. $a, -a, a$.

and, $a-0, a-0, 0-a$ i.e. $a, a, -a$

Let the direction ratios of a line be proportional to l, m, n. Suppose this line makes angles α, β, γ and δ with OP, AR, BS and CQ respectively.

Now, α is the angle between OP and the line whose direction ratios are proportional to l, m, n.

$$\therefore \quad \cos\alpha = \frac{a\,.\,l + a\,.\,m + a\,.\,n}{\sqrt{a^2+a^2+a^2}\,\sqrt{l^2+m^2+n^2}} \Rightarrow \cos\alpha = \frac{l+m+n}{\sqrt{3}\,.\,\sqrt{l^2+m^2+n^2}}$$

Since β is the angle between AR and the line with direction ratios proportional to l, m, n.

$$\therefore \quad \cos\beta = \frac{-a.l + a.m + a.n}{\sqrt{a^2+a^2+a^2}\sqrt{l^2+m^2+n^2}} \Rightarrow \cos\beta = \frac{-l+m+n}{\sqrt{3}.\sqrt{l^2+m^2+n^2}}$$

Similarly,

$$\cos\gamma = \frac{al - am + an}{\sqrt{a^2+a^2+a^2}\sqrt{l^2+m^2+n^2}} \Rightarrow \cos\gamma = \frac{l-m+n}{\sqrt{3}.\sqrt{l^2+m^2+n^2}}$$

and,

$$\cos\delta = \frac{al + am - an}{\sqrt{a^2+a^2+a^2}\sqrt{l^2+m^2+n^2}} \Rightarrow \cos\theta = \frac{l+m-n}{\sqrt{3}.\sqrt{l^2+m^2+n^2}}$$

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{(l+m+n)^2}{3(l^2+m^2+n^2)} + \frac{(-l+m+n)^2}{3(l^2+m^2+n^2)} + \frac{(l-m+n)^2}{3(l^2+m^2+n^2)} + \frac{(l+m-n)^2}{3(l^2+m^2+n^2)}$$

$$= \frac{1}{3(l^2+m^2+n^2)}\{(l+m+n)^2 + (-l+m+n)^2 + [l-m+n)^2 + (l+m-n)^2]$$

$$= \frac{1}{3(l^2+m^2+n^2)}\{4(l^2+m^2+n^2)\} = \frac{4}{3}.$$

EXERCISE 26.2

1. Find the angle between the vectors with direction ratios $1, -2, 1$ and $4, 3, 2$.
2. Find the angle between the vectors whose direction cosines are proportional to $2, 3, -6$ and $3, -4, 5$.
3. Find the direction cosines of the lines, connected by the relations : $l + m + n = 0$ and $2lm + 2ln - mn = 0$.
4. Find the angle between the lines whose direction cosines are given by the equations
   (i) $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$
   (ii) $2l - m + 2n = 0$ and $mn + nl + lm = 0$
   (iii) $l + 2m + 3n = 0$ and $3lm - 4ln + mn = 0$
5. Find the acute angle between the lines whose direction ratios are $2:3:6$ and $1:2:2$.
6. Show that the points $(2, 3, 4)$, $(-1, -2, 1)$, $(5, 8, 7)$ are collinear.
7. Show that the line through points $(4, 7, 8)$ and $(2, 3, 4)$ is parallel to the line through the points $(-1, -2, 1)$ and $(1, 2, 5)$.
8. Show that the line through the points $(1, -1, 2)$ and $(3, 4, -2)$ is perpendicular to the line through the points $(0, 3, 2)$ and $(3, 5, 6)$.
9. Show that the line joining the origin to the point $(2, 1, 1)$ is perpendicular to the line determined by the points $(3, 5, -1)$ and $(4, 3, -1)$.
10. Find the angle between the lines whose direction ratios are proportional to a, b, c and $b-c, c-a, a-b$.
11. If the coordinates of the points A, B, C, D are $(1, 2, 3)$, $(4, 5, 7)$, $(-4, 3, -6)$ and $(2, 9, 2)$, then find the angle between AB and CD.

ANSWERS

1. $\frac{\pi}{2}$ 2. $\cos^{-1}\left(-\frac{18\sqrt{2}}{35}\right)$ 3. $\pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}, \pm\frac{(-2)}{56}; \pm\left(\frac{-1}{\sqrt{6}}\right), \pm\frac{2}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}$

4. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{2}$ (iii) $\frac{\pi}{2}$ 5. $\cos^{-1}\left(\frac{20}{21}\right)$ 10. $\frac{\pi}{2}$ 11. 0

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Define direction cosines of a directed line.
2. What are the direction cosines of X-axis?
3. What are the direction cosines of Y-axis?
4. What are the direction cosines of Z-axis?
5. Write the distances of the point $(7, -2, 3)$ from XY, YZ and XZ-planes.
6. Write the distance of the point $(3, -5, 12)$ from X-axis?
7. Write the ratio in which YZ-plane divides the segment joining $P(-2, 5, 9)$ and $Q(3, -2, 4)$.
8. A line makes an angle of $60°$ with each of X-axis and Y-axis. Find the acute angle made by the line with Z-axis.
9. If a line makes angles α, β and γ with the coordinate axes, find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$.
10. Write the ratio in which the line segment joining (a, b, c) and $(-a, -c, -b)$ is divided by the xy-plane.
11. Write the inclination of a line with Z-axis, if its direction ratios are proportional to $0, 1, -1$.
12. Write the angle between the lines whose direction ratios are proportional to $1, -2, 1$ and $4, 3, 2$.
13. Write the distance of the point $P(x, y, z)$ from XOY plane.
14. Write the coordinates of the projection of point $P(x, y, z)$ on XOZ-plane.
15. Write the coordinates of the projection of the point $P(2, -3, 5)$ on Y-axis.
16. Find the distance of the point $(2, 3, 4)$ from the x-axis. [CBSE 2010]

ANSWERS

2. 1, 0, 0	3. 0, 1, 0	4. 0, 0, 1	5. 3, 7, 2	6. 13 units	
7. 2:3 internally	8. 45°	9. -1	10. $c:b$	11. $\frac{3\pi}{4}$	12. $\frac{\pi}{2}$
13. $\lvert z \rvert$	14. $(x, 0, z)$		15. $(0, -3, 0)$		16. 5

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. For every point $P(x, y, z)$ on the xy-plane,
   (a) $x = 0$ (b) $y = 0$ (c) $z = 0$ (d) none of these
2. For every point $P(x, y, z)$ on the x-axis (except the origin),
   (a) $x = 0, y = 0, z \neq 0$ (b) $x = 0, z = 0, y \neq 0$
   (c) $y = 0, z = 0, x \neq 0$ (d) none of these
3. A rectangular parallelopiped is formed by planes drawn through the points $(5, 7, 9)$ and $(2, 3, 7)$ parallel to the coordinate planes. The length of an edge of this rectangular parallelopiped is
   (a) 2 (b) 3 (c) 4 (d) all of these
4. A parallelopiped is formed by planes drawn through the points $(2, 3, 5)$ and $(5, 9, 7)$, parallel to the coordinate planes. The length of a diagonal of the parallelopiped is

(a) 7 (b) $\sqrt{38}$ (c) $\sqrt{155}$ (d) none of these

5. The xy-plane divides the line joining the points $(-1, 3, 4)$ and $(2, -5, 6)$
   (a) internally in the ratio $2:3$ (b) externally in the ratio $2:3$
   (c) internally in the ratio $3:2$ (d) externally in the ratio $3:2$

6. If the x-coordinate of a point P on the join of $Q(2, 2, 1)$ and $R(5, 1, -2)$ is 4, then its z-coordinate is
   (a) 2 (b) 1 (c) -1 (d) -2

7. The distance of the point $P(a, b, c)$ from the x-axis is
   (a) $\sqrt{b^2+c^2}$ (b) $\sqrt{a^2+c^2}$ (c) $\sqrt{a^2+b^2}$ (d) none of these

8. Ratio in which the xy-plane divides the join of $(1, 2, 3)$ and $(4, 2, 1)$ is
   (a) $3:1$ internally (b) $3:1$ externally
   (c) $1:2$ internally (d) $2:1$ externally.

9. If $P(3, 2, -4)$, $Q(5, 4, -6)$ and $R(9, 8, -10)$ are collinear, then R divides PQ in the ratio
   (a) $3:2$ internally (b) $3:2$ externally
   (c) $2:1$ internally (d) $2:1$ externally

10. $A(3, 2, 0)$, $B(5, 3, 2)$ and $C(-9, 6, -3)$ are the vertices of a triangle ABC. If the bisector of $\angle ABC$ meets BC at D, then coordinates of D are
    (a) $(19/8, 57/16, 17/16)$ (b) $(-19/8, 57/16, 17/16)$
    (c) $(19/8, -57/16, 17/16)$ (d) none of these

11. If O is the origin, $OP = 3$ with direction ratios proportional to $-1, 2, -2$ then the coordinates of P are
    (a) $(-1, 2, -2)$ (b) $(1, 2, 2)$
    (c) $(-1/9, 2/9, -2/9)$ (d) $(3, 6, -9)$

12. The angle between the two diagonals of a cube is
    (a) $30°$ (b) $45°$ (c) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\cos^{-1}\left(\frac{1}{3}\right)$

13. If a line makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is equal to
    (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{4}{3}$ (d) $\frac{8}{3}$

ANSWERS

1. (c) 2. (c) 3. (d) 4. (a) 5. (b) 6. (c) 7. (a) 8. (b)
9. (b) 10. (a) 11. (a) 12. (d) 13. (c)

SUMMARY

1. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in space, then
$$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

2. The distance of a point $P(x, y, z)$ from the origin O is given by
$$OP = \sqrt{x^2+y^2+z^2}$$

3. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, then the coordinates of a point dividing PQ internally in the ratio $m:n$ are

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$

If R divides PQ externally in the ratio $m:n$, then its coordinates are

$$\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$$

The coordinates of the mid-point of PQ are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

4. The line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is divided by
   (i) YZ-plane in the ratio $-x_1 : x_2$
   (ii) ZX-plane in the ratio $-y_1 : y_2$
   (iii) XY-plane in the ratio $-z_1 : z_2$

5. The coordinates of the centroid of the triangle formed by the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

6. The coordinates of the centroid of the tetrahedron formed by the points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ and (x_4, y_4, z_4) is

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

7. The distances of point $P(x, y, z)$ from x, y and z axes are

   $\sqrt{y^2+z^2}, \sqrt{z^2+x^2}$ and $\sqrt{x^2+y^2}$ respectively.

8. If a directed line segment OP makes angles α, β, γ with OX, OY and OZ respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are known as the direction cosines of OP and are generally denoted by l, m, n.
   Thus, we have $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$
   Direction cosines of PO are $-l, -m, -n$.
   If $OP = r$ and the coordinates of P are (x, y, z), then $x = lr$, $y = mr$, $z = nr$.

9. If l, m, n are direction cosines of a vector $\vec{r}$, then
   (i) $\vec{r} = |\vec{r}|(l\hat{i}+m\hat{j}+n\hat{k}) \Rightarrow \hat{r} = l\hat{i}+m\hat{j}+n\hat{k}$
   (ii) $l^2+m^2+n^2 = 1$
   (iii) Projections of $\vec{r}$ on the coordinates axes are

   $l|\vec{r}|, m|\vec{r}|, n|\vec{r}|$

   (iv) $|\vec{r}| = \sqrt{\text{Sum of the squares of projections of } \vec{r} \text{ on the coordinate axes}}$

10. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points such that the direction cosines of $\vec{PQ}$ are l, m, n. Then,

    $x_2 - x_1 = l|\vec{PQ}|$, $y_2 - y_1 = m|\vec{PQ}|$, $z_2 - z_1 = n|\vec{PQ}|$

    These are projections of $\vec{PQ}$ on X, Y and Z-axes respectively.

11. If l, m, n are direction cosines of a vector $\vec{r}$ and a, b, c are three numbers such that

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

Then, we say that the direction ratios of $\vec{r}$ are proportional to a, b, c.
Also, we have

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

12. If θ is the angle between two lines having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, then

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

(i) Lines are parallel, iff $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

(ii) Lines are perpendicular, iff $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

13. If θ is the angle between two lines whose direction ratios are proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then the angle θ between them is given by

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Lines are parallel, iff $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Lines are perpendicular, iff $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

14. The projection of the line segment joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ to the line having direction cosines l, m, n is

$$\left|(x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n\right|$$

15. The direction ratios of the line passing through points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$
Direction cosines of $\vec{PQ}$ are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}.$$

27

STRAIGHT LINE IN SPACE

27.1 INTRODUCTION

We know that in space a straight line is uniquely determined if either (i) coordinates of one point on it and its direction are given or (ii) coordinates of two points on it are given. In this chapter, we shall obtain the vector and Cartesian equations of a straight line under the above conditions.

27.2 VECTOR AND CARTESIAN EQUATIONS OF A LINE

THEOREM 1 *The vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\,\vec{b}$, where λ is scalar.*

PROOF Let O be the origin and let A be the fixed point with position vector $\vec{a}$. Then, $\vec{OA} = \vec{a}$.

Let $\vec{r}$ be the position vector of any point P on the line drawn through A and parallel to $\vec{b}$ as indicated in Fig. 27.1. Then, $\vec{OP} = \vec{r}$.

Since $\vec{AP}$ is parallel to $\vec{b}$. Therefore, $\vec{AP} = \lambda\,\vec{b}$ for some scalar λ.

Now,

$$\vec{AP} = \lambda\,\vec{b}$$

$$\Rightarrow \quad \vec{r} - \vec{a} = \lambda\,\vec{b}$$

$$\Rightarrow \quad \vec{r} = \vec{a} + \lambda\,\vec{b}$$

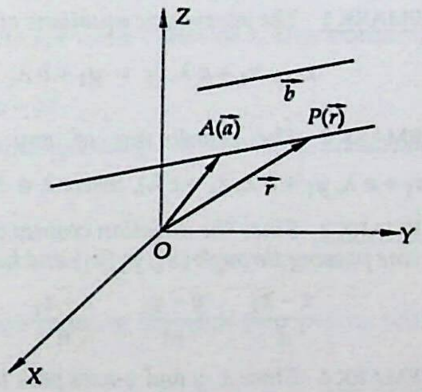

Fig. 27.1

Since every point on the line satisfies this equation and for each value of λ, this equation gives the position vector of a point P on the line.

Hence, the vector equation of a line is $\vec{r} = \vec{a} + \lambda\,\vec{b}$ Q.E.D.

REMARK 1 *In the above equation $\vec{r}$ is the position vector of any point $P(x, y, z)$ on the line. Therefore, $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$.*

REMARK 2 *The position vector of any point on the line is taken as $\vec{a} + \lambda\,\vec{b}$.*

ILLUSTRATION *Find the vector equation of a line which passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + \hat{j} - 2\hat{k}$.*

SOLUTION Here, $\vec{a} = 2\,\hat{i} - \hat{j} + 4\,\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - 2\,\hat{k}$. So, the vector equation of the required line is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

$\Rightarrow$ $\vec{r} = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + \hat{j} - 2\hat{k})$, where λ is a scalar

THEOREM 2 *The cartesian equation of a straight line passing through a fixed point (x_1, y_1, z_1) and having direction ratios proportional to a, b, c is given by*

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

PROOF We know that the vector equation of a line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{m}$ is given by

$$\vec{r} = \vec{a} + \lambda \vec{m} \quad \text{...(i)}$$

Here, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$

Putting the values of $\vec{r}, \vec{a}$ and $\vec{m}$ in (i), we obtain

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$$

$$\Rightarrow \quad x\hat{i} + y\hat{j} + z\hat{k} = (x_1 + \lambda a)\hat{i} + (y_1 + \lambda b)\hat{j} + (z_1 + \lambda c)\hat{k}$$

Comparing the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

$$x = x_1 + \lambda a,\ y = y_1 + \lambda b,\ z = z_1 + \lambda c \quad \text{...(ii)}$$

Eliminating the parameter λ from (ii), we get the following cartesian equation of the line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \text{...(iii)}$$

This is known as the *symmetrical form* of a line. Q.E.D.

REMARK 1 *The parametric equations of the line* $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ *are*

$x = x_1 + a\lambda,\ y = y_1 + b\lambda,\ z = z_1 + c\lambda$, *where λ is the parameter.*

REMARK 2 *The coordinates of any point on the line* $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ *are* $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, *where* $\lambda \in R$.

REMARK 3 *Since the direction cosines of a line are also direction ratios. Therefore, equation of a line passing through (x_1, y_1, z_1) and having direction cosines l, m, n is*

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

REMARK 4 *Since x, y and z-axes pass through the origin and have direction cosines 1, 0, 0; 0, 1, 0 and 0, 0, 1 respectively. Therefore, their equations are*

x-axis : $\frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z - 0}{0}$ or, $y = 0$ *and* $z = 0$

y-axis : $\frac{x - 0}{0} = \frac{y - 0}{1} = \frac{z - 0}{0}$ or, $x = 0$ *and* $z = 0$

z-axis : $\frac{x - 0}{0} = \frac{y - 0}{0} = \frac{z - 0}{1}$ or, $x = 0$ *and* $y = 0$.

THEOREM 3 *The vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is*

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

PROOF Let O be the origin and A and B be the given points with position vectors $\vec{a}$ and $\vec{b}$ respectively. Let $\vec{r}$ be the position vector of any point P on the line passing through the points A and B. Then,

$$\vec{OP} = \vec{r}, \ \vec{OA} = \vec{a} \text{ and } \vec{OB} = \vec{b}.$$

Since $\vec{AP}$ is collinear with $\vec{AB}$.

$\therefore \quad \vec{AP} = \lambda \vec{AB}$ for some scalar λ

$\Rightarrow \quad \vec{OP} - \vec{OA} = \lambda(\vec{OB} - \vec{OA})$

$\Rightarrow \quad \vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$

$\Rightarrow \quad \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

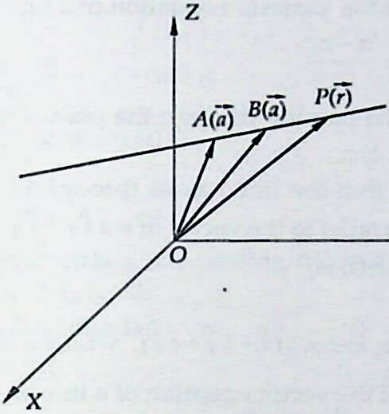

Fig. 27.2

Since every point on the line satisfies this equation for each value of λ, this equation gives the position vector of a point P on the line.

Hence, the vector equation of the line is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

THEROEM 4 *The cartesian equations of a line passing through two given points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *are given by*

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

PROOF We know that the vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \qquad \text{...(i)}$$

Here, $\vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}, \ \vec{b} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}$

Since $\vec{r}$ is the position vector of any point $P(x, y, z)$ on the line. Therefore, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Putting the values of $\vec{r}, \vec{a}$ and $\vec{b}$ in (i), we obtain

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}) + \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

$\Rightarrow \quad (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$

$\Rightarrow \quad x - x_1 = \lambda(x_2 - x_1), \ y - y_1 = \lambda(y_2 - y_1)$ and $z - z_1 = \lambda(z_2 - z_1)$

[On equating coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$]

$\Rightarrow \quad \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1} = \lambda$ [Eliminating λ]

Hence, the cartesian equations of the line passing through (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

27.2.1 REDUCTION OF CARTESIAN FORM OF THE EQUATION OF A LINE TO VECTOR FORM AND VICE-VERSA

The cartesian equations of a line can be reduced to vector form and vice-versa as discussed below.

CARTESIAN TO VECTOR Let the cartesian equation of a line be

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \quad \text{...(i)}$$

This is the equation of a line passing through the point $A(x_1, y_1, z_1)$ and its direction ratios are proportional to a, b, c.

In vector form this means that the line passes through point having position vector $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and is parallel to the vector $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$.

Thus, the vector form of line (i) is

$$\vec{r}=\vec{a}+\lambda\vec{m}$$

or, $\vec{r}=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k})$, where λ is a parameter.

VECTOR TO CARTESIAN Let the vector equation of a line be

$$\vec{r}=\vec{a}+\lambda\vec{m} \quad \text{...(ii)}$$

where $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$, $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$ and λ is a parameter.

In order to reduce equation (ii) to cartesian form we put $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and equate the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$ as discussed below.

Putting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$ in (ii), we obtain

$$x\hat{i}+y\hat{j}+z\hat{k}=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k})$$

On equating coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

$$x=x_1+a\lambda,\ y=y_1+b\lambda,\ z=z_1+c\lambda$$

$$\Rightarrow \quad \frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda$$

ILLUSTRATIVE EXAMPLES

Type I **ON FINDING THE VECTOR EQUATION OF A LINE SATISFYING THE GIVEN CONDITIONS AND REDUCING IT TO CARTESIAN FORM.**

Formulae to be used : (i) $\vec{r}=\vec{a}+\lambda\vec{b}$ *(ii)* $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$.

EXAMPLE 1 *Find the vector equation of a line which passes through the point with position vector $2\hat{i}-\hat{j}+4\hat{k}$ and is in the direction of $\hat{i}+\hat{j}-2\hat{k}$. Also, reduce it to cartesian form.*

SOLUTION We know that the vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to the vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$.

Here, $\vec{a}=2\hat{i}-\hat{j}+4\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}-2\hat{k}$. So, the vector equation of the required line is

$$\vec{r}=(2\hat{i}-\hat{j}+4\hat{k})+\lambda(\hat{i}+\hat{j}-2\hat{k}) \quad \text{...(i)}$$

where λ is a parameter.

Reduction to cartesian form : Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (i), we obtain

$$x\hat{i}+y\hat{j}+z\hat{k} = (2\hat{i}-\hat{j}+4\hat{k}) + \lambda(\hat{i}+\hat{j}-2\hat{k})$$

$$\Rightarrow \quad x\hat{i}+y\hat{j}+z\hat{k} = (2+\lambda)\hat{i}+(-1+\lambda)\hat{j}+(4-2\lambda)\hat{k}$$

On equating coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

$$x = 2+\lambda,\ y = -1+\lambda,\ z = 4-2\lambda$$

$$\Rightarrow \quad x-2 = \lambda,\ y+1 = \lambda,\ \frac{z-4}{-2} = \lambda$$

Eliminating λ, we have

$$\frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2}$$

Hence, the cartesian form of equation (i) is

$$\frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2}$$

ALITER The equation (i) represents a line passing through a point $(2, -1, 4)$ and has direction ratios proportional to $1, 1, -2$.

So, the cartesian form of its equation is

$$\frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2} \qquad \left[\text{Using}: \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z=z_1}{c}\right]$$

EXAMPLE 2 *Find the vector equation of the line through $A(3, 4, -7)$ and $B(1, -1, 6)$. Find also, its cartesian equations.*

SOLUTION We know that the vector equation of a line passing through the points having position vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \text{ where } \lambda \text{ is a scalar.}$$

Here, $\vec{a} = 3\hat{i}+4\hat{j}-7\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+6\hat{k}$. So, the vector equation of the line passing through $A(3, 4, -7)$ and $B(1, -1, 6)$ is

$$\vec{r} = (3\hat{i}+4\hat{j}-7\hat{k}) + \lambda[(\hat{i}-\hat{j}+6\hat{k}) - (3\hat{i}+4\hat{j}-7\hat{k})]$$

$$\text{or,} \quad \vec{r} = (3\hat{i}+4\hat{j}-7\hat{k}) + \lambda(-2\hat{i}-5\hat{j}+13\hat{k}) \qquad \text{...(i)}$$

where λ is a parameter.

Reduction to cartesian form : Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (i), we obtain

$$x\hat{i}+y\hat{j}+z\hat{k} = (3\hat{i}+4\hat{j}-7\hat{k}) + \lambda(-2\hat{i}-5\hat{j}+13\hat{k})$$

$$\text{or,} \quad x\hat{i}+y\hat{j}+z\hat{k} = (3-2\lambda)\hat{i}+(4-5\lambda)\hat{j}+(-7+13\lambda)\hat{k}$$

On equating coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

$$x = 3-2\lambda,\ y = 4-5\lambda,\ z = -7+13\lambda$$

$$\text{or,} \quad \frac{x-3}{-2} = \lambda,\ \frac{y-4}{-5} = \lambda,\ \frac{z+7}{13} = \lambda$$

Eliminating λ, we have

$$\frac{x-3}{-2} = \frac{y-4}{-5} = \frac{z+7}{13}$$

Hence, the cartesian form of the equation (i) is

$$\frac{x-3}{-2} = \frac{y-4}{-5} = \frac{z+7}{13}$$

EXAMPLE 3 *The points A (4, 5, 10), B (2, 3, 4) and C (1, 2, – 1) are three vertices of a parallelogram ABCD. Find vector and cartesian equations for the sides AB and BC and find the coordinates of D.* **[CBSE 2010]**

SOLUTION The line AB passes through A (4, 5, 10) and B (2, 3, 4) having position vectors $\vec{a} = 4\hat{i} + 5\hat{j} + 10\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ respectively. So, vector equation of AB is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

or, $$\vec{r} = 4\hat{i} + 5\hat{j} + 10\hat{k} + \lambda[(2\hat{i} + 3\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + 10\hat{k})]$$

or, $$\vec{r} = 4\hat{i} + 5\hat{j} + 10\hat{k} + \lambda(-2\hat{i} - 2\hat{j} - 6\hat{k})$$

or, $$\vec{r} = 4i + 5\hat{j} + 10\hat{k} + \mu(\hat{i} + \hat{j} + 3\hat{k}), \text{ where } \mu = -2\lambda \quad \text{...(i)}$$

Cartesian equation of line (i) is

$$\frac{x-4}{1} = \frac{y-5}{1} = \frac{z-10}{3}$$

Since BC passes through the points B (2, 3, 4) and C (1, 2, – 1) having position vector $\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$ respectively. Therefore, vector equation of BC is

$$\vec{r} = \vec{b} + \lambda(\vec{c} - \vec{b})$$

or, $$\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + \mu[(\hat{i} + 2\hat{j} - \hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k})]$$

or, $$\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + \mu(-\hat{i} - \hat{j} - 5\hat{k}$$

or, $$\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + \nu(\hat{i} + \hat{j} + 5\hat{k}), \text{ where } \nu = -\mu \quad \text{...(ii)}$$

Cartesian equation of line (ii) is

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{5}$$

Suppose the coordinates of D are (x, y, z). Since $ABCD$ is a parallelogram, the diagonals AC and BD bisect each other. Therefore, AC and BD must have the same mid-point.

The coordinates of the mid-point of AC are $P\left(\frac{4+1}{2}, \frac{5+2}{2}, \frac{10-1}{2}\right)$ i.e. $P\left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$ and the coordinates of the mid-point of BD are $Q\left(\frac{2+x}{2}, \frac{3+y}{2}, \frac{4+z}{2}\right)$. Since P and Q coincide. Therefore,

$$\frac{2+x}{2} = \frac{5}{2}, \frac{3+y}{2} = \frac{7}{2}, \frac{4+z}{2} = \frac{9}{2} \Rightarrow x = 3, y = 4, z = 5.$$

Thus, the coordinates of D are (3, 4, 5).

EXAMPLE 4 *Find the vector equation of a line passing through a point with position vector* $2\hat{i} - \hat{j} + \hat{k}$, *and parallel to the line joining the points* $-\hat{i} + 4\hat{j} + \hat{k}$ *and* $\hat{i} + 2\hat{j} + 2\hat{k}$. *Also, find the cartesian equivalent of this equation.* **[CBSE 2003]**

SOLUTION Let A, B, C be the points with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$ respectively.

We have to find the equation of a line passing through the point A and parallel to $\vec{BC}$

We have,

$$\vec{BC} = \text{Position vector of C} - \text{Position vector of B}$$

$$\Rightarrow \quad \vec{BC} = (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 4\hat{j} + \hat{k}) = 2\hat{i} - 2\hat{j} + \hat{k}$$

We know that the equation of a line passing through a point $\vec{a}$ and parallel to $\vec{b}$ is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

Here, $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$.
So, the equation of the required line is

$$\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda (2\hat{i} - 2\hat{j} + \hat{k}) \quad ...(i)$$

Reduction to cartesian form : Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (i), we obtain

$$x\hat{i} + y\hat{j} + z\hat{k} = (2 + 2\lambda)\hat{i} + (-1 - 2\lambda)\hat{j} + (1 + \lambda)\hat{k}$$

$$\Rightarrow \quad x = 2 + 2\lambda, \; y = -1 - 2\lambda, \; z = 1 + \lambda$$

$\Rightarrow \quad \dfrac{x-2}{2} = \dfrac{y+1}{-2} = \dfrac{z-1}{1}$, which is the cartesian equivalent of equation (i).

Type II **ON FINDING THE CARTESIAN EQUATION OF A LINE SATISFYING THE GIVEN CONDITIONS AND REDUCING IT TO VECTOR FORM.**

Formulae to be used :

(i) $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ (ii) $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

EXAMPLE 5 *Find the cartesian equation of a line passing through the points A* (2, − 1, 3) *and B* (4, 2, 1). *Also, reduce it to vector form.*

SOLUTION We know that the equations of a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

So, the equations of the required line are given by

$$\frac{x-2}{4-2} = \frac{y-(-1)}{2-(-1)} = \frac{z-3}{1-3} \text{ or, } \frac{x-2}{2} = \frac{y+1}{3} = \frac{z-3}{-2} \quad ...(i)$$

Reduction to vector form : We have,

$$\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-3}{-2} = \lambda \text{ (say)}$$

$$\Rightarrow \quad x = 2\lambda + 2, y = 3\lambda - 1, z = -2\lambda + 3$$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point on the line.

Then, $\vec{r} = (2\lambda + 2)\hat{i} + (3\lambda - 1)\hat{j} + (-2\lambda + 3)\hat{k}$

$\Rightarrow \quad \vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda (2\hat{i} + 3\hat{j} - 2\hat{k})$

This is the required vector form.

EXAMPLE 6 *The cartesian equations of a line are* $6x - 2 = 3y + 1 = 2z - 2$. *Find its direction ratios and also find vector equation of the line.* [CBSE 2003]

SOLUTION Recall that in the symmetrical form of a line the coefficients of x, y and z are unity. Therefore, to put the given line in symmetric form, we must make the coefficients of x, y and z as unity.

We have,

$$6x - 2 = 3y + 1 = 2z - 2$$

$$\Rightarrow \quad 6\left(x-\frac{1}{3}\right)=3\left(y+\frac{1}{3}\right)=2(z-1)$$

$$\Rightarrow \quad \frac{x-\frac{1}{3}}{1}=\frac{y+\frac{1}{3}}{2}=\frac{z-1}{3} \qquad \left[\begin{array}{l}\text{Dividing throughout}\\ \text{by the l.c.m. of 6, 3, 2 i.e. 6}\end{array}\right]$$

This shows that the given line passes through (1/3, – 1/3, 1) and has direction ratios proportional to 1, 2, 3.

In vector form this means that the line passes through the point having position vector $\vec{a}=\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}$ and is parallel to the vector $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$. Therefore, its vector equation is

$$\vec{r}=\left(\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}\right)+\lambda(\hat{i}+2\hat{j}+3\hat{k}).$$

EXAMPLE 7 *Find the direction cosines of the line* $\frac{x-2}{2}=\frac{2y-5}{-3}, z=-1$. *Also, find the vector equation of the line.*

SOLUTION The given line is

$$\frac{x-2}{2}=\frac{2y-5}{-3}, z=-1$$

$$\Rightarrow \quad \frac{x-2}{2}=\frac{2y-5}{-3}=\frac{z+1}{0}$$

$$\Rightarrow \quad \frac{x-2}{2}=\frac{y-5/2}{-3/2}=\frac{z+1}{0}$$

This shows that the given line passes through the point (2, 5/2,– 1) and has direction ratios proportional to 2, – 3/2, 0. So, its direction cosines are

$$\frac{2}{\sqrt{2^2+\left(-\frac{3}{2}\right)^2+0^2}}, \frac{-3/2}{\sqrt{2^2+\left(-\frac{3}{2}\right)^2+0^2}}, \frac{0}{\sqrt{2^2+\left(-\frac{3}{2}\right)^2+0^2}}$$

or, $\frac{2}{5/2}, \frac{-3/2}{5/2}, 0$ or, $\frac{4}{5}, -\frac{3}{5}, 0$

Thus given line passes through the point having position vector $\vec{a}=2\hat{i}+\frac{5}{2}\hat{j}-\hat{k}$ and is parallel to the vector $\vec{b}=2\hat{i}-\frac{3}{2}\hat{j}+0\hat{k}$.

So, its vector equation is

$$\vec{r}=\left(2\hat{i}+\frac{5}{2}\hat{j}-\hat{k}\right)+\lambda\left(2\hat{i}-\frac{3}{2}\hat{j}+0\hat{k}\right)$$

Type III **ON CHECKING COLLINEARITY OF THREE POINTS**

EXAMPLE 8 *Show that the points whose position vectors are* $5\hat{i}+5\hat{k}, 2\hat{i}+\hat{j}+3\hat{k}$ *and* $-4\hat{i}+3\hat{j}-\hat{k}$ *are collinear.*

SOLUTION Let the given points be P, Q and R and let their position vectors be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then,

$$\vec{a}=5\hat{i}+5\hat{k}, \vec{b}=2\hat{i}+\hat{j}+3\hat{k} \text{ and } \vec{c}=-4\hat{i}+3\hat{j}-\hat{k}$$

The equation of the line passing through P and Q is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

or, $$\vec{r} = (5\hat{i} + 5\hat{k}) + \lambda(-3\hat{i} + \hat{j} - 2\hat{k}) \quad \text{...(i)}$$

If points P, Q and R are collinear, then point R must satisfy equation (i).

Replacing $\vec{r}$ by $\vec{c} = -4\hat{i} + 3\hat{j} - \hat{k}$ in (i), we get

$$-4\hat{i} + 3\hat{j} - \hat{k} = 5\hat{i} + 5\hat{k} + \lambda(-3\hat{i} + \hat{j} - 2\hat{k})$$

$\Rightarrow \quad -4 = 5 - 3\lambda,\ 3 = \lambda$ and $-1 = 5 - 2\lambda$ \quad [On equating coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$]

These three equations are consistent i.e. they give the same value of λ.

Hence, points P, Q and R are collinear.

ALITER We have,

$$\vec{PQ} = -3\hat{i} + \hat{j} - 2\hat{k} \text{ and } \vec{QR} = -6\hat{i} + 2\hat{j} - 4\hat{k}$$

Clearly, $2\vec{PQ} = \vec{QR}$

$\Rightarrow \quad \vec{PQ} \parallel \vec{QR}$

Hence, points P, Q and R are collinear.

EXAMPLE 9 *If the points $A(-1, 3, 2)$, $B(-4, 2, -2)$ and $C(5, 5, \lambda)$ are collinear, find the value of λ.*

SOLUTION *The equation of the line passing through $A(-1, 3, 2)$ and $B(-4, 2, -2)$ is*

$$\frac{x+1}{-4+1} = \frac{y-3}{2-3} = \frac{z-2}{-2-2}$$

$$\Rightarrow \quad \frac{x+1}{-3} = \frac{y-3}{-1} = \frac{z-2}{-4}$$

$$\Rightarrow \quad \frac{x+1}{3} = \frac{y-3}{1} = \frac{z-2}{4} \quad \text{...(i)}$$

If the points $A(-1, 3, 2)$, $B(-4, 2, -2)$ and $C(5, 5, \lambda)$ are collinear, then the coordinates of C must satisfy equation (i). Therefore,

$$\frac{5+1}{3} = \frac{5-3}{1} = \frac{\lambda-2}{4}$$

$$\Rightarrow \quad 2 = 2 = \frac{\lambda-2}{4} \Rightarrow \frac{\lambda-2}{4} = 2 \Rightarrow \lambda = 10.$$

Type IV ON FINDING A POINT ON A LINE

EXAMPLE 10 *Find the point on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance of $3\sqrt{2}$ from the piont (1, 2, 3).* **[CBSE 2008]**

SOLUTION The coordinates of any point on the line

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$$

are given by

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \lambda$$

$$\Rightarrow \quad x+2 = 3\lambda,\ y+1 = 2\lambda,\ z-3 = 2\lambda$$

$$\Rightarrow \quad x = 3\lambda - 2,\ y = 2\lambda - 1,\ z = 2\lambda + 3 \quad \text{...(i)}$$

So, let the coordinates of the desired point are $(3\lambda - 2, 2\lambda - 1, 2\lambda + 3)$. The distance between this point and $(1, 2, 3)$ is $3\sqrt{2}$.

$$\therefore \quad \sqrt{(3\lambda-2-1)^2+(2\lambda-1-2)^2+(2\lambda+3-3)^2} = 3\sqrt{2}$$

$$\Rightarrow \quad 9(\lambda-1)^2+(2\lambda-3)^2+4\lambda^2 = 18$$

$$\Rightarrow \quad 17\lambda^2 - 30\lambda = 0$$

$$\Rightarrow \quad \lambda = 0, \lambda = \frac{30}{17}$$

Substituting the values of λ in (i), we obtain that the coordinates of the desired point are $(-2, -1, 3)$ and $\left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17}\right)$.

EXERCISE 27.1

1. Find the vector equation of a line which is parallel to the vector $2\hat{i} - \hat{j} + 3\hat{k}$ and which passes through the point $(5, -2, 4)$. Also, reduce it to cartesian form.
2. A line passes through the point with position vector $2\hat{i} - 3\hat{j} + 4\hat{k}$ and is in the direction of $3\hat{i} + 4\hat{j} - 5\hat{k}$. Find equations of the line in vector and cartesian form.
3. $ABCD$ is a parallelogram. The position vectors of the points A, B and C are respectively, $4\hat{i} + 5\hat{j} - 10\hat{k}$, $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-\hat{i} + 2\hat{j} + \hat{k}$. Find the vector equation of the line BD. Also, reduce it to cartesian form.
4. Find in vector form as well as in cartesian form, the equation of the line passing through the points $A(1, 2, -1)$ and $B(2, 1, 1)$.
5. Find the vector equation for the line which passes through the point $(1, 2, 3)$ and parallel to the vector $\hat{i} - 2\hat{j} + 3\hat{k}$. Reduce the corresponding equation in cartesian from.
6. Find the vector equation of a line passing through $(2, -1, 1)$ and parallel to the line whose equations are $\frac{x-3}{2} = \frac{y+1}{7} = \frac{z-2}{-3}$.
7. The cartesian equations of a line are
$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$
Find a vector equation for the line. [NCERT]
8. Find the cartesian equation of a line passing through $(1, -1, 2)$ and parallel to the line whose equations are
$$\frac{x-3}{1} = \frac{y-1}{2} = \frac{z+1}{-2}$$
Also, reduce the equation obtained in vector form.
9. Find the direction cosines of the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.
Also, reduce it to vector form.
10. The cartesian equations of a line are $x = ay + b$, $z = cy + d$. Find its direction ratios and reduce it to vector form.
11. Find the vector equation of a line passing through the point with position vector $\hat{i} - 2\hat{j} - 3\hat{k}$ and parallel to the line joining the points with position vectors $\hat{i} - \hat{j} + 4\hat{k}$ and $2\hat{i} + \hat{j} + 2\hat{k}$. Also, find the cartesian equivalent of this equation.

12. Find the points on the line $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}$ at a distance of 5 units from the point $P(1, 3, 3)$. **[CBSE 2010]**
13. Show that the points whose position vectors are $-2\hat{i}+3\hat{j}$, $\hat{i}+2\hat{j}+3\hat{k}$ and $7\hat{i}-\hat{k}$ are collinear.
14. Find the cartesian and vector equations of a line which passes through the point $(1, 2, 3)$ and is parallel to the line $\frac{-x-2}{1}=\frac{y+3}{7}=\frac{2z-6}{3}$. **[CBSE 2004]**
15. The cartesian equations of a line are $3x+1=6y-2=1-z$. Find the fixed point through which it passes, its direction ratios and also its vector equation. **[CBSE 2004]**

ANSWERS

1. $\vec{r}=5\hat{i}-2\hat{j}+4\hat{k}+\lambda(2\hat{i}-\hat{j}+3\hat{k})$; $\frac{x-5}{2}=\frac{y+2}{-1}=\frac{z-4}{3}$
2. $\vec{r}=2\hat{i}-3\hat{j}+4\hat{k}+\lambda(3\hat{i}+4\hat{j}-5\hat{k})$; $\frac{x-2}{3}=\frac{y+3}{4}=\frac{z-4}{-5}$
3. $\vec{r}=2\hat{i}-3\hat{j}+4\hat{k}+\lambda(\hat{i}-13\hat{j}+17\hat{k})$; $\frac{x-2}{1}=\frac{y+3}{-13}=\frac{z-4}{17}$
4. $\vec{r}=(\hat{i}+2\hat{j}-\hat{k})+\lambda(\hat{i}-\hat{j}+2\hat{k})$; $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z+1}{2}$
5. $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(\hat{i}-2\hat{j}+3\hat{k})$; $\frac{x-1}{1}=\frac{y-2}{-2}=\frac{z-3}{3}$
6. $\vec{r}=2\hat{i}-\hat{j}+\hat{k}+\lambda(2\hat{i}+7\hat{j}-3\hat{k})$
7. $\vec{r}=(5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}+2\hat{k})$
8. $\frac{x-1}{1}=\frac{y+1}{2}=\frac{z-2}{-2}$; $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\lambda(\hat{i}+2\hat{j}-2\hat{k})$
9. $-\frac{2}{7}, \frac{6}{7}, -\frac{3}{7}$; $\vec{r}=(4\hat{i}+0\hat{j}+\hat{k})+\lambda(-2\hat{i}+6\hat{j}-3\hat{k})$
10. DRS : $a, 1, c$; $\vec{r}=(b\hat{i}+0\hat{j}+d\hat{k})+\lambda(a\hat{i}+\hat{j}+c\hat{k})$
11. $\vec{r}=(\hat{i}-2\hat{j}-3\hat{k})+\lambda(\hat{i}+2\hat{j}-2\hat{k})$; $\frac{x-1}{1}=\frac{y+2}{2}=\frac{z+3}{-2}$
12. $(4, 3, 7), (-8, -5, -1)$
14. $\frac{x-1}{-2}=\frac{y-2}{14}=\frac{z-3}{3}$ $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(-27+14\hat{j}+3\hat{k})$
15. $\left(-\frac{1}{3}, \frac{1}{3}, 1\right)$; $2, 1, -6$; $\vec{r}=-\frac{1}{3}\hat{i}+\frac{1}{3}\hat{j}+\hat{k}+\lambda(2\hat{i}+\hat{j}-6\hat{k})$

27.3 ANGLE BETWEEN TWO LINES

VECTOR FORM Let the vector equations of the two lines be

$$\vec{r}=\vec{a_1}+\lambda\vec{b_1} \text{ and } \vec{r}=\vec{a_2}+\mu\vec{b_2}.$$

These two lines are parallel to the vectors $\vec{b_1}$ and $\vec{b_2}$ respectively. Therefore, angle between these two lines is equal to the angle between $\vec{b_1}$ and $\vec{b_2}$. Thus, if θ is the angle between the given lines, then

$$\cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|}$$

Condition of perpendicularity: If the lines $\vec{b_1}$ and $\vec{b_2}$ are perpendicular. Then,

$$\vec{b_1}\cdot\vec{b_2}=0$$

Condition of parallelism : If the lines are parallel, then $\vec{b_1}$ and $\vec{b_2}$ are parallel.

$\therefore \quad \vec{b_1}=\lambda\,\vec{b_2}$ for some scalar λ

CARTESIAN FORM Let the cartesian equations of the two lines be

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \quad \text{...(i)}$$

and, $$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2} \quad \text{...(ii)}$$

Direction ratios of line (i) are proportional to a_1, b_1, c_1.

$\therefore \quad \vec{m_1}$ = Vector parallel to line (i) $= a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$

Direction ratios of line (ii) are proportional to a_2, b_2, c_2.

$\therefore \quad \vec{m_2}$ = Vector parallel to line (ii) $= a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$

Let θ be the angle between (i) and (ii). Then, θ is also the angle between $\vec{m_1}$ and $\vec{m_2}$.

$$\therefore \quad \cos\theta = \frac{\vec{m_1}\cdot\vec{m_2}}{|\vec{m_1}|\,|\vec{m_2}|} \Rightarrow \cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}$$

Condition of perpendicularity : If the lines are perpendicular, then

$$\vec{m_1}\cdot\vec{m_2}=0 \Rightarrow a_1a_2+b_1b_2+c_1c_2=0$$

Condition of parallelism : If the lines are parallel, then $\vec{m_1}$ and $\vec{m_2}$ are parallel.

$\therefore \quad \vec{m_1}=\lambda\,\vec{m_2}$ for some scalar λ

$$\Rightarrow \quad \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

ILLUSTRATIVE EXAMPLES

Type I ON FINDING THE ANGLE BETWEEN TWO LINES

Formula to be used : $\cos\theta = \dfrac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|}$ *or,* $\cos\theta = \dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}$

EXAMPLE 1 *Find the angle between the lines*

$$\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k}) \text{ and } \vec{r}=(5\hat{j}-2\hat{k})+\mu(3\hat{i}+2\hat{j}+6\hat{k}).$$

SOLUTION Let θ be the angle between the given lines. The given lines are parallel to the vectors $\vec{b_1}=\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b_2}=3\hat{i}+2\hat{j}+6\hat{k}$ respectively. So, the angle θ between them is given by

$$\cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|} = \frac{(\hat{i}+2\hat{j}+2\hat{k})\cdot(3\hat{i}+2\hat{j}+6\hat{k})}{|\hat{i}+2\hat{j}+2\hat{k}|\,|3\hat{i}+2\hat{j}+6\hat{k}|}$$

$$\Rightarrow \quad \cos\theta = \frac{3+4+12}{\sqrt{1+4+4}\,\sqrt{9+4+36}} = \frac{19}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{21}\right).$$

EXAMPLE 2 *Find the angle between the lines*

$$\frac{x-2}{3} = \frac{y+1}{-2},\ z = 2 \text{ and } \frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}.$$

SOLUTION The given equations are not in the standard form. The equations of the given lines can be written as

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-2}{0} \quad \text{...(i)}$$

and, $$\frac{x-1}{1} = \frac{y+3/2}{3/2} = \frac{z+5}{2} \quad \text{...(ii)}$$

Let $\vec{b_1}$ and $\vec{b_2}$ be vectors parallel to (i) and (ii) respectively. Then,

$$\vec{b_1} = 3\hat{i}-2\hat{j}+0\hat{k} \text{ and } \vec{b_2} = \hat{i}+\frac{3}{2}\hat{j}+2\hat{k}.$$

If θ is the angle between the given lines, then

$$\cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|} = \frac{(3)(1)+(-2)(3/2)+(0)(2)}{\sqrt{3^2+(-2)^2+0^2}\,\sqrt{1^2+(3/2)^2+2^2}} = 0$$

$$\Rightarrow \quad \theta = \pi/2.$$

EXAMPLE 3 *Prove that the line $x = ay+b$, $z = cy+d$ and $x = a'y+b'$, $z = c'y+d'$ are perpendicular if $aa'+cc'+1 = 0$.*

SOLUTION The equations of the given lines are not in symmetrical form. We first put them in symmetrical form.

Equations of first line are $x = ay+b$, $z = cy+d$. These equations can be written as

$$\frac{x-b}{a} = y,\ \frac{z-d}{c} = y \text{ or, } \frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c} \quad \text{...(i)}$$

Similarly equations $x = a'y+b'$, $z = c'y+d'$ can be written as

$$\frac{x-b'}{a'} = \frac{y-0}{1} = \frac{z-d'}{c'} \quad \text{...(ii)}$$

Lines (i) and (ii) are parallel to the vectors $\vec{m_1} = a\hat{i}+\hat{j}+c\hat{k}$ and $\vec{m_2} = a'\hat{i}+\hat{j}+c'\hat{k}$ respectively.

If lines (i) and (ii) are perpendicular, then vectors $\vec{m_1}$ and $\vec{m_2}$ are perpendicular

$$\Rightarrow \quad \vec{m_1}\cdot\vec{m_2} = 0$$

$$\Rightarrow \quad (a\hat{i}+\hat{j}+c\hat{k})\cdot(a'\hat{i}+\hat{j}+c'\hat{k}) = 0$$

$$\Rightarrow \quad aa'+1+cc' = 0.$$

EXAMPLE 4 *Find the angle between two lines having direction ratios 1, 1, 2 and $(\sqrt{3}-1), (-\sqrt{3}-1), 4$*

SOLUTION Let $\vec{m_1}$ and $\vec{m_2}$ be vectors parallel to the two given lines. Then, angle between the two given lines is same as the angle between $\vec{m_1}$ and $\vec{m_2}$.

Now,

$$\vec{m_1} = \text{Vector parallel to the line with direction ratio } 1, 1, 2 = \hat{i}+\hat{j}+2\hat{k}$$

and, $\quad \vec{m_2}$ = Vector parallel to the line with direction ratios $\sqrt{3}-1, -\sqrt{3}-1, 4$

$$= (\sqrt{3}-1)\hat{i} - (\sqrt{3}+1)\hat{j} + 4\hat{k}$$

Let θ be the angle between the lines. Then,

$$\cos\theta = \frac{\vec{m_1}\cdot\vec{m_2}}{|\vec{m_1}|\,|\vec{m_2}|} = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)+8}{\sqrt{1+1+4}\,\sqrt{(\sqrt{3}-1)^2+(\sqrt{3}+1)^2+16}}$$

$$\Rightarrow \quad \cos\theta = \frac{6}{\sqrt{6}\,\sqrt{24}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

Type II FINDING THE EQUATION OF A LINE PARALLEL TO A GIVEN LINE AND PASSING THROUGH A GIVEN POINT

Formulae to be used : (i) $\vec{r} = \vec{a} + \lambda\vec{b}$ $\qquad$ *(ii)* $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$

EXAMPLE 5 *Find the equation of a line passing through a point (2, – 1, 3) and parallel to the line* $\vec{r} = (\hat{i}+\hat{j}) + \lambda(2\hat{i}+\hat{j}-2\hat{k})$.

SOLUTION The given line is parallel to the vector $2\hat{i}+\hat{j}-2\hat{k}$ and the required line is parallel to the given line. So, required line is parallel to the vector $2\hat{i}+\hat{j}-2\hat{k}$. It is given that the required line passes through the point (2, – 1, 3). So, the equation of the required line is

$$\vec{r} = (2\hat{i}-\hat{j}+3\hat{k}) + \mu(2\hat{i}+\hat{j}-2\hat{k})$$

EXAMPLE 6 *Find the equation of a line passing through (1, – 1, 0) and parallel to the line*

$$\frac{x-2}{3} = \frac{2y+1}{2} = \frac{5-z}{1}$$

SOLUTION The equation of the given line is

$$\frac{x-2}{3} = \frac{2y+1}{2} = \frac{5-z}{1}$$

This can be written as

$$\frac{x-2}{3} = \frac{y+1/2}{1} = \frac{z-5}{-1}$$

Direction ratios of this line are 3, 1, – 1. So, the direction ratios of the parallel line are proportional to 3, 1, – 1.

The required line passes through (1, – 1, 0) and its direction ratios are proportional to 3, 1, – 1. So, its equation is

$$\frac{x-1}{3} = \frac{y+1}{1} = \frac{z-0}{-1}.$$

Type III FINDING THE EQUATION OF A LINE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO TWO GIVEN LINES

Result to be used: A line passing through a point having position vector $\vec{\alpha}$ and perpendicular to the lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is parallel to $\vec{b_1}\times\vec{b_2}$. So, its vector equation is

$$\vec{r} = \vec{\alpha} + \lambda(\vec{b_1}\times\vec{b_2})$$

Following algorithm may be used to find the equation of a line passing through a given point and perpendicular to the given lines.

ALGORITHM

STEP I *Obtain the point through which the line passes. Let its position vector be $\vec{\alpha}$.*

STEP II *Obtain the vectors parallel to the two given lines. Let the vectors be $\vec{b_1}$ and $\vec{b_2}$.*

STEP III *Obtain $\vec{b_1} \times \vec{b_2}$*

STEP IV *Write the required line as $\vec{r} = \vec{\alpha} + \lambda (\vec{b_1} \times \vec{b_2})$.*

EXAMPLE 7 *Find the equation of the line passing through the point $(-1, 3, -2)$ and perpendicular to the lines*

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \text{ and } \frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$$ **[CBSE 2005]**

SOLUTION Let the direction ratios of the required line be a, b, c. Since it is perpendicular to the two given lines. Therefore,

$$a + 2b + 3c. = 0 \quad \text{...(i)}$$

and, $$-3a + 2b + 5c = 0 \quad \text{...(ii)}$$

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{a}{4} = \frac{b}{-14} = \frac{c}{8} \text{ or, } \frac{a}{2} = \frac{b}{-7} = \frac{c}{4} = k \text{ (say)}$$

Thus, the required line passes through $(-1, 3, -2)$ and has direction ratios proportional to $2, -7, 4$. So, its equation is

$$\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

EXAMPLE 8 *A line passes through $(2, -1, 3)$ and is perpendicular to the line $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda (2\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu (\hat{i} + 2\hat{j} + 2\hat{k})$. Obtain its equation.*

SOLUTION The required line is perpendicular to the lines which are parallel to vectors $\vec{b_1} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$ respectively.

So, it is parallel to the vector $\vec{b} = \vec{b_1} \times \vec{b_2}$.

Now, $$\vec{b} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\hat{i} - 3\hat{j} + 6\hat{k}$$

Thus, the required line passes through the point $(2, -1, 3)$ and is parallel to the vector $\vec{b} = -6\hat{i} - 3\hat{j} + 6\hat{k}$.

So, its vector equation is

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(-6\hat{i} - 3\hat{j} + 6\hat{k})$$

or, $$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \mu(2\hat{i} + \hat{j} - 2\hat{k}), \text{ where } \mu = -2\lambda.$$

Type IV **ON PERPENDICULARITY OF TWO LINES**

EXAMPLE 9 *Find the value of λ so that the lines*

$$\frac{1-x}{3} = \frac{7y-14}{2\lambda} = \frac{z-3}{2} \text{ and } \frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5} \text{ are at right angle.}$$ **[NCERT]**

SOLUTION The equations of the given lines are

$$\frac{1-x}{3} = \frac{7y-14}{2\lambda} = \frac{z-3}{2} \text{ and } \frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$$

These equations may be re-written in standard form as follows:

$$\frac{x-1}{-3}=\frac{y-2}{\frac{2\lambda}{7}}=\frac{z-3}{2} \text{ and } \frac{x-1}{\frac{-3\lambda}{7}}=\frac{y-5}{7}=\frac{z-6}{-5}$$

If these lines are perpendicular, then

$$-3\times\frac{-3\lambda}{7}+\frac{2\lambda}{7}\times 1+2\times -5=0$$

$$\Rightarrow \quad \frac{9\lambda}{7}+\frac{2\lambda}{7}-10=0 \Rightarrow \frac{11\lambda}{7}-10=0 \Rightarrow \lambda=\frac{70}{11}.$$

EXERCISE 27.2

1. Find the angle between the following pairs of lines:
   (i) $\vec{r}=(4\hat{i}-\hat{j})+\lambda(\hat{i}+2\hat{j}-2\hat{k})$ and $\vec{r}=\hat{i}-\hat{j}+2\hat{k}-\mu(2\hat{i}+4\hat{j}-4\hat{k})$
   (ii) $\vec{r}=(3\hat{i}+2\hat{j}-4\hat{k})+\lambda(\hat{i}+2\hat{j}+2\hat{k})$ and $\vec{r}=(5\hat{j}-2\hat{k})+\mu(3\hat{i}+2\hat{j}+6\hat{k})$
   (iii) $\vec{r}=\lambda(\hat{i}+\hat{j}+2\hat{k})$ and $\vec{r}=2\hat{j}+\mu[(\sqrt{3}-1)\hat{i}-(\sqrt{3}+1)\hat{j}+4\hat{k}]$
2. Find the angle between the following pairs of lines :
   (i) $\frac{x+4}{3}=\frac{y-1}{5}=\frac{z+3}{4}$ and $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$
   (ii) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{-3}$ and $\frac{x+3}{-1}=\frac{y-5}{8}=\frac{z-1}{4}$
   (iii) $\frac{5-x}{-2}=\frac{y+3}{1}=\frac{1-z}{3}$ and $\frac{x}{3}=\frac{1-y}{-2}=\frac{z+5}{-1}$
   (iv) $\frac{x-2}{3}=\frac{y+3}{-2}, z=5$ and $\frac{x+1}{1}=\frac{2y-3}{3}=\frac{z-5}{2}$
   (v) $\frac{x-5}{1}=\frac{2y+6}{-2}=\frac{z-3}{1}$ and $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-6}{5}$
3. Find the angle between the pairs of lines with direction ratios proportional to
   (i) 5, – 12, 13 and – 3, 4, 5
   (ii) 2, 2, 1 and 4, 1, 8
   (iii) 1, 2, – 2 and – 2, 2, 1
   (iv) a, b, c and $b-c, c-a, a-b$. [NCERT]
4. Find the angle between two lines, one of which has direction ratios 2, 2, 1 while the other one is obtained by joining the points (3, 1, 4) and (7, 2, 12).
5. Find the equation of the line passing through the point (1, 2, – 4) and parallel to the line $\frac{x-3}{4}=\frac{y-5}{2}=\frac{z+1}{3}$.
6. Find the equations of the line passing through the point (– 1, 2, 1) and parallel to the line
$$\frac{2x-1}{4}=\frac{3y+5}{2}=\frac{2-z}{3}$$
7. Find the equation of the line passing through the point (2, – 1, 3) and parallel to the line $\vec{r}=(\hat{i}-2\hat{j}+\hat{k})+\lambda(2\hat{i}+3\hat{j}-5\hat{k})$.
8. Find the equations of the line passing through the point (2, 1, 3) and perpendicular to the lines

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3} \text{ and } \frac{x}{-3}=\frac{y}{2}=\frac{z}{5}.$$

9. Find the equation of the line passing through the point $\hat{i}+\hat{j}-3\hat{k}$ and perpendicular to the lines $\vec{r}=\hat{i}+\lambda(2\hat{i}+\hat{j}-3\hat{k})$ and $\vec{r}=(2\hat{i}+\hat{j}-\hat{k})+\mu(\hat{i}+\hat{j}+\hat{k})$.
10. Find the equation of the line passing through the point $(1,-1,1)$ and perpendicular to the lines joining the points $(4,3,2)$, $(1,-1,0)$ and $(1,2,-1)$, $(2,1,1)$.
11. Determine the equations of the line passing through the point $(1,2,-4)$ and perpendicular to the two lines
$$\frac{x-8}{8}=\frac{y+9}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$
12. Show that the lines
$$\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1} \text{ and } \frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$
are perpendicular to each other.
13. Find the vector equation of the line passing through the point $(2,-1,-1)$ which is parallel to the line $6x-2=3y+1=2z-2$.
14. If the lines $\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$ and $\frac{x-1}{3\lambda}=\frac{y-1}{1}=\frac{z-6}{-5}$ are perpendicular, find the value of λ. **[NCERT, CBSE 2009]**
15. If the coordinates of the points A, B, C, D be $(1,2,3)$, $(4,5,7)$, $(-4,3,-6)$ and $(2,9,2)$ respectively, then find the angle between the lines AB and CD. **[NCERT]**
16. Find the value of λ so that the following lines are perpendicular to each other.
$$\frac{x-5}{5\lambda+2}=\frac{2-y}{5}=\frac{1-z}{-1}, \frac{x}{1}=\frac{2y+1}{4\lambda}=\frac{1-z}{-3}$$ **[CBSE 2009]**

ANSWERS

1. (i) $0°$ (ii) $\cos^{-1}\left(\frac{19}{21}\right)$ (iii) $\frac{\pi}{3}$
2. (i) $\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$ (ii) $\cos^{-1}\left(\frac{10}{9\sqrt{22}}\right)$ (iii) $\cos^{-1}\left(\frac{11}{14}\right)$ (iv) $\frac{\pi}{2}$ (v) $\frac{\pi}{2}$
3. (i) $\cos^{-1}\left(\frac{1}{65}\right)$ (ii) $\cos^{-1}\left(\frac{2}{3}\right)$ (iii) $\frac{\pi}{2}$ (iv) $\frac{\pi}{2}$
4. $\cos^{-1}\left(\frac{2}{3}\right)$
5. $\frac{x-1}{4}=\frac{y-2}{2}=\frac{z+4}{3}$
6. $\frac{x+1}{2}=\frac{y-2}{2/3}=\frac{z-1}{-3}$
7. $\vec{r}=(2\hat{i}-\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}-5\hat{k})$
8. $\frac{x-2}{2}=\frac{y-1}{-7}=\frac{z-3}{4}$
9. $\vec{r}=(\hat{i}+\hat{j}-3\hat{k})+\lambda(4\hat{i}-5\hat{j}+\hat{k})$
10. $\frac{x-1}{10}=\frac{y+1}{-4}=\frac{z-1}{-7}$
11. $\frac{x-1}{24}=\frac{y-2}{61}=\frac{z+4}{112}$.
13. $\vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\lambda(\hat{i}+2\hat{j}+3\hat{k})$
14. $\lambda=-\frac{10}{7}$
15. 0.
16. 1

27.4 INTERSECTION OF TWO LINES

To determine whether two lines intersect or not and in case they intersect the following algorithm is used to find their point of intersection.

ALGORITHM Let the two lines be

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \qquad ...(i)$$

and, $$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2} \qquad ...(ii)$$

STEP I *Write the coordinates of general points on (i) and (ii). The coordinates of general points on (i) and (ii) are given by*

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}=\lambda \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}=\mu \text{ respectively,}$$

i.e. $(a_1\lambda+x_1, b_1\lambda+y_1, c_1\lambda+z_1)$ *and* $(a_2\mu+x_2, b_2\mu+y_2, c_2\mu+z_2)$

STEP II *If the line (i) and (ii) intersect, then they have a common point.*

$\therefore\ a_1\lambda+x_1=a_2\mu+x_2,\ b_1\lambda+y_1=b_2\mu+y_2$ *and* $c_1\lambda+z_1=c_2\mu+z_2$

STEP III *Solve any two of the equations in* λ *and* μ *obtained in step II. If the values of* λ *and* μ *satisfy the third equation, then the lines (i) and (ii) intersect. Otherwise they do not intersect.*

STEP IV *To obtain the coordinates of the point of intersection, substitute the value* λ *(or* μ*) in the coordinates of general point(s) obtained in step I.*

ALGORITHM FOR VECTOR FORM Let the two lines be

$$\vec{r}=(a\hat{i}+a_2\hat{j}+a_3\hat{k})+\lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \qquad ...(i)$$

and, $$\vec{r}=(a'_1\hat{i}+a'_2\hat{j}+a'_3\hat{k})+\mu(b'_1\hat{i}+b'_2\hat{j}+b'_3\hat{k}) \qquad ...(ii)$$

STEP I *Since* $\vec{r}$ *in the equation of a line denotes the position vector of an arbitrary point on it. Therefore, position vectors of arbitrary points on (i) and (ii) are*

$$(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})+\lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$

and, $(a'_1\hat{i}+a'_2\hat{j}+a'_3\hat{k})+\mu(b'_1\hat{i}+b'_2\hat{j}+b'_3\hat{k})$ *respectively*

STEP II *If the lines (i) and (ii) intersect, then they have a common point. So,*

$$(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})+\lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})=(a'_1\hat{i}+a'_2\hat{j}+a'_3\hat{k})+\mu(b'_1\hat{i}+b'_2\hat{j}+b'_3\hat{k})$$

$$\Rightarrow (a_1+\lambda b_1)\hat{i}+(a_2+\lambda b_2)\hat{j}+(a_3+\lambda b_3)\hat{k}=(a'_1+\mu b'_1)\hat{i}+(a'_2+\mu b'_2)\hat{j}+(a'_3+\mu b'_3)\hat{k}$$

$$\Rightarrow a_1+\lambda b_1=a'_1+\mu b'_1,\ a_2+\lambda b_2=a'_2+\mu b'_2 \text{ and } a_3+\lambda b_3=a'_3+\mu b'_3$$

STEP III *Solve any two of the equations in* λ *and* μ *obtained in step II. If the values of* λ *and* μ *satisfy the third equation, then the two lines intersect. Otherwise they do not.*

STEP IV *To obtain the position vector of the point of intersection, substitute the value of* λ *(or* μ*) in (i) (or (ii)).*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Show that the line* $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ *and* $\frac{x-4}{5}=\frac{y-1}{2}=z$ *intersect. Find their point of intersection.* **[CBSE 2004, 2005]**

SOLUTION We have,

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda \quad \text{(say)}$$

$\Rightarrow \quad x=2\lambda+1, y=3\lambda+2$ and $z=4\lambda+3$

So, the coordinates of a general point on this line are $(2\lambda+1, 3\lambda+2, 4\lambda+3)$.

The equation of second line is

$$\frac{x-4}{5}=\frac{y-1}{2}=\frac{z-0}{1}=\mu \text{ (say)}$$

$\Rightarrow$ $x=5\mu+4,\ y=2\mu+1,\ z=\mu$

So, the coordinates of a general point on this line are $(5\mu+4, 2\mu+1, \mu)$.

If the lines intersect, then they have a common point. So, for some values of λ and μ, we must have

$$2\lambda+1 = 5\mu+4,\ 3\lambda+2 = 2\mu+1 \text{ and } 4\lambda+3 = \mu$$

or, $2\lambda-5\mu = 3,\ 3\lambda-2\mu = -1,\ 4\lambda-\mu = -3.$

Solving first two of these two equations, we get

$\lambda=-1$ and $\mu=-1$.

Since $\lambda=-1$ and $\mu=-1$ satisfy the third equation. So, the given lines intersect.

Putting $\lambda=-1$ in $(2\lambda+1, 3\lambda+2, 4\lambda+3)$, the coordinates of the required point of intersection are $(-1,-1,-1)$.

EXAMPLE 2 *Show that the lines* $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$ *and* $\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$ *do not intersect.* **[CBSE 2002]**

SOLUTION We have,

$$\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}=\lambda \quad \text{(say)}$$

$\Rightarrow$ $x=3\lambda+1,\ y=2\lambda-1,\ z=5\lambda+1.$

So, the coordinates of a general point on this line are $(3\lambda+1, 2\lambda-1, 5\lambda+1)$.

The equation of the second line is

$$\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}=\mu. \text{ (say)}$$

$\Rightarrow$ $x=4\mu-2,\ y=3\mu+1,\ z=-2\mu-1$

So, the coordinates of a general point on this line are

$(4\mu-2,\ 3\mu+1,\ -2\mu-1)$.

If the line intersect, then they have a common point. So, for some values of λ and μ, we must have

$$3\lambda+1 = 4\mu-2,\ 2\lambda-1 = 3\mu+1 \text{ and } 5\lambda+1 = -2\mu-1.$$

$\Rightarrow$ $3\lambda-4\mu = -3$ ...(i)

$2\lambda-3\mu = 2$ ...(ii)

and, $5\lambda+2\mu = -2$ ...(iii)

Solving (i) and (ii), we obtain $\lambda=-17$ and $\mu=-12$.

These values of λ and μ do not satisfy the third equation.

Hence, the given lines do not intersect.

EXAMPLE 3 *Show that the lines*

$$\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j}) \text{ and } \vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})$$

intersect. Find their point of intersection.

SOLUTION The position vectors of arbitrary points on the given lines are

$$(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})=(3\lambda+1)\hat{i}+(1-\lambda)\hat{j}-\hat{k}$$

and, $$(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})=(2\mu+4)\hat{i}+0\hat{j}+(3\mu-1)\hat{k} \text{ respectively.}$$

If the lines intersect, then they have a common point. So, for some values of λ and μ, we must have

$$(3\lambda+1)\hat{i}+(1-\lambda)\hat{j}-\hat{k} = (2\mu+4)\hat{i}+0\hat{j}+(3\mu-1)\hat{k}$$

$$\Rightarrow \quad 3\lambda+1 = 2\mu+4,\ 1-\lambda = 0 \text{ and } -1 = 3\mu-1$$

[On equating coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$]

Solving last two of these two equations, we get $\lambda = 1$ and $\mu = 0$.

These values of λ and μ satisfy the third equation.

So, the given lines intersect.

Putting $\lambda = 1$ in first line, we get $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+(3\hat{i}-\hat{j})=4\hat{i}+0\hat{j}-\hat{k}$ as the position vector of the point of interaction.

Thus, the coordinates of the point of intersection are $(4, 0, -1)$.

EXERCISE 27.3

1. Show that the lines $\frac{x}{1}=\frac{y-2}{2}=\frac{z+3}{3}$ and $\frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4}$ intersect and find their point of intersection.
2. Show that the lines $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$ and $\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$ do not intersect.
3. Show that the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect. Find their point of intersection.
4. Prove that the lines through $A(0,-1,-1)$ and $B(4,5,1)$ intersects the line through $C(3,9,4)$ and $D(-4,4,4)$. Also find their point of intersection.
5. Prove that the line $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})$ and $\vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})$ intersect and find their point of intersection.
6. Determine whether the following pair of lines intersect or not :
   (i) $\vec{r}=(\hat{i}-\hat{j})+\lambda(2\hat{i}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j})+\mu(\hat{i}+\hat{j}-\hat{k})$
   (ii) $\frac{x-1}{2}=\frac{y+1}{3}=z,\ \frac{x+1}{5}=\frac{y-2}{1};\ z=2$
   (iii) $\frac{x-1}{3}=\frac{y-1}{-1}=\frac{z+1}{0},\ \frac{x-4}{2}=\frac{y-0}{0}=\frac{z+1}{3}$
   (iv) $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5},\ \frac{x-8}{7}=\frac{y-4}{1}=\frac{3-5}{3}$ [CBSE 2002]

ANSWERS

1. $(2, 6, 3)$ 3. $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$ 4. $(10, 14, 4)$ 5. $(4, 0, -1)$

6. (i) No (ii) No (iii) Yes (iv) Yes.

27.5 PERPENDICULAR DISTANCE OF A LINE FROM A POINT

CARTESIAN FORM Let $P(\alpha, \beta, \gamma)$ be a given point and let $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ be a given line .

Let L be the foot of the perpendicular drawn from $P(\alpha, \beta, \gamma)$ on the line $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$

Let the coordinates of L be

$$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$$

Then, direction ratios of PL are proportional to $x_1 + a\lambda - \alpha, y_1 + b\lambda - \beta, z_1 + c\lambda - \gamma$

Direction ratio of AB are proportional to a, b, c.

Since PL is perpendicular to AB. Therefore,

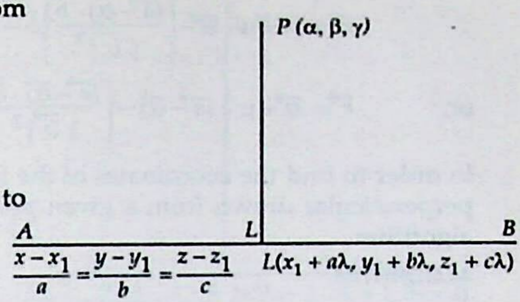

Fig. 27.3

$$(x_1 + a\lambda - \alpha)\,a + (y_1 + b\lambda - \beta)\,b + (z_1 + c\lambda - \gamma)\,c = 0$$

$$\Rightarrow \quad \lambda = -\frac{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2}$$

Putting this value of λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, we obtain coordinates of L.

Now, using distance formula we can obtain the length PL.

VECTOR FORM Let $P(\vec{\alpha})$ be a point and $\vec{r} = \vec{a} + \lambda\vec{b}$ be the vector equation of a line.

Let L be the foot of the perpendicular drawn from $P(\vec{\alpha})$ on the line $\vec{r} = \vec{a} + \lambda\vec{b}$.

Since $\vec{r}$ denotes the position vector of any point on the line $\vec{r} = \vec{a} + \lambda\vec{b}$. So, let the position vector of L be $\vec{a} + \lambda\vec{b}$. Then,

$$\vec{PL} = \vec{a} + \lambda\vec{b} - \vec{\alpha} = \vec{a} - \vec{\alpha} + \lambda\vec{b}$$

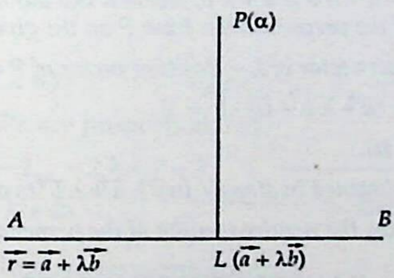

Fig. 27.4

Since $\vec{PL}$ is perpendicular to the line which is parallel to $\vec{b}$. Therefore,

$$\vec{PL} \perp \vec{b}$$

$$\Rightarrow \quad \vec{PL} \cdot \vec{b} = 0$$

$$\Rightarrow \quad (\vec{a} - \vec{\alpha} + \lambda\vec{b}) \cdot \vec{b} = 0$$

$$\Rightarrow \quad (\vec{a} - \vec{\alpha}) \cdot \vec{b} + \lambda(\vec{b} \cdot \vec{b}) = 0$$

$$\Rightarrow \quad \lambda = -\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}$$

$\therefore$ Position vector of L is $\vec{a} + \lambda\vec{b} = \vec{a} - \left(\dfrac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}$

Now we can obtain $\vec{PL}$ and length PL is the magnitude of $\vec{PL}$.

Since PL passes through P and L. Therefore, equation of the perpendicular line PL is

$$\vec{r} = \vec{\alpha} + \mu \left\{ \vec{a} - \left(\frac{(\vec{a} - \alpha) \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b} - \vec{\alpha} \right\}$$

or, $$\vec{r} = \vec{\alpha} + \mu \left\{ (\vec{a} - \vec{\alpha}) - \left(\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b} \right\}$$

In order to find the coordinates of the foot of the perpendicular and the length of the perpendicular drawn from a given point on a given line we may use the following algorithm.

ALGORITHM

CARTESIAN FORM *Let P (α, β, γ) be the given point, and let the given line be*

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

STEP I *Write the coordinates of a general point on the given line. The coordinates of general point on the line are $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, where λ is a parameter. Assume that this point L is the foot of the perpendicular drawn from P on the given line.*

STEP II *Write direction ratios of PL.*

STEP III *Apply the condition of perpendicularity of the given line and PL.*

STEP IV *Obtain the value of λ from step III.*

STEP V *Substitute λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$ to obtain the coordinates of L.*

STEP VI *Obtain PL by using distance formula.*

VECTOR FORM *Let P $(\vec{\alpha})$ be the given point, and let $\vec{r} = \vec{a} + \lambda \vec{b}$ be the given line.*

STEP I *Write the position vector of a general point on the given line. The position vector of a general point on $\vec{r} = \vec{a} + \lambda \vec{b}$ is $\vec{a} + \lambda \vec{b}$, where λ is a parameter. Assume that this point L is required foot of the perpendicular from P on the given line.*

STEP II *Obtain $\vec{PL}$ = Position vector of L – Position vector of P $= \vec{a} + \lambda \vec{b} - \vec{\alpha}$.*

STEP III *Put* $\vec{PL} \cdot \vec{b} = 0$ i.e. $(\vec{a} + \lambda \vec{b} - \vec{\alpha}) \cdot \vec{b} = 0$

STEP IV *Obtain λ from step III.*

STEP V *Put the value of λ obtained in step IV in $\vec{r} = \vec{a} + \lambda \vec{b}$ to obtain the position vector of L.*

STEP VI *Find $|\vec{PL}|$ to obtain the required length of the perpendicular.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the foot of the perpendicular from the point (0, 2, 3) on the line* $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$. *Also, find the length of the perpendicular.*

SOLUTION Let L be the foot of the perpendicular drawn from the point P (0, 2, 3) to the given line.

The coordinates of a general point on

$$\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3} \text{ are given by } \frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3} = \lambda$$

i.e., $x = 5\lambda - 3,\ y = 2\lambda + 1,\ z = 3\lambda - 4.$

Let the coordinates of L be

$(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$ ...(i)

$\therefore$ Direction ratios of PL are proportional to

$$5\lambda - 3 - 0,\ 2\lambda + 1 - 2,\ 3\lambda - 4 - 3$$

i.e. $\quad 5\lambda - 3,\ 2\lambda - 1,\ 3\lambda - 7.$

Direction ratios of the given line are proportional to 5, 2, 3.

Since PL is perpendicular to the given line.

$\therefore\ 5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0 \Rightarrow \lambda = 1.$

Putting $\lambda = 1$ in (i), the coordinates of L are (2, 3, −1)

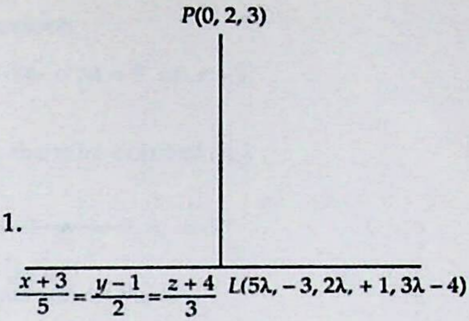

Fig. 27.5

$\therefore$ Length of the perpendicular from P on the given line

$$= PL = \sqrt{(2-0)^2 + (3-2)^2 + (-1-3)^2} = \sqrt{21} \text{ units.}$$

EXAMPLE 2 *Find the length of the perpendicular from the point (1, 2, 3) to the line* $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$.

SOLUTION Let L be the foot of the perpendicular drawn from the point P (1, 2, 3) to the given line.

The coordinated of a general point on $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ are given by

$$\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2} = \lambda$$

i.e. $\quad x = 3\lambda + 6, y = 2\lambda + 7, z = -2\lambda + 7$

Let the coordinates of L be

$(3\lambda + 6, 2\lambda + 7, -2\lambda + 7)$ ...(i)

$\therefore$ Direction ratios of PL are proportional to

$3\lambda + 6 - 1,\ 2\lambda + 7 - 2,,\ -2\lambda + 7 - 3$

i.e. $\quad 3\lambda + 5, 2\lambda + 5,\ -2\lambda + 4.$

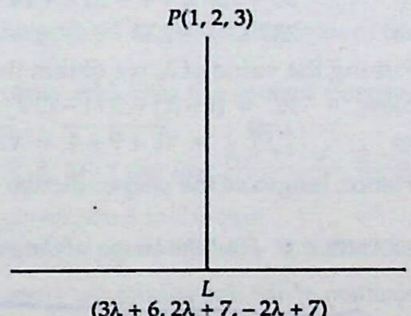

Fig. 27.6

Direction ratios of the given line are proportional to 3, 2, −2.

Since PL is perpendicular to the given line. Therefore,

$\therefore\quad 3(3\lambda + 5) + 2(2\lambda + 5) + (-2)(-2\lambda + 4) = 0 \Rightarrow \lambda = -1.$

Putting $\lambda = -1$ in (i), we obtain the coordinates of L as (3, 5, 9)

$\therefore\quad PL = \sqrt{(3-1)^2 + (5-2)^2 + (9-3)^2} = 7$ unit

Hence, the required length of the perpendicular is 7 units.

EXAMPLE 3 *Find the foot of the perpendicular drawn from the point* $2\hat{i} - \hat{j} + 5\hat{k}$ *to the line* $\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$. *Also, find the length of the perpendicular.*

SOLUTION Let L be the foot of the perpendicular drawn from P $(2\hat{i} - \hat{j} + 5\hat{k})$ on the line

$$\vec{r} = 11\hat{i} - 2\hat{j} - 8\hat{k} + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k}).$$

Let the position vector of L be

$$11\hat{i} - 2\hat{j} - 8\hat{k} + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k}) = (11 + 10\lambda)\hat{i} + (-2 - 4\lambda)\hat{j} + (-8 - 11\lambda)\hat{k}.$$

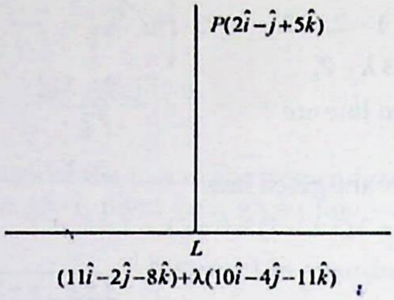

Fig. 27.7

Then, $\vec{PL}$ = Position vector of L – Position vector of P

$\Rightarrow \quad \vec{PL} = [(11+10\lambda)\hat{i}+(-2-4\lambda)\hat{j}+(-8-11\lambda)\hat{k}]-[2\hat{i}-\hat{j}+5\hat{k}]$

$\Rightarrow \quad \vec{PL} = (9+10\lambda)\hat{i}+(-1-4\lambda)\hat{j}+(-13-11\lambda)\hat{k}$

Since $\vec{PL}$ is perpendicular to the given line which is parallel to $\vec{b}=10\hat{i}-4\hat{j}-11\hat{k}$

$\therefore \quad \vec{PL} \perp \vec{b}$

$\Rightarrow \quad \vec{PL}\cdot\vec{b}=0$

$\Rightarrow \quad [(9+10\lambda)\hat{i}+(-1-4\lambda)\hat{j}+(-13-11\lambda)\hat{k}]\cdot(10\hat{i}-4\hat{j}-11\hat{k})=0$

$\Rightarrow \quad 10(9+10\lambda)-4(-1-4\lambda)-11(-13-11\lambda)=0$

$\Rightarrow \quad 90+100\lambda+4+16\lambda+143+121\lambda=0$

$\Rightarrow \quad 237\lambda=-237 \Rightarrow \lambda=-1$

Putting the value of λ, we obtain the position vector of L as $\hat{i}+2\hat{j}+3\hat{k}$

Now, $\quad \vec{PL} = (i+2\hat{j}+3\hat{k})-(2\hat{i}-\hat{j}+5\hat{k}) = -\hat{i}+3\hat{j}-2\hat{k}$

$\Rightarrow \quad |\vec{PL}| = \sqrt{1+9+4} = \sqrt{14}.$

Hence, length of the perpendicular from P on the give line is 14 units.

EXAMPLE 4 *Find the image of the point* (1, 6, 3) *in the line* $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$. *Also, write the equation of the line joining the given point and its image and find the length of the segment joining the given point and its image.* **[CBSE 2010]**

SOLUTION Let Q be the image of point P (1, 6, 3) in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ and M be the foot of perpendicular drawn from P to this line. Then, $PM = MQ$.

Let the coordinates of M be given by

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}=r$$

The coordinates of M are $(r, 2r+1, 3r+2)$.

The direction ratios of M are proportional to $r-1, 2r-5, 3r-1$.

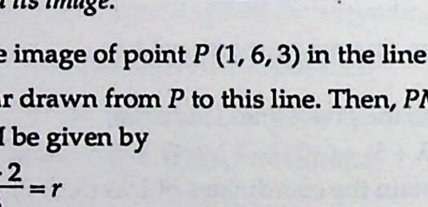

Fig. 27.8

Since PM is perpendicular to the given line. Therefore,

$$1(r-1)+2(2r-5)+3(3r-1) = 0 \Rightarrow 14r-14=0 \Rightarrow r=1.$$

So, the coordinates of M are $(1, 3, 5)$.
Let (x_1, y_1, z_1) be the coordinates of Q. Since M is the mid-point of PQ.

$$\therefore \quad \frac{x_1+1}{2}=1, \frac{y_1+6}{2}=3, \frac{z_1+3}{2}=5 \Rightarrow x_1 = 1, y_1 = 0, z_1 = 7$$

Thus, the coordinates of Q are $(1, 0, 7)$. So, the equation of PQ is

$$\frac{x-1}{1-1} = \frac{y-6}{0-6} = \frac{z-3}{7-3} \text{ or, } \frac{x-1}{0} = \frac{y-6}{-6} = \frac{z-3}{4}$$

and, $$PQ = \sqrt{(1-1)^2+(6-0)^2+(3-7)^2} = 2\sqrt{13}$$

EXERCISE 27.4

1. Find the perpendicular distance of the point $(3, -1, 11)$ from the line $\frac{x}{2}=\frac{y-2}{-3}=\frac{z-3}{4}$.
2. Find the perpendicular distance of the point $(1, 0, 0)$ from the line $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$. Also, find the coordinates of the foot of the perpendicular. [CBSE 2005]
3. Find the foot of the perpendicular drawn from the point $A(1, 0, 3)$ to the joint of the points $B(4, 7, 1)$ and $C(3, 5, 3)$.
4. $A(1, 0, 4)$, $B(0, -11, 3)$, $C(2, -3, 1)$ are three points and D is the foot of perpendicular from A on BC. Find the coordinates of D.
5. Find the foot of perpendicular from the point $(2, 3, 4)$ to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line.
6. Find the equation of the perpendicular drawn from the point $P(2, 4, -1)$ to the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$.
   Also, write down the coordinates of the foot of the perpendicular from P.
7. Find the length of the perpendicular drawn from the point $(5, 4, -1)$ to the line $\vec{r}=\hat{i}+\lambda(2\hat{i}+9\hat{j}+5\hat{k})$.
8. Find the foot of the perpendicular drawn from the point $\hat{i}+6\hat{j}+3\hat{k}$ to the line $\vec{r}=\hat{j}+2\hat{k}+\lambda(\hat{i}+2\hat{j}+3\hat{k})$. Also find the length of the perpendicular
9. Find the equation of the perpendicular drawn from the point $P(-1, 3, 2)$ to the line $\vec{r}=(2\hat{j}+3\hat{k})+\lambda(2\hat{i}+\hat{j}+3\hat{k})$. Also find the coordinates of the foot of the perpendicular from P.
10. Find the foot of the perpendicular from $(0, 2, 7)$ on the line
$$\frac{x+2}{-1}=\frac{y-1}{3}=\frac{z-3}{-2}$$
11. Find the foot of the perpendicular from $(1, 2, -3)$ to the line
$$\frac{x+1}{2}=\frac{y-3}{-2}=\frac{z}{-1}$$
12. Find the equation of line passing through the points $A(0, 6, -9)$ and $B(-3, -6, 3)$. If D is the foot of perpendicular drawn from a point $C(7, 4, -1)$ on the line AB, then find the coordinates of the point D and the equation of line CD. [CBSE 2010]

ANSWERS

1. $\sqrt{53}$
2. $2\sqrt{6}, (3, -4, -2)$
3. $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$
4. $\left(\frac{22}{9}, \frac{-11}{9}, \frac{5}{9}\right)$
5. $\left(\frac{170}{49}, \frac{78}{49}, \frac{10}{49}\right), 3\sqrt{\frac{101}{49}}$
6. $(-4, 1, -3), \frac{x-2}{-6} = \frac{y-4}{-3} = \frac{z+1}{-2}$
7. $\sqrt{\frac{2109}{110}}$
8. $\hat{i} + 3\hat{j} + 5\hat{k}, \sqrt{13}$
9. $\vec{r} = (-\hat{i} + 3\hat{j} + 2\hat{k}) + \lambda(3\hat{i} - 9\hat{j} + \hat{k}), \left(-\frac{4}{7}, \frac{12}{7}, \frac{15}{7}\right)$
10. $\left(-\frac{5}{2}, \frac{5}{2}, 2\right)$
11. $(1, 1, -1)$
12. $\frac{x}{1} = \frac{y-6}{4} = \frac{z+9}{-4}; \frac{x-7}{4} = \frac{y-4}{1} = \frac{z+1}{2}$
13. $(1, 0, 7)$

27.6 SHORTEST DISTANCE BETWEEN TWO STRAIGHT LINES

SKEW LINES *Two straight line in space which are neither parallel nor intersecting are called skew lines.*

Thus, the skew lines are those lines which do not lie in the same plane.

LINE OF SHORTEST DISTANCE *If l_1 and l_2 are two skew-lines, then there is one and only one line perpendicular to each of lines l_1 and l_2 which is known as the line of shortest distance.*

SHORTEST DISTANCE *The shortest distance between two lines l_1 and l_2 is the distance PQ between the points P and Q where the lines of shortest distance intersects the two given lines.*

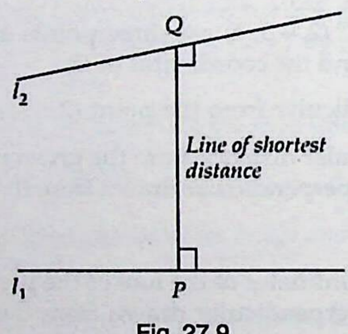

Fig. 27.9

If two lines intersect then the shortest distance between them is zero. If two lines are parallel then the shortest distance between them is the distance between the two lines.

27.6.1 SHORTEST DISTANCE BETWEEN TWO SKEW LINES (Vector Form)

Let l_1 and l_2 be two lines whose equations are

$$l_1 : \vec{r} = \vec{a_1} + \lambda\, \vec{b_1} \text{ and } l_2 : \vec{r} = \vec{a_2} + \mu\, \vec{b_2} \text{ respectively.}$$

Clearly, l_1 and l_2 pass through the points A and B with position vectors $\vec{a_1}$ and $\vec{a_2}$ respectively and are parallel to the vectors $\vec{b_1}$ and $\vec{b_2}$ respectively.

Let $\vec{PQ}$ be the shortest distance vector between l_1 and l_2. Then, $\vec{PQ}$ is perpendicular to both l_1 and l_2 which are parallel to $\vec{b_1}$ and $\vec{b_2}$ respectively. Therefore, $\vec{PQ}$ is perpendicular to both $\vec{b_1}$ and $\vec{b_2}$. But, $\vec{b_1} \times \vec{b_2}$ is perpendicular to both $\vec{b_1}$ and $\vec{b_2}$. Therefore, $\vec{PQ}$ is parallel to $\vec{b_1} \times \vec{b_2}$.

Let $\hat{\eta}$ be a unit vector along $\vec{PQ}$. Then,

$$\hat{n} = \pm \frac{\vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|}$$

From Fig. 27.9, we have

$$PQ = \text{Projection of } \vec{AB} \text{ on } \vec{PQ}$$

$$\Rightarrow \quad PQ = \vec{AB} \cdot \hat{\eta}$$

$$\Rightarrow \quad PQ = \pm (\vec{a_2} - \vec{a_1}) \cdot \left\{ \frac{\vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|} \right\}$$

$$\Rightarrow \quad PQ = \pm \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}$$

$$\Rightarrow \quad PQ = \pm \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}$$

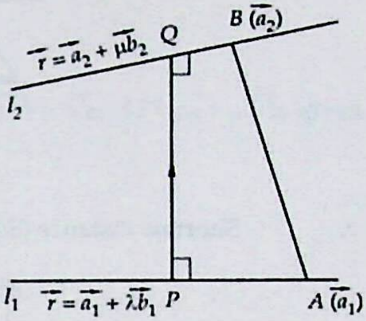

Fig. 27.10

Since the distance PQ is to be taken as positive.

$$\therefore \quad PQ = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

Condition for two given lines to intersect : If the lines $\vec{r} = \vec{a_1} + \lambda \, \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \, \vec{b_2}$ intersect, then the shortest distance between them is zero.

$$\therefore \quad \left| \frac{(\vec{b_1} \times \vec{b_2} \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right| = 0$$

$$\Rightarrow \quad (\vec{b_1} \times \vec{b_2} \cdot (\vec{a_2} - \vec{a_1}) = 0$$

27.6.2 SHORTEST DISTANCE BETWEEN TWO SKEW LINES (Cartesian Form)

Let the two skew lines be

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

The vector equations of these two lines are

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1} \text{ and } \vec{r} = \vec{a_2} + \mu \, \vec{b_2} \quad \text{respectively,}$$

where

$$\vec{a_1} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}, \; \vec{a_2} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k},$$

$$\vec{b_1} = l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k} \text{ and, } \vec{b_2} = l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k}.$$

We have,

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = (m_1 n_2 - m_2 n_1) \hat{i} + (l_2 n_1 - l_1 n_2) \hat{j} + (l_1 m_2 - l_2 m_1) \hat{k}$$

$$\Rightarrow \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(m_1 n_2 - m_2 n_1)^2 + (l_2 n_1 - l_1 n_2)^2 + (l_1 m_2 - l_2 m_1)^2}$$

$$\text{and, } \quad (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

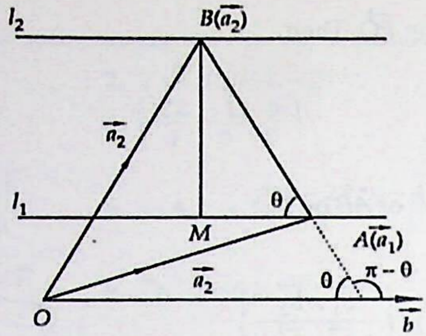

Fig. 27.11

$\therefore$ Shortest distance (S.D.) $= \left|\dfrac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}\right|$

$$\Rightarrow \quad \text{S.D.} = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_1 & n_2 \end{vmatrix}}{\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - l_1 n_2)^2 + (l_1 m_2 - l_2 m_1)^2}}$$

Condition for two given lines to intersect : If the lines $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$ intersect, then the shortest distance between them is zero.

$$\therefore \quad \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_1 & n_2 \end{vmatrix}}{\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

27.6.3 SHORTEST DISTANCE BETWEEN TWO PARALLEL LINES

Let l_1 and l_2 be two parallel lines whose equations are $l_1 : \vec{r} = \vec{a_1} + \lambda \vec{b}$ and $l_2 : \vec{r} = \vec{a_2} + \mu \vec{b}$ respectively.

Clearly, l_1 and l_2 pass through the points A and B with position vectors $\vec{a_1}$ and $\vec{a_2}$ respectively and both are parallel to the vector $\vec{b}$.

Let BM be perpendicular from B on l_1. Then, BM is the shortest distance between l_1 and l_2.

Let θ be the angle between AB and line l_1. Then, angle between $\vec{AB}$ and $\vec{b}$ is $\pi - \theta$.

In triangle ABM, we have

$$\sin\theta = \frac{BM}{AB}$$

$$\Rightarrow \quad BM = AB \sin\theta = |\vec{AB}| \sin\theta \qquad \text{...(i)}$$

Now,

$$|\vec{AB}\times\vec{b}| = |\vec{AB}|\ |\vec{b}|\sin(\pi-\theta)$$

$$\Rightarrow\quad |\vec{AB}\times\vec{b}| = |\vec{AB}|\ |\vec{b}|\sin\theta$$

$$\Rightarrow\quad |\vec{AB}\times\vec{b}| = (|\vec{AB}|\sin\theta)\ |\vec{b}|$$

$$\Rightarrow\quad |\vec{AB}\times\vec{b}| = BM\ |\vec{b}| \qquad \text{[Using (i)]}$$

$$\therefore\quad BM = \frac{|\vec{AB}\times\vec{b}|}{|\vec{b}|} = \frac{|(\vec{a_2}-\vec{a_1})\times\vec{b}|}{|\vec{b}|}$$

Thus, the shortest distance between the parallel lines $\vec{r}=\vec{a_1}+\lambda\vec{b}$ and $\vec{r}=\vec{a_2}+\mu\vec{b}$ is given by

$$d = \left|\frac{(\vec{a_2}-\vec{a_1})\times\vec{b}}{|\vec{b}|}\right|$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the shortest distance between the lines*

$$\vec{r}=(4\hat{i}-\hat{j})+\lambda(\hat{i}+2\hat{j}-3\hat{k}) \text{ and } \vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+4\hat{j}-5\hat{k})$$

SOLUTION We know that the shortest distance between the lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ is given by

$$d = \left|\frac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$$

Comparing the given equations with the equations $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ respectively, we have

$$\vec{a_1}=4\hat{i}-\hat{j},\ \vec{a_2}=\hat{i}-\hat{j}+2\hat{k},\ \vec{b_1}=\hat{i}+2\hat{j}-3\hat{k} \text{ and } \vec{b_2}=2\hat{i}+4\hat{j}-5\hat{k}$$

Now, $\quad \vec{a_2}-\vec{a_1} = -3\hat{i}+0\hat{j}+2\hat{k}$

and, $\quad \vec{b_1}\times\vec{b_2} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 4 & -5\end{vmatrix} = 2\hat{i}-\hat{j}+0\hat{k}$

$\therefore\quad (\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = (-3\hat{i}+0\hat{j}+2\hat{k})\cdot(2\hat{i}-\hat{j}+0\hat{k}) = -6+0+0 = -6$

and, $\quad |\vec{b_1}\times\vec{b_2}| = \sqrt{4+1+0} = \sqrt{5}$

$$\therefore\quad \text{Shortest distance} = \left|\frac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right| = \left|\frac{-6}{\sqrt{5}}\right| = \frac{6}{\sqrt{5}}$$

EXAMPLE 2 *Find the shortest distance between the lines*

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$

SOLUTION The equations of two given lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad \text{...(i)} \quad \text{and} \quad \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5} \quad \text{...(ii)}$$

Line (i) passes through (1, 2, 3) and has direction ratios proportional to 2, 3, 4. So, its vector equation is

$$\vec{r}=\vec{a_1}+\lambda\vec{b_1} \quad \text{...(iii)}$$

where, $\quad \vec{a_1}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b_1}=2\hat{i}+3\hat{j}+4\hat{k}$.

Line (ii) passes through $(2, 4, 5)$ and has direction ratio proportional to 3, 4, 5. So, its vector equation is

$$\vec{r} = \vec{a_2} + \mu \vec{b_2} \qquad \text{...(iv)}$$

where, $\vec{a_2} = 2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{b_2} = 3\hat{i} + 4\hat{j} + 5\hat{k}$.

The shortest distance between the lines (iii) and (iv) is given by

$$\text{S.D.} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| \qquad \text{...(v)}$$

We have, $\vec{a_2} - \vec{a_1} = (2\hat{i} + 4\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = \hat{i} + 2\hat{j} + 2\hat{k}$

and, $\vec{b_2} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\hat{i} + 2\hat{j} - \hat{k}$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{1 + 4 + 1} = \sqrt{6}$

and, $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (-\hat{i} + 2\hat{j} - \hat{k}) = -1 + 4 - 2 = 1.$

Substituting the values of $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})$ and $|\vec{b_1} \times \vec{b_2}|$ in (v), we obtain S.D. $= 1/\sqrt{6}$

EXAMPLE 3 *By computing the shortest distance determine whether the following pairs of lines intersect or not :*

(i) $\vec{r} = (\hat{i} - \hat{j}) + \lambda (2\hat{i} + \hat{k})$; $\vec{r} = 2\hat{i} - \hat{j} + \mu (\hat{i} - \hat{j} - \hat{k})$

(ii) $\dfrac{x-1}{2} = \dfrac{y+1}{3} = z$; $\dfrac{x+1}{5} = \dfrac{y-2}{1}$; $z = 2$.

SOLUTION (i) Let the vector equations of two given lines be $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ respectively. Then,

$$\vec{a_1} = \hat{i} - \hat{j}, \vec{b_1} = 2\hat{i} + \hat{k}, \vec{a_2} = 2\hat{i} - \hat{j} \text{ and } \vec{b_2} = \hat{i} - \hat{j} - \hat{k}.$$

$\therefore \quad \vec{a_2} - \vec{a_1} = (2\hat{i} - \hat{j}) - (\hat{i} - \hat{j}) = \hat{i}$

and, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} = \hat{i} + 3\hat{j} - 2\hat{k}$

$\Rightarrow \quad (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = \hat{i} \cdot (\hat{i} + 3\hat{j} - 2\hat{k}) = 1 + 0 + 0 = 1$

Since $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) \neq 0$. So, the given lines do not intersect.

(ii) The equation of two given lines are

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-0}{1} \quad \text{...(i)} \quad \text{and,} \quad \frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0} \qquad \text{...(ii)}$$

Line (i) passes through the point $(1, -1, 0)$ and has direction ratios proportional to 2, 3, 1. So, its vector equation is

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1},$$

where $\vec{a_1} = \hat{i} - \hat{j}$ and $\vec{b_1} = 2\hat{i} + 3\hat{j} + \hat{k}$.

Line (ii) passes through the point $(-1, 2, 2)$ and has direction ratios proportional to 5, 1, 0. So, its vector equation is

$$\vec{r} = \vec{a_2} + \mu \vec{b_2},$$

where $\vec{a_2} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b_2} = 5\hat{i} + \hat{j} + 0\hat{k}$

We have, $\vec{a_2} - \vec{a_1} = (-\hat{i} + 2\hat{j} + 2\hat{k}) - (\hat{i} - \hat{j}) = -2\hat{j} + 3\hat{j} + 2\hat{k}$

and, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 5 & 1 & 0 \end{vmatrix} = -\hat{i} + 5\hat{j} - 13\hat{k}$

$\therefore \quad (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (-2\hat{i} + 3\hat{j} + 2\hat{k}) \cdot (-\hat{i} + 5\hat{j} - 13\hat{k})$

$$= 2 + 15 - 26 = -9 \neq 0.$$

Hence, given lines do not intersect.

EXAMPLE 4 *Find the shortest distance between the lines whose vector equations are*

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$$

and, $\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu(4\hat{i} + 6\hat{j} + 8\hat{k})$ **[CBSE 2008]**

SOLUTION The given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \quad \text{...(i)}$$

and, $\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 4\hat{k})$ ...(ii)

Equation (ii) can re-written as

$$\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu'(2\hat{i} + 3\hat{j} + 4\hat{k}) \quad \text{...(iii)}$$

where $\mu' = 2\mu$.

These two lines passes through the points having position vectors $\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{a_2} = 2\hat{i} + 4\hat{j} + 5\hat{k}$ respectively and both are parallel to the vector $\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$.

$\therefore \quad$ Shortest distance $= \dfrac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|}$ ...(iv)

We have,

$$(\vec{a_2} - \vec{a_1}) \times \vec{b} = (\hat{i} + 2\hat{j} + 2\hat{k}) \times (2\hat{i} + 3\hat{j} + 4\hat{k})$$

$\Rightarrow \quad (\vec{a_2} - \vec{a_1}) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = (8-6)\hat{i} - (4-4)\hat{j} + (3-4)\hat{k} \quad = 2\hat{i} - 0\hat{j} - \hat{k}$

$\Rightarrow \quad |(\vec{a_2} - \vec{a_1}) \times \vec{b}| = \sqrt{4+0+1} = \sqrt{5}$ and $|\vec{b}| = \sqrt{4+9+16} = \sqrt{29}$

Substituting the values of $|(\vec{a_2} - \vec{a_1}) \times \vec{b}|$ and $|\vec{b}|$ in (iv), we get

Shortest distance $= \dfrac{\sqrt{5}}{\sqrt{29}}$.

EXERCISE 27.5

1. Find the shortest distance between the following pairs of lines whose vector equations are:
   (i) $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$
   and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$
   (ii) $\vec{r} = (3\hat{i} + 5\hat{j} + 7\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 7\hat{k})$
   and $\vec{r} = -\hat{i} - \hat{j} - \hat{k} + \mu(7\hat{i} - 6\hat{j} + \hat{k})$
   (iii) $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$
   and $\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} + 5\hat{k}$
   (iv) $\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-t)\hat{k}$

and $\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$ [NCERT, CBSE 2002]

(v) $\vec{r} = (\lambda - 1)\hat{i} + (\lambda + 1)\hat{j} - (1+\lambda)\hat{k}$

and $\vec{r} = (1-\mu)\hat{i} + (2\mu - 1)\hat{j} + (\mu + 2)\hat{k}$

(vi) $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \lambda(2\hat{i} - 5\hat{j} + 2\hat{k})$; $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

[CBSE 2008]

(vii) $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$; $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$ [NCERT]

2. Find the shortest distance between the following pairs of lines whose Cartesian equations are :

(i) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-5}{5}$

(ii) $\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{3} = \frac{y-2}{1}; z = 2$

(iii) $\frac{x-1}{-1} = \frac{y+2}{1} = \frac{z-3}{-2}$ and $\frac{x-1}{1} = \frac{y+1}{2} = \frac{z+1}{-2}$

(iv) $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ [CBSE 2008]

3. By computing the shortest distance determine whether the following pairs of lines intersect or not :

(i) $\vec{r} = (\hat{i} - \hat{j}) + \lambda(2\hat{i} + \hat{k})$; $\vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} + \hat{j} - \hat{k})$

(ii) $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$; $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$

(iii) $\frac{x-1}{2} = \frac{y+1}{3} = z$; $\frac{x+1}{5} = \frac{y-2}{1}$; $z = 2$

(iv) $\frac{x-5}{4} = \frac{y-7}{-5} = \frac{z+3}{-5}$; $\frac{x-8}{7} = \frac{y-7}{1} = \frac{z-5}{3}$

4. Find the shortest distance between the following pairs of parallel lines whose equations are :

(i) $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(-\hat{i} + \hat{j} - \hat{k})$

(ii) $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(4\hat{i} - 2\hat{j} + 2\hat{k})$

5. Find the equations of the lines joining the following pairs of vertices and then find the shortest distance between the lines

(i) (0, 0, 0) and (1, 0, 2) (ii) (1, 3, 0) and (0, 3, 0).

6. Write the vector equations of the following lines and hence determine the distance between them

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}; \frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}$$ [CBSE 2010]

ANSWERS

1. (i) $\sqrt{270}$ (ii) $\frac{512}{\sqrt{3968}}$ (iii) $\frac{1}{\sqrt{6}}$ (iv) $\frac{7}{3\sqrt{2}}$ (v) $\frac{5}{\sqrt{2}}$ (vi) $\frac{3}{\sqrt{2}}$ (vii) $\frac{10}{\sqrt{59}}$

2. (i) $\frac{1}{\sqrt{6}}$ (ii) $\sqrt{\frac{3}{59}}$ (iii) $\frac{8}{\sqrt{29}}$ (iv) $2\sqrt{29}$ 3. (i) No (ii) Yes (iii) No (iv) No

4. (i) $\sqrt{\frac{78}{3}}$ (ii) $\frac{\sqrt{11}}{\sqrt{6}}$ 5. $\frac{x}{1} = \frac{y}{0} = \frac{z}{2}; \frac{x-1}{-1} = \frac{y-3}{0} = \frac{z}{0}$; 3 units

6. $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$; $\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$, S.D. $= \frac{\sqrt{293}}{7}$ units

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the cartesian and vector equations of X-axis.
2. Write the cartesian and vector equations of Y-axis.
3. Write the cartesian and vector equations of Z-axis.
4. Write the vector equation of a line passing through a point having position vector $\vec{\alpha}$ and parallel to vector $\vec{\beta}$.
5. Cartesian equations of a line AB are $\frac{2x-1}{2}=\frac{4-y}{7}=\frac{z+1}{2}$
   Write the direction ratios of the line parallel to AB.
6. Write the direction cosines of the line whose cartesian equations are
   $6x-2=3y+1=2z-4$.
7. Write the direction cosines of the line $\frac{x-2}{2}=\frac{2y-5}{-3}$, $z=2$.
8. Write the coordinate axis to which the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-1}{0}$ is perpendicular.
9. Write the angle between the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z-2}{1}$ and $\frac{x-1}{1}=\frac{y}{2}=\frac{z-1}{3}$.
10. Write the direction cosines of the line whose cartesian equations are $2x=3y=-z$.
11. Write the angle between the lines $2x=3y=-z$ and $6x=-y=-4z$.
12. Write the value of λ for which the lines $\frac{x-3}{-3}=\frac{y+2}{2\lambda}=\frac{z+4}{2}$ and $\frac{x+1}{3\lambda}=\frac{y-2}{1}=\frac{z+6}{-5}$ are perpendicular to each other.
13. Write the formula for the shortest distance between the lines $\vec{r}=\vec{a_1}+\lambda\vec{b}$ and $\vec{r}=\vec{a_2}+\mu\vec{b}$.
14. Write the condition for the lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ to be intersecting.
15. The cartesian equation of a line AB is $\frac{2x-1}{\sqrt{3}}=\frac{y+2}{2}=\frac{z-3}{3}$. Find the direction cosines of a line parallel to AB. **[CBSE 2008]**

ANSWERS

	Cartesian equation	*Vector equation*
1.	$\frac{x-0}{1}=\frac{y-0}{0}-\frac{z-0}{0}$	$\vec{r}=\lambda\hat{i}$
2.	$\frac{x-0}{0}=\frac{y-0}{1}=\frac{z-0}{0}$,	$\vec{r}=\lambda\hat{j}$
3.	$\frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{1}$,	$\vec{r}=\lambda\hat{k}$

4. $\vec{r}=\vec{\alpha}+\lambda\vec{\beta}$ 5. $\frac{1}{\sqrt{54}},\frac{-7}{\sqrt{54}},\frac{2}{\sqrt{54}}$ 6. $\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}$ 7. $\frac{4}{5},\frac{-3}{5},0$

8. Z-axis 9. 90° 10. $\frac{3}{7},\frac{2}{7},-\frac{6}{7}$ 11. 90°

12. $\frac{-10}{7}$ 13. $\frac{|(\vec{a_2}-\vec{a_1})\times\vec{b}|}{|\vec{b}|}$ 14. $(\vec{b_1}\times\vec{b_2})\cdot(\vec{a_2}-\vec{a_1})=0$ 15. $\frac{\sqrt{3}}{\sqrt{55}},\frac{4}{\sqrt{55}},\frac{6}{\sqrt{55}}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The angle between the straight lines

$$\frac{x+1}{2}=\frac{y-2}{5}=\frac{z+3}{4} \text{ and } \frac{x-1}{1}=\frac{y+2}{2}=\frac{z-3}{-3} \text{ is}$$

(a) $45°$ (b) $30°$ (c) $60°$ (d) $90°$

2. The lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and $\frac{x-1}{-2}=\frac{y-2}{-4}=\frac{z-3}{-6}$ are

(a) coincident (b) skew (c) intersecting (d) parallel

3. The direction ratios of the line perpendicular to the lines

$$\frac{x-7}{2}=\frac{y+17}{-3}=\frac{z-6}{1} \text{ and, } \frac{x+5}{1}=\frac{y+3}{2}=\frac{z-4}{-2}$$

are proportional to

(a) $4, 5, 7$ (b) $4, -5, 7$ (c) $4, -5, -7$ (d) $-4, 5, 7$

4. The angle between the lines

$$\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{2} \text{ and, } \frac{x-1}{-\sqrt{3}-1}=\frac{y-1}{\sqrt{3}-1}=\frac{z-1}{4} \text{ is}$$

(a) $\cos^{-1}\left(\frac{1}{65}\right)$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

5. The direction ratios of the line $x-y+z-5=0=x-3y-6$ are proportional to

(a) $3, 1, -2$ (b) $2, -4, 1$

(c) $\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$ (d) $\frac{2}{\sqrt{41}}, \frac{-4}{\sqrt{41}}, \frac{1}{\sqrt{41}}$

6. The perpendicular distance of the point $P(1, 2, 3)$ from the line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$ is

(a) 7 (b) 5 (c) 0 (d) none of these

7. The equation of the line passing through the points

$$a_1\hat{i}+a_2\hat{j}+a_3\hat{k} \text{ and } b_1\hat{i}+b_2\hat{j}+b_3\hat{k} \text{ is}$$

(a) $\vec{r}=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})+\lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$

(b) $\vec{r}=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})-t(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$

(c) $\vec{r}=a_1(1-t)\hat{i}+a_2(1-t)\hat{j}+a_3(1-t)\hat{k}+t(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$

(d) none of these

8. If a line makes angles α, β and γ with the axes respectively, then $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=$

(a) -2 (b) -1 (c) 1 (d) 2

9. If the direction ratios of a line are proportional to $1, -3, 2$, then its direction cosines are

(a) $\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}$ (b) $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

(c) $-\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}$ (d) $-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}$

10. If a line makes angle $\frac{\pi}{3}$ and $\frac{\pi}{4}$ with x-axis and y-axis respectively, then the angle made by the line with z-axis is

(a) $\pi/2$ (b) $\pi/3$ (c) $\pi/4$ (d) $5\pi/12$

11. The projections of a line segment on X, Y, Z axes are 12, 4, 3. The length and direction cosines of the line segment are

(a) $13; \frac{12}{13}, \frac{4}{13}, \frac{3}{13}$ (b) $19; \frac{12}{19}, \frac{4}{19}, \frac{3}{19}$

(c) $11; \frac{12}{11}, \frac{14}{11}, \frac{3}{11}$ (d) none of these

12. The lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x-1}{-2} = \frac{y-2}{-4} = \frac{z-3}{-6}$ are

(a) parallel (b) intersecting (c) skew (d) coincident

13. The straight line $\frac{x-3}{3} = \frac{y-2}{1} = \frac{z-1}{0}$ is

(a) parallel to x-axis (b) parallel to y-axis

(c) parallel to z-axis (d) perpendicular to z-axis

14. The shortest distance between the lines

$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and, $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is

(a) $\sqrt{30}$ (b) $2\sqrt{30}$ (c) $5\sqrt{30}$ (d) $3\sqrt{30}$

ANSWERS

1. (d)	2. (a)	3. (a)	4. (c)	5. (a)	6. (a)	7. (c)	8. (d)
9. (a)	10. (b)	11. (a)	12. (a)	13. (d)	14. (d)		

SUMMARY

1. Two non-parallel planes always intersect in a straight line. Thus, if $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are equations of two non-parallel planes, then these two equations taken together represent a line i.e.,

$$a_1 x + b_1 y + c_1 z + d_1 = 0 = a_2 x + b_2 y + c_2 z + d_2$$

is the equation of a line.
This is known as an un-symmetrical form of a line.

2. The equations of a line passing through a point (x_1, y_1, z_1) and having direction cosines (or direction ratios) l, m, n are given by

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

The coordinates of an arbitrary point on this line are $(x_1 + lr, y_1 + mr, z_1 + nr)$, where r is a parameter.
This is known as symmetrical form of a line.

3. The vector equation of a line passing through a point having position vector $\vec{a}$ and parallel to vector $\vec{b}$ is

$\vec{r} = \vec{a} + \lambda \vec{b}$, where λ is a parameter.

4. The equations of a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

5. The vector equation of a line passing through points having position vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \text{ where } \lambda \text{ is a parameter.}$$

6. If l, m, n are the direction cosines of the line of intersection of planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$, then

$$a_1 l + b_1 m + c_1 n = 0$$
$$a_2 l + b_2 m + c_2 n = 0$$
$$\therefore \quad \frac{l}{b_1 c_2 - b_2 c_1} = \frac{m}{c_1 a_2 - c_2 a_1} = \frac{n}{a_1 b_2 - a_2 b_1}$$

$\Rightarrow$ l, m, n are proportional to $b_1 c_2 - b_2 c_1,\ c_1 a_2 - c_2 a_1,\ a_1 b_2 - a_2 b_1$

7. (i) The length of the perpendicular from a point $P(\vec{\alpha})$ on the line $\vec{r} = \vec{a} + \lambda \vec{b}$ is given by

$$\sqrt{|\vec{\alpha} - \vec{a}|^2 - \left\{\frac{(\vec{\alpha} - \vec{a}) \cdot \vec{b}}{|\vec{b}|}\right\}^2}$$

(ii) The length of the perpendicular from a point $P(x_1, y_1, z_1)$ on the line $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$ is given by

$$\sqrt{\{(a-x_1)^2 + (b-y_1)^2 + (c-z_1)^2\} - \{(a-x_1)\, l + (b-y_1)\, m + (c-z_1)\, n\}^2},$$

where, l, m, n are direction cosines of the line.

8. Two straight lines in space are said to be skew lines if they are neither parallel nor intersecting.

9. If l_1 and l_2 are two skew lines, then a line perpendicular to each of lines l_1 and l_2 is known as the line of shortest distance.
If the line of shortest distance intersects lines l_1 and l_2 at P and Q respectively, then the distance PQ between points P and Q is known as the shortest distance between l_1 and l_2.

10. The shortest distance between lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is given by

$$d = \left|\frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right|$$

11. The shortest distance between the lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \text{ and, } \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

is given by
$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{d = \sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}}$$

12. The shortest distance between parallel lines $\vec{r} = \vec{a_1} + \lambda \vec{b}$ and $\vec{r} = \vec{a_2} + \mu \vec{b}$ is given by

$$d = \left|\frac{(\vec{a_2} - \vec{a_1}) \times \vec{b}}{|\vec{b}|}\right|$$

13. Lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ are intersecting lines, if $(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1}) = 0$

28

THE PLANE

28.1 PLANE

DEFINITION *A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface. In other words, every point on the line segment joining any two points lies on the plane.*

THEOREM *Prove that every first degree equation in x, y and z represents a plane i.e., $ax + by + cz + d = 0$ is the general equation of a plane.*

PROOF Let $ax + by + cz + d = 0$ ...(i)

be a first degree equation in x, y and z.

In order to prove that the equation (i) represents a plane, it is sufficient to show that every point on the line segment joining any two points on the surface represented by it lies on it.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points on the surface represented by (i). Then,

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{...(ii)}$$

and, $$ax_2 + by_2 + cz_2 + d = 0 \quad \text{...(iii)}$$

Let R be an arbitrary point on the line segment joining P and Q. Suppose R divides PQ in the ratio $\lambda : 1$. Then, coordinates of R are

$$\left(\frac{x_1 + \lambda x_2}{\lambda + 1}, \frac{y_1 + \lambda y_2}{\lambda + 1}, \frac{z_1 + \lambda z_2}{\lambda + 1}\right)$$

We have to prove that R lies on the surface represented by the equation (i) for all values of λ. For this it is sufficient to show that R satisfies equation (i).

We have,

$$a\left(\frac{x_1 + \lambda x_2}{\lambda + 1}\right) + b\left(\frac{y_1 + \lambda y_2}{\lambda + 1}\right) + c\left(\frac{z_1 + \lambda z_2}{\lambda + 1}\right) + d$$

$$= \frac{1}{\lambda + 1}\left[(ax_1 + by_1 + cz_1 + d) + \lambda(ax_2 + by_2 + cz_2 + d)\right]$$

$$= \frac{1}{\lambda + 1}\left[0 + \lambda\, 0\right] = 0 \quad \text{[Using (ii) and (iii)]}$$

This shows that the point $R\left(\frac{x_1 + \lambda x_2}{\lambda + 1}, \frac{y_1 + \lambda y_2}{\lambda + 1}, \frac{z_1 + \lambda z_2}{\lambda + 1}\right)$ satisfies equation (i).

Since R is an arbitrary point on the line segment joining P and Q. Therefore, every point on PQ lies on the surface represented by equation (i).

Hence, equation (i) represents a plane.

Q.E.D.

REMARK *The general equation of a plane is $ax + by + cz + d = 0$. To determine a plane satisfying some given conditions we will have to find the values of constants a, b, c and d. It seems that there are four unknowns viz. a, b, c and d in the equation $ax + by + cz + d = 0$. But, there are only three unknowns, because the equation $ax + by + cz + d = 0$ can be written as*

$$\left(\frac{a}{d}\right)x + \left(\frac{b}{d}\right)y + \left(\frac{c}{d}\right)z + 1 = 0 \text{ or, } Ax + By + Cz + 1 = 0.$$

Thus, to find a plane we must have three conditions to find the values of A, B and C.

28.2 EQUATIONS OF A PLANE PASSING THROUGH A GIVEN POINT

THEOREM *The general equation of a plane passing through a point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b and c are constants.*

PROOF The general equation of a plane is

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

If it passes through (x_1, y_1, z_1), then

$$ax_1 + by_1 + cz_1 + d = 0 \Rightarrow d = -(ax_1 + by_1 + cz_1)$$

Substituting the value of d in (i), we obtain

$$ax + by + cz - (ax_1 + by_1 + cz_1) = 0$$

$$\Rightarrow \quad a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

This is the general equation of a plane passing through a given point (x_1, y_1, z_1).

Q.E.D.

In order to find the equation of a plane passing through three given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), we may use the following algorithm.

ALGORITHM

STEP I *Write the equation of a plane passing through (x_1, y_1, z_1) as*

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \text{...(i)}$$

STEP II *If the plane (i) passes through (x_2, y_2, z_2) and (x_3, y_3, z_3), then*

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \quad \text{...(ii)}$$

$$\text{and, } \quad a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0 \quad \text{...(iii)}$$

STEP III *Solve equations (ii) and (iii) obtained in step II by cross-multiplication.*

STEP IV *Substitute the values of a, b and c, obtained in step III, in equation (i) in step I to get the required plane.*

REMARK *On eliminating a, b, c from equations (i), (ii) and (iii), we get*

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

as the equation of the plane passing through three given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

ILLUSTRATIVE EXAMPLES

Type I ON FINDING THE EQUATION OF A PLANE PASSING THROUGH THREE GIVEN POINTS

EXAMPLE 1 *Find the equation of the plane through the points A (2, 2, – 1), B (3, 4, 2) and C (7, 0, 6).*

SOLUTION The general equation of a plane passing through (2, 2, − 1) is

$$a(x-2)+b(y-2)+c(z+1)=0 \quad \text{...(i)}$$

It will pass through $B(3,4,2)$ and $C(7,0,6)$ if

$$a(3-2)+b(4-2)+c(2+1)=0 \Rightarrow a+2b+3c=0 \quad \text{...(ii)}$$

and, $$a(7-2)+b(0-2)+c(6+1)=0 \Rightarrow 5a-2b+7c=0 \quad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we have

$$\frac{a}{14+6}=\frac{b}{15-7}=\frac{c}{-2-10}$$

$$\Rightarrow \quad \frac{a}{5}=\frac{b}{2}=\frac{c}{-3}=\lambda \text{ (say)}$$

$$\Rightarrow \quad a=5\lambda,\ b=2\lambda \text{ and } c=-3\lambda.$$

Substituting the values of a, b and c in (i), we get

$$5\lambda(x-2)+2\lambda(y-2)-3\lambda(z+1)=0$$

$$\Rightarrow \quad 5(x-2)+2(y-2)-3(z+1)=0$$

$\Rightarrow \quad 5x+2y-3z=17,$ which is the required equation of the plane.

ALITER The equation of the plane passing through points (2, 2, − 1), (3, 4, 2) and (7, 0, 6) is given by

$$\begin{vmatrix} x-2 & y-2 & z+1 \\ 3-2 & 4-2 & 2+1 \\ 7-2 & 0-2 & 6+1 \end{vmatrix}=0$$

or, $$\begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix}=0$$

or, $$(x-2)(14+6)-(y-2)(7-15)+(z+1)(-2-10)=0$$

or, $$20(x-2)+8(y-2)-12(z+1)=0$$

or, $$20x+8y-12z-68=0$$

or, $$5x+2y-3z=17.$$

EXAMPLE 2 *Find the equation of the plane through the points $P(1,1,0)$, $Q(1,2,1)$ and $R(-2,2,-1)$.*

SOLUTION The general equation of a plane passing through $P(1,1,0)$ is

$$a(x-1)+b(y-1)+c(z-0)=0 \quad \text{...(i)}$$

It will pass through $Q(1,2,1)$ and $R(-2,2,-1)$, if

$$a\cdot 0+b\cdot 1+c\cdot 1=0 \quad \text{...(ii)}$$

and, $$a(-3)+b\cdot 1+c(-1)=0 \quad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we have

$$\frac{a}{(1)(-1)-(1)(1)}=\frac{b}{(1)(-3)-0(-1)}=\frac{c}{(0)(1)-(1)(-3)}$$

$$\Rightarrow \quad \frac{a}{-2}=\frac{b}{-3}=\frac{c}{3}=\lambda \text{ (say)}$$

$$\Rightarrow \quad a=-2\lambda,\ b=-3\lambda \text{ and } c=3\lambda$$

Substituting the values of a, b and c in (i), we get

$$-2\lambda(x-1)-3\lambda(y-1)+3\lambda z=0$$

$$\Rightarrow \quad -2(x-1)-3(y-1)+3z=0$$

$$\Rightarrow \quad 2x+3y-3z-5=0$$

EXAMPLE 3 *If from a point P (a, b, c) perpendiculars PA and PB are drawn to yz and zx-planes, find the equation of the plane OAB.*

SOLUTION The coordinates of A and B are $(0, b, c)$ and $(a, 0, c)$ respectively.

The equation of the plane passing through $O\,(0, 0, 0)$, $A\,(0, b, c)$ and $B\,(a, 0, c)$ is given by

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 0-0 & b-0 & c-0 \\ a-0 & 0-0 & c-0 \end{vmatrix} = 0$$

$$\Rightarrow \quad bcx + acy - abz = 0 \Rightarrow \frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 0.$$

Type II ON PROVING COPLANARITY OF FOUR POINTS

In order to prove the coplanarity of four points, we may use the following algorithm.

ALGORIGHM

STEP I *Find the equation of a plane passing through any three out of given four points.*

TEP II *Show that the fourth point satisfies the equation in Step I.*

AMPLE 4 *Show that the four points (0, –1, –1), (–4, 4, 4), (4, 5, 1) and (3, 9, 4) are coplanar. ıd the equation of the plane containing them.*

LUTION The equation of a plane passing through $(0, -1, -1)$ is

$$a\,(x-0) + b\,(y+1) + c\,(z+1) = 0 \qquad \text{...(i)}$$

ıf it passes through $(-4, 4, 4)$ and $(4, 5, 1)$, then

$$a\,(-4) + b\,(5) + c\,(5) = 0 \qquad \text{...(ii)}$$

and, $\quad a\,(4) + b\,(6) + c\,(2) = 0$

or, $\quad a\,(2) + b\,(3) + c\,(1) = 0 \qquad \text{...(iii)}$

Solving (ii) and (iii) by cross-multiplication, we obtain

$$\frac{a}{5-15} = \frac{b}{10+4} = \frac{c}{-12-10}$$

$$\Rightarrow \quad \frac{a}{-5} = \frac{b}{7} = \frac{c}{-11} = \lambda \text{ (say)}$$

$$\Rightarrow \quad a = -5\lambda,\ b = 7\lambda \text{ and } c = -11\lambda$$

Substituting the values of a, b and c in (ii), we get

$$-5\lambda x + 7\lambda\,(y+1) - 11\lambda\,(z+1) = 0$$

$$\Rightarrow \quad -5x + 7y + 7 - 11z - 11 = 0$$

$$\Rightarrow \quad 5x - 7y + 11z + 4 = 0 \qquad \text{...(iv)}$$

Clearly, the fourth point viz. $(3, 9, 4)$ satisfies this equation. Hence, the given points are coplanar. The equation of the plane containing the given points is

$$5x - 7y + 11z + 4 = 0.$$

EXERCISE 28.1

1. Find the equation of the plane passing through the following points:
   (i) $(2, 1, 0)$, $(3, -2, -2)$ and $(3, 1, 7)$
   (ii) $(-5, 0, -6)$, $(-3, 10, -9)$ and $(-2, 6, -6)$
   (iii) $(1, 1, 1)$, $(1, -1, 2)$ and $(-2, -2, 2)$
   (iv) $(2, 3, 4)$, $(-3, 5, 1)$ and $(4, -1, 2)$
   (v) $(0, -1, 0)$, $(3, 3, 0)$ and $(1, 1, 1)$ [CBSE 2004]

2. Show that the four point $(0, -1, -1)$, $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ are coplanar and find the equation of the common plane.
3. Show that the following points are coplanar:
   (i) $(0, -1, 0)$, $(2, 1, -1)$, $(1, 1, 1)$ and $(3, 3, 0)$
   (ii) $(0, 4, 3)$, $(-1, -5, -3)$, $(-2, -2, 1)$ and $(1, 1, -1)$

ANSWERS

1. (i) $7x + 3y - z = 17$, (ii) $2x - y - 2z - 2 = 0$, (iii) $x - 3y - 6z + 8 = 0$, (iv) $x + y - z = 1$, (v) $4x - 3y + 2z = 3$. 2. $5x - 7y + 11z + 4 = 0$.

28.3 INTERCEPT FORM OF A PLANE

THEOREM *The equation of a plane intercepting lengths a, b and c with x-axis, y-axis and z-axis respectively is*

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

PROOF Let O be the origin and let OX, OY and OZ be the coordinate axes. Suppose a plane meets the coordinate axes OX, OY and OZ at A, B and C respectively such that $OA = a$, $OB = b$ and $OC = c$. Then, the coordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

The equation of a plane passing through $A\,(a, 0, 0)$ is

$$P\,(x - a) + Q\,(y - 0) + R\,(z - 0) = 0 \quad \text{...(i)}$$

If the plane in (i) passes through $B\,(0, b, 0)$ and $C\,(0, 0, c)$, then

$$P\,(0 - a) + Q\,(b - 0) + R\,(0 - 0) = 0$$

$$\Rightarrow \quad P\,(-a) + Q\,(b) + R\,(0) = 0 \quad \text{...(ii)}$$

and, $$P\,(0 - a) + Q\,(0 - 0) + R\,(c - 0) = 0$$

$$\Rightarrow \quad P\,(-a) + Q\,(0) + R\,(c) = 0 \quad \text{...(iii)}$$

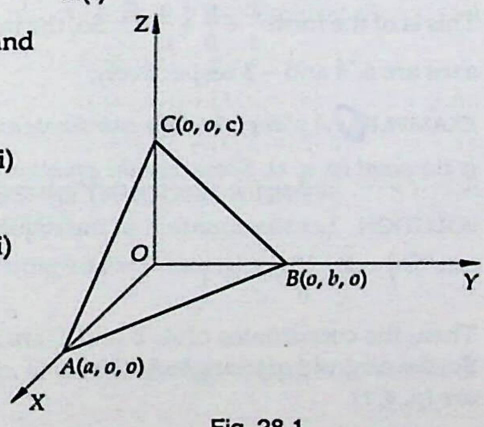

Fig. 28.1

Solving (ii) and (iii) by cross-multiplication, we have

$$\frac{P}{bc} = \frac{Q}{ac} = \frac{R}{ab} = \lambda \text{ (say)}$$

$$\Rightarrow \quad P = \lambda\,(bc),\ Q = \lambda\,(ac) \text{ and } R = \lambda\,(ab)$$

Substituting the values of P, Q and R in (i), we get the required equation of the plane as

$$\lambda\,(bc)\,(x - a) + \lambda\,(ac)\,(y - 0) + \lambda\,(ab)\,(z - 0) = 0$$

$$\Rightarrow \quad bcx - abc + acy + abz = 0$$

$$\Rightarrow \quad bcx + acy + abz = abc$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Q.E.D.

NOTE 1 *The above equation is known as the intercept form of the plane, because the plane intercepts lengths a, b and c with x, y and z-axis respectively.*

NOTE 2 *To determine the intercepts made by a plane with the coordinate axes we proceed as follows:*

For x-intercept : Put $y = 0$, $z = 0$ in the equation of the plane and obtain the value of x. The value of x is the intercept on x-axis.

For y-intercept : Put $x = 0, z = 0$ in the equation of the plane and obtain the value of y. The value of y is the intercept on y-axis

For z-intercept : Put $x = 0$, $y = 0$ in the equations of the plane and obtain the value of z. The value of z is the intercept on z-axis.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Write the equation of the plane whose intercepts on the coordinate axes are -4, 2 and 3.*

SOLUTION We know that the equation of a plane whose intercepts on the coordinate axes are a, b and c respectively, is given by

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$$

Here, $a = -4$, $b = 2$, and $c = 3$. So, the equation of the required plane is

$$\frac{x}{-4}+\frac{y}{2}+\frac{z}{3} = 1 \Rightarrow -3x+6y+4z = 12.$$

EXAMPLE 2 *Reduce the equation of the plane $2x + 3y - 4z = 12$ to intercept form and find its intercepts on the coordinate axes.*

SOLUTION The equation of the given plane is

$$2x+3y-4z = 12 \Rightarrow \frac{2x}{12}+\frac{3y}{12}-\frac{4z}{12} = 1 \Rightarrow \frac{x}{6}+\frac{y}{4}+\frac{z}{-3} = 1$$

This is of the form $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. So, the intercepts made by the plane with the coordinate axes are 6, 4 and -3 respectively.

EXAMPLE 3 *A plane meets the coordinate axes in A, B, C such that the centroid of triangle ABC is the point (p, q, r). Show that the equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r} = 3$.* [HSB 1991C]

SOLUTION Let the equation of the required plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \qquad \text{...(i)}$$

Then, the coordinates of A, B and C are $A\ (a, 0, 0)$, $B\ (0, b, 0)$ and $C(0, 0, c)$ respectively. So, the centroid of triangle ABC is $(a/3,\ b/3,\ c/3)$. But, the coordinates of the centroid are (p, q, r).

$$\therefore \qquad p = \frac{a}{3},\ q = \frac{b}{3} \text{ and } r = \frac{c}{3} \Rightarrow a = 3p,\ b = 3q \text{ and } c = 3r$$

Substituting the values of a, b and c in (i), we get the required plane as

$$\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r} = 1 \Rightarrow \frac{x}{p}+\frac{y}{q}+\frac{z}{r} = 3.$$

EXAMPLE 4 *A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the three coordinate axes is constant. Show that the plane passes through a fixed point.*

SOLUTION Let the equation of the plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \qquad \text{...(i)}$$

This plane cuts intercepts of lengths a, b and c on the coordinate axes. It is given that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \lambda, \text{ where } \lambda \text{ is a constant}$$

$$\Rightarrow \quad \frac{1}{\lambda a}+\frac{1}{\lambda b}+\frac{1}{\lambda c}=1 \Rightarrow \frac{1}{a}\left(\frac{1}{\lambda}\right)+\frac{1}{b}\left(\frac{1}{\lambda}\right)+\frac{1}{c}\left(\frac{1}{\lambda}\right)=1$$

This shows that the plane (i) passes through the fixed points $(1/\lambda, 1/\lambda, 1/\lambda)$.

EXERCISE 28.2

1. Write the equation of the plane whose intercepts on the coordinate axes are 2, -3 and 4.
2. Reduce the equations of the following planes in intercept form and find its intercepts on the coordinate axes:
   (i) $4x+3y-6z-12=0$, (ii) $2x+3y-z=6$, (iii) $2x-y+z=5$.
3. Find the equation of a plane which meets the axes in A, B and C, given that the centroid of the triangle ABC is the point (α, β, γ).
4. Find the equation of the plane passing through the point (2, 4, 6) and making equal intercepts on the coordinate axes.
5. A plane meets the coordinate axes at A, B and C respectively such that the centroid of triangle ABC is $(1, -2, 3)$. Find the equation of the plane.

ANSWERS

1. $6x-4y+3z=12$
2. (i) $\frac{x}{3}+\frac{y}{4}+\frac{z}{-2}=1;\ 3, 4, -2,$ (ii) $\frac{x}{3}+\frac{y}{2}+\frac{z}{-6}=1;\ 3, 2, -6,$
   (iii) $\frac{x}{5/2}+\frac{y}{-5}+\frac{z}{5}=1;\ \frac{5}{2}, -5, 5$
3. $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$ 4. $x+y+z=12$ 5. $6x-3y+2z=18$.

28.4 VECTOR EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND NORMAL TO A GIVEN VECTOR

THEOREM *The vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to vector $\vec{n}$ is*

$$(\vec{r}-\vec{a})\cdot\vec{n}=0 \text{ or, } \vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

PROOF Suppose the plane π passes through a point having a position vector $\vec{a}$ (see Fig. 28.2) and is normal to the vector $\vec{n}$.

Let $\vec{r}$ be the position vector of an arbitrarily chosen point P on the plane π. Then, $\vec{OP}=\vec{r}$.

Now, $\vec{AP}$ lies in the plane and $\vec{n}$ is normal to the plane π.

$\therefore \quad \vec{AP}\perp\vec{n}$

$\Rightarrow \quad \vec{AP}\cdot\vec{n}=0$

$\Rightarrow \quad (\vec{r}-\vec{a})\cdot\vec{n}=0$

Since $\vec{r}$ is the position vector of an arbitrary point on the plane. So, the vector equation of the plane is

$$(\vec{r}-\vec{a})\cdot\vec{n}=0 \qquad \text{...(i)}$$

Equation (i) can also be written as

$$\vec{r}\cdot\vec{n}-\vec{a}\cdot\vec{n}=0 \text{ or, } \vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

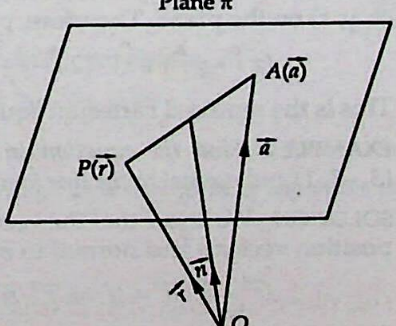

Fig. 28.2

NOTE 1 *It is to note here that vector equation of a plane means a relation involving the position vector $\vec{r}$ of an arbitrary point on the plane.*

NOTE 2 *The above equation can also be written as $\vec{r} \cdot \vec{n} = d$, where $d = \vec{a} \cdot \vec{n}$.*
This is known as the scalar product form of a plane.

REDUCTION TO CARTESIAN FORM

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$.

Then, $\vec{r} - \vec{a} = (x - a_1)\hat{i} + (y - a_2)\hat{j} + (z - a_3)\hat{k}$.

Substituting the values of $(\vec{r} - \vec{a})$ and $\vec{n}$ in equation (i), we get

$$[(x - a_1)\hat{i} + (y - a_2)\hat{j} + (z - a_3)\hat{k}] \cdot (n_1\hat{i} + n_2\hat{j} + n_3\hat{k}) = 0$$

$$\Rightarrow \quad (x - a_1) n_1 + (y - a_2) n_2 + (z - a_3) n_3 = 0 \quad ...(ii)$$

This is the cartesian equation of a plane passing through (a_1, a_2, a_3).

Note that the coefficients of x, y and z in equation (ii) are n_1, n_2 and n_3 respectively which are direction ratios of vector $\vec{n}$ normal to the plane.

Thus, the coefficient of x, y and z in the cartesian equation of a plane are the direction ratios of normal to the plane.

For example, the direction ratios of a vector normal to the plane $2x + y - 2z - 5 = 0$ are $2, 1, -2$ and hence a vector normal to the plane is $2\hat{i} + \hat{j} - 2\hat{k}$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the vector equation of a plane passing through a point having position vector $2\hat{i} + 3\hat{j} - 4\hat{k}$ and perpendicular to the vector $2\hat{i} - \hat{j} + 2\hat{k}$. Also, reduce it to cartesian form.*

SOLUTION We know that the vector equation of a plane passing through a point $\vec{a}$ and normal to $\vec{n}$ is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \Rightarrow \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

Here, $\vec{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{n} = 2\hat{i} - \hat{j} + 2\hat{k}$.

So, the equation of the required plane is

$$\vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = (2\hat{i} + 3\hat{j} - 4\hat{k}) \cdot (2\hat{i} - \hat{j} + 2\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 4 - 3 - 8$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = -7 \quad ...(i)$$

Reduction to cartesian form : Since $\vec{r}$ denotes the position vector of an arbitrary point (x, y, z) on the plane. Therefore, putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (i), we obtain

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = -7 \Rightarrow 2x - y + 2z = -7$$

This is the required cartesian equation of the plane.

EXAMPLE 2 *Find the equation in cartesian form of the plane passing through the point $(3, -3, 1)$ and normal to the line joining the points $(3, 4, -1)$ and $(2, -1, 5)$.*

SOLUTION We know that the vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to $\vec{n}$ is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \Rightarrow \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} \quad ...(i)$$

Since the given plane passes through the point $(3, -3, 1)$ and is normal to the line joining $A(3, 4, -1)$ and $B(2, -1, 5)$. Therefore,

$$\vec{a} = 3\hat{i} - 3\hat{j} + \hat{k}$$

and, $\vec{n} = \vec{AB} = (2\hat{i} - \hat{j} + 5\hat{k}) - (3\hat{i} + 4\hat{j} - \hat{k}) = -\hat{i} - 5\hat{j} + 6\hat{k}$

Substituting $\vec{a} = 3\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{n} = -\hat{i} - 5\hat{j} + 6\hat{k}$ in equation (i), we obtain

$$\vec{r} \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = (3\hat{i} - 3\hat{j} + \hat{k}) \cdot (-\hat{i} - 5\hat{j} + 6\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = -3 + 15 + 6$$

$$\Rightarrow \quad \vec{r} \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = 18$$

This is the vector equation of the required plane.

The cartesian equation is given by

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = 18 \qquad [\text{Putting } \vec{r} = x\hat{i} + y\hat{j} + 3\hat{k}]$$

$$\Rightarrow \quad -x - 5y + 6z = 18$$

$$\Rightarrow \quad x + 5y - 6z + 18 = 0$$

EXAMPLE 3 *The foot of perpendicular drawn from the origin to the plane is $(4, -2, -5)$. Find the equation of the plane.*

SOLUTION The required plane passes through the point $P(4, -2, -5)$ and is perpendicular to $\vec{OP}$.

$$\therefore \quad \vec{a} = 4\hat{i} - 2\hat{j} - 5\hat{k} \text{ and } \vec{n} = \vec{OP} = 4\hat{i} - 2\hat{j} - 5\hat{k}.$$

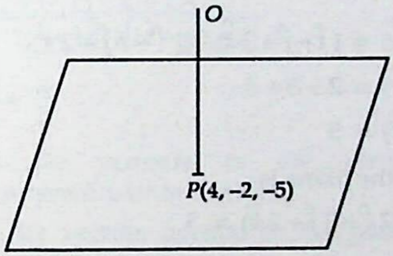

Fig. 28.3

So, the equation of the plane is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (4\hat{i} - 2\hat{j} - 5\hat{k}) = (4\hat{i} - 2\hat{j} - 5\hat{k}) \cdot (4\hat{i} - 2\hat{j} - 5\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (4\hat{i} - 2\hat{j} - 5\hat{k}) = 16 + 4 + 25$$

$$\Rightarrow \quad \vec{r} \cdot (4\hat{i} - 2\hat{j} - 5\hat{k}) = 45 \qquad \text{...(i)}$$

Reduction to cartesian form : Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (i), we obtain

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (4\hat{i} - 2\hat{j} - 5\hat{k}) = 45 \Rightarrow 4x - 2y - 5z = 45$$

This is the cartesian equation of the required plane.

EXAMPLE 4 *Find the vector equation of the plane whose cartesian form of equation is $3x - 4y + 2z = 5$.*

SOLUTION The equation of the given plane is

$$3x - 4y + 2z = 5$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3\hat{i} - 4\hat{j} + 2\hat{k}) = 5$$

$\Rightarrow \quad \vec{r} \cdot (3\hat{i} - 4\hat{j} + 2\hat{k}) = 5$

This is the vector form of the equation of the given plane.

EXAMPLE 5 *Find a normal vector to the plane $2x - y + 2z = 5$. Also, find a unit vector normal to the plane.*

SOLUTION We know that the coefficient of x, y and z respectively in the cartesian equation of a plane determine the direction ratios of a vector normal to the plane. Therefore, direction ratios of a vector $\vec{n}$ normal to the given plane are proportional to $2, -1, 2$.

$\therefore \quad \vec{n} = 2\hat{i} - \hat{j} + 2\hat{k}$

Thus, a unit vector normal to the plane is given by

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i} - \hat{j} + 2\hat{k})$$

EXAMPLE 6 *Find the equation of the plane passing through the point $(1, -1, 2)$ having $2, 3, 2$ as direction ratios of normal to the plane.*

SOLUTION Here the plane passes through the point having position vector $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and is normal to the vector $\vec{n} = 2\hat{i} + 3\hat{j} + 2\hat{k}$. So, the vector equation of the plane is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} + 2\hat{k}) = (\hat{i} - \hat{j} + 2\hat{k}) \cdot (2\hat{i} + 3\hat{j} + 2\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} + 2\hat{k}) = 2 - 3 + 4$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} + 2\hat{k}) = 3$$

The cartesian equation of the plane is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} + 2\hat{k}) = 3 \qquad [\text{Putting } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$$

$$\Rightarrow \quad 2x + 3y + 2z = 3$$

EXAMPLE 7 *Let $\vec{n}$ be a vector of magnitude $2\sqrt{3}$ such that it makes equal acute angles with the coordinate axes. Find the vector and cartesian forms of the equation of a plane passing through $(1, -1, 2)$ and normal to $\vec{n}$.*

SOLUTION Let α, β and γ be the angles made by $\vec{n}$ with ox, oy and oz respectively. Then,

$$\alpha = \beta = \gamma \Rightarrow \cos\alpha = \cos\beta = \cos\gamma \Rightarrow l = m = n,$$

where l, m, n are direction cosines of $\vec{n}$.

But, $\quad l^2 + m^2 + n^2 = 1$

$$\therefore \quad l = m = n \Rightarrow 3l^2 = 1 \Rightarrow l = \frac{1}{\sqrt{3}} \qquad [\because \alpha \text{ is acute } \therefore \cos\alpha = l > 0]$$

Thus, $\quad \vec{n} = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right) \qquad [\text{Using } \vec{r} = |\vec{r}|\,(l\hat{i} + m\hat{j} + n\hat{k})]$

$$\Rightarrow \quad \vec{n} = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

The required plane passes through a point $(1, -1, 2)$ having position vector $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and is normal to $\vec{n}$. So, its vector equation is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r}\cdot(2\hat{i}+2\hat{j}+2\hat{k}) = (\hat{i}-\hat{j}+2\hat{k})\cdot(2\hat{i}+2\hat{j}+2\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(2\hat{i}+2\hat{j}+2\hat{k}) = 2-2+4$$

$$\Rightarrow \quad \vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 2.$$

The cartesian equation of this plane is

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k}) = 2 \Rightarrow x+y+z=2.$$

EXAMPLE 8 *Find the angle between the normals to the planes $2x - y + z = 6$ and $x+y+2z = 7$.*

SOLUTION Let $\vec{n_1}$ and $\vec{n_2}$ be vectors normal to the planes $2x - y + z = 6$ and $x+y+2z=7$.

Direction ratios of normal to $2x-y+z=6$ are proportional to $2, -1, 1$

So, $\quad \vec{n_1} = 2\hat{i}-\hat{j}+\hat{k}$

Direction ratios of normal to $x+y+2z=7$ are proportional to $1, 1, 2$

So, $\quad \vec{n_2} = \hat{i}+\hat{j}+2\hat{k}$

Let θ be the angle between the normals $\vec{n_1}$ and $\vec{n_2}$. Then,

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\ |\vec{n_2}|}$$

$$\Rightarrow \quad \cos\theta = \frac{2\times1+(-1)\times1+1\times2}{\sqrt{2^2+(-1)^2+1^2}\ \sqrt{1^2+1^2+2^2}}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

EXAMPLE 9 *Show that the normals to the planes $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=3$ and $\vec{r}\cdot(3\hat{i}+2\hat{j}-\hat{k})+5=0$ are perpendicular to each other.*

SOLUTION Let $\vec{n_1}$ and $\vec{n_2}$ be vectors normal to the planes $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k}) = 3$ and, $\vec{r}\cdot(3\hat{i}+2\hat{j}-\hat{k}) = 0$ respectively.

Then, $\quad \vec{n_1} = \hat{i}-\hat{j}+\hat{k}$ and $\vec{n_2} = 3\hat{i}+2\hat{j}-\hat{k}$.

Now, $\quad \vec{n_1}\cdot\vec{n_2} = (\hat{i}-\hat{j}+\hat{k})\cdot(3\hat{i}+2\hat{j}-\hat{k}) = 3-2-1 = 0$

Hence, $\quad \vec{n_1} \perp \vec{n_2}$.

EXAMPLE 10 *Find the angles at which the normal vector to the plane $4x+8y+z=5$ is inclined to the coordinate axes.*

SOLUTION Let $\vec{n}$ be a vector normal to the plane. Since direction ratios of normal to the plane are proportional to 4, 8, 1. Therefore, $\vec{n} = 4\hat{i}+8\hat{j}+\hat{k}$

Direction cosines of $\vec{n}$ are

$$\frac{4}{\sqrt{4^2+8^2+1^2}}, \frac{8}{\sqrt{4^2+8^2+1^2}}, \frac{1}{\sqrt{4^2+8^2+1^2}} \text{ or, } \frac{4}{9}, \frac{8}{9}, \frac{1}{9}$$

Let α, β, γ be the angles made by $\vec{r}$ with x-axis, y-axis and z-axis respectively. Then,

$$\cos\alpha = \frac{4}{9}, \cos\beta = \frac{8}{9} \text{ and } \cos\gamma = \frac{1}{9}$$

$$\therefore \quad \alpha = \cos^{-1}\left(\frac{4}{9}\right), \beta = \cos^{-1}\left(\frac{8}{9}\right) \text{ and } \gamma = \cos^{-1}\left(\frac{1}{9}\right).$$

EXAMPLE 11 *A vector $\vec{n}$ of magnitude 8 units is inclined to x-axis at 45°, y-axis at 60° and an acute angle with z-axis. If a plane passes through a point $(\sqrt{2}, -1, 1)$ and is normal to $\vec{n}$, find its equation in vector form.*

SOLUTION Let γ be the angle made by $\vec{n}$ with z-axis. Then direction cosines of $\vec{n}$ are

$$l = \cos 45° = \frac{1}{\sqrt{2}},\ m = \cos 60° = \frac{1}{2} \text{ and } n = \cos\gamma.$$

$$\therefore \quad l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \quad \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + n^2 = 1$$

$$\Rightarrow \quad n^2 = \frac{1}{4} \Rightarrow n = \frac{1}{2} \qquad [\because \gamma \text{ is acute } \therefore n = \cos\gamma > 0]$$

We have, $|\vec{n}| = 8$.

$$\therefore \quad \vec{n} = |\vec{n}|\,(l\,\hat{i} + m\,\hat{j} + n\,\hat{k})$$

$$\Rightarrow \quad \vec{n} = 8\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}\right) = 4\sqrt{2}\,\hat{i} + 4\hat{j} + 4\hat{k}$$

The required plane passes through the point $(\sqrt{2}, -1, 1)$ having position vector $\vec{a} = \sqrt{2}\,\hat{i} - \hat{j} + \hat{k}$.

So, its vector equation is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (4\sqrt{2}\,\hat{i} + 4\hat{j} + 4\hat{k}) = (\sqrt{2}\,\hat{i} - \hat{j} + \hat{k}) \cdot (4\sqrt{2}\,\hat{i} + 4\hat{j} + 4\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (4\sqrt{2}\,\hat{i} + 4\hat{j} + 4\hat{k}) = 8 - 4 + 4$$

$$\Rightarrow \quad \vec{r} \cdot (4\sqrt{2}\,\hat{i} + 4\hat{j} + 4\hat{k}) = 8$$

$$\Rightarrow \quad \vec{r} \cdot (\sqrt{2}\,\hat{i} + \hat{j} + \hat{k}) = 2.$$

EXERCISE 28.3

1. Find the vector equation of a plane passing through a point having position vector $2\hat{i} - \hat{j} + \hat{k}$ and perpendicular to the vector $4\hat{i} + 2\hat{j} - 3\hat{k}$.
2. Find the cartesian form of equation of a plane whose vector equation is
   (i) $\vec{r} \cdot (12\hat{i} - 3\hat{j} + 4\hat{k}) + 5 = 0$ (ii) $\vec{r} \cdot (-\hat{i} + \hat{j} + 2\hat{k}) = 9$
3. Find the vector equations of the coordinates planes.
4. Find the vector equation of each one of following planes:
   (i) $2x - y + 2z = 8$ (ii) $x + y - z = 5$ (iii) $x + y = 3$
5. Find the vector and cartesian equations of a plane passing through the point $(1, -1, 1)$ and normal to the line joining the points $(1, 2, 5)$ and $(-1, 3, 1)$.
6. If $\vec{n}$ is a vector of magnitude $\sqrt{3}$ and is equally inclined with an acute angle with the coordinate axes. Find the vector and cartesian forms of equation of a plane which passes through $(2, 1, -1)$ and is normal to $\vec{n}$.
7. The foot of the perpendicular drawn from the origin to a plane is $(12, -4, 3)$. Find the equation of the plane.
8. Find the equation of the plane passing through the point $(2, 3, 1)$ having 5, 3, 2 as direction ratios of normal to the plane.
9. If the axes are rectangular and P is the point $(2, 3, -1)$, find the equation of the plane through P at right angles to OP.

10. Find the intercepts made on the coordinate axes by the plane $2x+y-2z=3$ and find also the direction cosines of the normal to the plane.
11. A plane passes through the point $(1,-2,5)$ and is perpendicular to the line joining the origin to the point $3\hat{i}+\hat{j}-\hat{k}$. Find the vector and cartesian forms of the equation of the plane.
12. Find the equation of the plane that bisects the line joining $(1,2,3)$ and $(3,4,5)$ and is at right angle to the line.
13. Show that the normals to the following pairs of planes are perpendicular to each other:
    (i) $x-y+z-2=0$ and $3x+2y-z+4=0$
    (ii) $\vec{r}\cdot(2\hat{i}-\hat{j}+3\hat{k})=5$ and $\vec{r}\cdot(2\hat{i}-2\hat{j}-2\hat{k})=5$.
14. Show that the normal vector to the plane $2x+2y+2z=3$ is equally inclined with the coordinate axes.
15. Find a vector of magnitude 26 units normal to the plane $12x-3y+4z=1$.
16. If the line drawn from $(4,-1,2)$ meets a plane at right angles at the point $(-10,5,4)$, find the equation of the plane.
17. Find the equation of the plane which bisects the line joining the points $(-1,2,3)$ and $(3,-5,6)$ at right angles.

ANSWERS

1. $\vec{r}\cdot(4\hat{i}+2\hat{j}-3\hat{k})=3$
2. (i) $12x-3y+4z+5=0$ (ii) $-x+y+2z=9$
3. $\vec{r}\cdot\hat{i}=0,\ \vec{r}\cdot\hat{j}=0,\ \vec{r}\cdot\hat{k}=0$
4. (i) $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=8$ (ii) $\vec{r}\cdot(\hat{i}+\hat{j}-\hat{k})=5$
   (iii) $\vec{r}\cdot(\hat{i}+\hat{j})=3$.
5. $\vec{r}\cdot(2\hat{i}-\hat{j}+4\hat{k})=7$, $2x-y+4z=7$
6. $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$, $x+y+z=2$
7. $12x-4y+3z=169$
8. $5x+3y+2z=21$
9. $2x+3y-z=14$
10. $\frac{3}{2}, 3, -\frac{3}{2}; \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$
11. $\vec{r}\cdot(3\hat{i}+\hat{j}-\hat{k})=-4$, $3x+y-z=-4$
12. $x+y+z=9$
15. $24\hat{i}-6\hat{j}+8\hat{k}$
16. $7x-3y-z+89=0$
17. $4x-7y+3z-28=0$

HINTS TO SELECTED PROBLEM

3. XY-plane passes through the origin and is perpendicular to z-axis.
   So, its vector equation is $(\vec{r}-\vec{0})\cdot\hat{k}=0$ or, $\vec{r}\cdot\hat{k}=0$.
   Similarly, the equations of YZ and XZ-planes are $\vec{r}\cdot\hat{i}=0$ and $\vec{r}\cdot\hat{j}=0$ respectively.

28.5 EQUATION OF A PLANE IN NORMAL FORM

VECTOR FORM

THEOREM 1 *The vector equation of a plane normal to unit vector $\hat{n}$ and at a distance d from the origin is $\vec{r}\cdot\hat{n}=d$.*

PROOF Let O be the origin and let ON be the perpendicular from O to the given plane π such that $ON=d$. Let $\hat{n}$ be a unit vector along ON. Then, $\vec{ON}=d\hat{n}$. So, the position vector of N is $d\hat{n}$. Let $\vec{r}$ be the position vector of an arbitrary point P on the plane. Then,

$$\overrightarrow{NP} \perp \overrightarrow{ON}$$

$$\Rightarrow \quad \overrightarrow{NP} \perp \hat{n}$$

$$\Rightarrow \quad \overrightarrow{NP} \cdot \hat{n} = 0$$

$$\Rightarrow \quad (\vec{r} - d\hat{n}) \cdot \hat{n} = 0 \qquad [\because \overrightarrow{NP} = \vec{r} - d\hat{n}]$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} - d\,(\hat{n} \cdot \hat{n}) = 0$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} - d\,|\,\hat{n}\,|^2 = 0$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} - d = 0 \qquad [\because |\,\hat{n}\,| = 1]$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} = d.$$

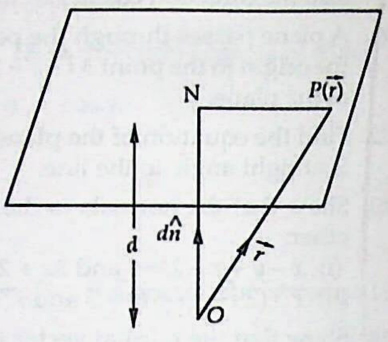

Fig. 28.4

Since $\vec{r}$ denotes the position vector of an arbitrary point on the plane π. Thus, the required equation of the plane is $\vec{r} \cdot \hat{n} = d$.

REMARK 1 *The vector equation of ON is $\vec{r} = \vec{0} + \lambda\,\hat{n}$ and the position vector of N is $d\,\hat{n}$ as it is at a distance d from the origin O.*

CARTESIAN FORM

THEOREM 2 *If l, m, n are direction cosines of the normal to a given plane which is at a distance p from the origin, then the equation of the plane is $lx + my + nz = p$.*

PROOF We know that the vector equation of a plane at a distance p from the origin and normal to unit vector $\hat{n}$ is

$$\vec{r} \cdot \hat{n} = p$$

Here, $\hat{n} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$.

So, the cartesian equation of the plane is

$$(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}) \cdot (l\,\hat{i} + m\,\hat{j} + n\,\hat{k}) = p$$

$$\Rightarrow \quad lx + my + nz = p$$

NOTE 1 *The equation $\vec{r} \cdot \vec{n} = d$ is in normal form if $\vec{n}$ is a unit vector and in such a case d denotes the distance of the plane from the origin. If $\vec{n}$ is not a unit vector, then to reduce the equation $\vec{r} \cdot \vec{n} = d$ to normal form we divide both sides by $|\,\vec{n}\,|$ to obtain*

$$\vec{r} \cdot \frac{\vec{n}}{|\,\vec{n}\,|} = \frac{d}{|\,\vec{n}\,|} \Rightarrow \vec{r} \cdot \hat{n} = \frac{d}{|\,\vec{n}\,|}$$

NOTE 2 *To reduce the cartesian equation $ax + by + cz + d = 0$ to normal form, we proceed as follows:*

STEP I *Keep terms containing x, y and z on LHS and shift the constant term d on the RHS.*

STEP II *If the constant term on RHS is not positive make it positive by multiplying both sides by −1.*

STEP III *Divide each term on two sides by*

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{(\text{Coeff. of } x)^2 + (\text{Coeff. of } y)^2 + (\text{Coeff. of } z)^2}.$$

The coefficients of x, y and z in the equation so obtained will be the direction cosines of the normal to the plane and the RHS will be the distance of the plane from the origin.

REMARK 2 *The cartesian equation of the normal to the plane $lx + my + nz = p$ drawn from the origin is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and the coordinates of the foot N of the perpendicular drawn from the origin O are given by $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = p$ i.e. (lp, mp, np).*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the vector equation of a plane at a distance of 5 units from the origin and has $\hat{i}$ as the unit vector normal to it.*

SOLUTION Here, $\hat{n} = \hat{i}$ and $d = 5$.

So, the vector equation of the plane is $\vec{r} \cdot \hat{i} = 5$.

EXAMPLE 2 *Find the vector equation of a plane which is at a distance of 8 units from the origin and which is normal to the vector $2\hat{i} + \hat{j} + 2\hat{k}$.*

SOLUTION Here, $d = 8$ and $\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$.

$$\therefore \qquad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Hence, the required equation of the plane is

$$\vec{r} \cdot \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = 8 \Rightarrow \vec{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 24.$$

EXAMPLE 3 *Reduce the equation $\vec{r} \cdot (3\hat{i} - 4\hat{j} + 12\hat{k}) = 5$ to normal form and hence find the length of perpendicular from the origin to the plane.*

SOLUTION The given equation is

$$\vec{r} \cdot (3\hat{i} - 4\hat{j} + 12\hat{k}) = 5.$$

$$\Rightarrow \qquad \vec{r} \cdot \vec{n} = 5, \text{ where } \vec{n} = 3\hat{i} - 4\hat{j} + 12\hat{k}.$$

Since $|\vec{n}| = \sqrt{3^2 + (-4)^2 + 12^2} = 13 \neq 1$. Therefore, the given equation is not in normal form. To reduce it to normal form, we divide both sides by $|\vec{n}|$ i.e.,

$$\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{5}{|\vec{n}|}$$

$$\Rightarrow \qquad \vec{r} \cdot \left(\frac{3}{13}\hat{i} - \frac{4}{13}\hat{j} + \frac{12}{13}\hat{k}\right) = \frac{5}{13}$$

This is the normal form of the equation of given plane. The length of the perpendicular from the origin is $\frac{5}{13}$.

EXAMPLE 4 *Reduce the equation of the plane $x - 2y - 2z = 12$ to normal form and hence find the length of the perpendicular from the origin to the plane. Also, find the direction cosines of the normal to the plane.*

SOLUTION The given equation of the plane is $x - 2y - 2z = 12$ ...(i)

Dividing (i) throughout by $\sqrt{1^2 + (-2)^2 + (-2)^2}$ i.e. by 3, we get

$$\frac{x}{3} - \frac{2}{3}y - \frac{2}{3}z = 4$$

This is the normal form of the equation of the given plane. Clearly, the length of the perpendicular from the origin to the plane is 4 units and the direction cosines of the normal to the plane are $\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$.

EXAMPLE 5 *Find the vector equation of a plane which is at a distance of 6 units from the origin and has 2, –1, 2 as the direction ratios of a normal to it. Also, find the coordinates of the foot of the normal drawn from the origin.*

SOLUTION Let $\vec{n}$ be a vector normal to the plane. Then the direction ratios of $\vec{n}$ are proportional to 2, – 1, 2.

$$\therefore \quad \vec{n} = 2\hat{i} - \hat{j} + 2\hat{k} \Rightarrow |\vec{n}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$$

$$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

The required plane is at a distance of 6 units from the origin. So, its equation is

$$\vec{r} \cdot \hat{n} = 6 \Rightarrow \vec{r} \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = 6$$

The position vector of the foot of the normal drawn from the origin is

$$d\,\hat{n} = 6\left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

So, the coordinates of the foot of the normal are (4, – 2, 4).

EXAMPLE 6 *Find the coordinates of the foot of the perpendicular drawn from the origin to the plane* $2x - 3y + 4z - 6 = 0$.

SOLUTION The equation of the plane is

$$2x - 3y + 4z - 6 = 0$$

$$\Rightarrow \quad 2x - 3y + 4z = 6$$

$$\Rightarrow \quad \frac{2}{\sqrt{29}}x - \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}} \qquad \left[\text{Dividing through out by } \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{29}\right]$$

This is the normal form of the given plane. It is evident from this equation that the direction cosines of the normal drawn from the origin to the given plane are

$$l = \frac{2}{\sqrt{29}}, \; m = -\frac{3}{\sqrt{29}}, \; n = \frac{4}{\sqrt{29}}$$

and the plane is at a distance of $d = \frac{6}{\sqrt{29}}$ units from the origin.

The coordinates of the foot of the perpendicular drawn from the origin are

$$(ld, md, nd) \text{ i.e., } \left(\frac{12}{29}, -\frac{18}{29}, \frac{24}{29}\right).$$

EXAMPLE 7 *Find the direction cosines of perpendicular from the origin to the plane* $\vec{r} \cdot (2\hat{i} - 3\hat{j} - 6\hat{k}) + 5 = 0$.

SOLUTION To find the direction cosines of perpendicular from the origin to the given plane we first reduce it to normal form.

The equation of the given plane is

$$\vec{r} \cdot (2\hat{i} - 3\hat{j} - 6\hat{k}) + 5 = 0$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} - 3\hat{j} - 6\hat{k}) = -5$$

$$\Rightarrow \quad \vec{r} \cdot (-2\hat{i} + 3\hat{j} + 6\hat{k}) = 5$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = 5, \text{ where } \vec{n} = -2\hat{i} + 3\hat{j} + 6\hat{k} \qquad \text{...(i)}$$

Now, $\quad |\vec{n}| = \sqrt{(-2)^2 + 3^2 + 6^2} = 7$

Dividing (i) throughout by $|\vec{n}| = 7$, we get

$$\vec{r}\cdot\frac{\vec{n}}{|\vec{n}|} = \frac{5}{|\vec{n}|} \Rightarrow \vec{r}\cdot\left(-\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}\right) = \frac{5}{7}$$

So, the direction cosines of the normal to the plane are $-\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$.

EXAMPLE 8 *Find the equation of the plane passing through the point (– 1, 2, 1) and perpendicular to the line joining the points (– 3, 1, 2) and (2, 3, 4). Find also the perpendicular distance of the origin from this plane.*

SOLUTION The required plane passes through the point $(-1, 2, 1)$ having position vector $\vec{a} = -\hat{i}+2\hat{j}+\hat{k}$ and is perpendicular to the line joining the points $A(-3, 1, 2)$ and $B(2, 3, 4)$. Therefore, a vector normal to the plane is given by

$$\vec{n} = \vec{AB} = (2\hat{i}+3\hat{j}+4\hat{k}) - (-3\hat{i}+\hat{j}+2\hat{k}) = 5\hat{i}+2\hat{j}+2\hat{k}$$

We know that the vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to vector $\vec{n}$ is given by

$$(\vec{r}-\vec{a})\cdot\vec{n} = 0 \Rightarrow \vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

Therefore, the equation of the required plane is

$$\vec{r}\cdot(5\hat{i}+2\hat{j}+2\hat{k}) = (-\hat{i}+2\hat{j}+\hat{k})\cdot(5\hat{i}+2\hat{j}+2\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(5\hat{i}+2\hat{j}+2\hat{k}) = -5+4+2$$

$$\Rightarrow \quad \vec{r}\cdot(5\hat{i}+2\hat{j}+2\hat{k}) = 1. \qquad \text{...(i)}$$

To find the distance of this plane from the origin, we reduce its equation to normal form.

We have, $|\vec{n}| = \sqrt{5^2+2^2+2^2} = \sqrt{33}$

Dividing (i) throughout by $|\vec{n}| = \sqrt{33}$, we get

$$\vec{r}\cdot\left(\frac{5}{\sqrt{33}}\hat{i}+\frac{2}{\sqrt{33}}\hat{j}+\frac{2}{\sqrt{33}}\hat{k}\right) = \frac{1}{\sqrt{33}}$$

So, the perpendicular distance of the origin from the plane is $\frac{1}{\sqrt{33}}$.

28.6 VECTOR EQUATION OF A PLANE PASSING THROUGH THREE GIVEN POINTS

Let A, B, and C be three points on a plane π having their position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then, vectors $\vec{AB}$ and $\vec{AC}$ are in the plane π. Therefore, $\vec{AB}\times\vec{AC}$ is a vector perpendicular to the plane π.

We have,

$$\vec{AB}\times\vec{AC} = (\vec{b}-\vec{a})\times(\vec{c}-\vec{a})$$

$$\Rightarrow \quad \vec{AB}\times\vec{AC} = \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$$

Thus, the plane π passes through the point A with position vector $\vec{a}$ and is normal to vector

$$\vec{n} = \vec{AB}\times\vec{AC} = \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$$

So, the vector equation of the plane π is

$$(\vec{r}-\vec{a})\cdot\vec{n} = 0$$

or, $$(\vec{r}-\vec{a})\cdot(\vec{AB}\times\vec{AC}) = 0$$

or, $$(\vec{r}-\vec{a})\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}) = 0$$

ILLUSTRATION 1 *Find the vector equation of the plane passing through the points $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$. Also, find the cartesian equation of the plane.*

SOLUTION The required plane passes through the point $A(2, 2, -1)$ whose position vector is $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and is normal to the vector $\vec{n}$ given by

$$\vec{n} = \vec{AB} \times \vec{AC}$$

We have,

$$\vec{AB} = (3\hat{i} + 4\hat{j} + 2\hat{k}) - (2\hat{i} + 2\hat{j} - \hat{k}) = \hat{i} + 2\hat{j} + 3\hat{k}$$

and, $$\vec{AC} = (7\hat{i} + 0\hat{j} + 6\hat{k}) - (2\hat{i} + 2\hat{j} - \hat{k}) = 5\hat{i} - 2\hat{j} + 7\hat{k}$$

$$\therefore \quad \vec{n} = \vec{AB} \times \vec{AC}$$

$$\Rightarrow \quad \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix}$$

$$\Rightarrow \quad \vec{n} = (14 + 6)\,\hat{i} - (7 - 15)\,\hat{j} + (-2 - 10)\,\hat{k} \quad = 20\hat{i} + 8\hat{j} - 12\hat{k}$$

The vector equation of the required plane is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (20\hat{i} + 8\hat{j} - 12\hat{k}) = (2\hat{i} + 2\hat{j} - \hat{k}) \cdot (20\hat{i} + 8\hat{j} - 12\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (20\hat{i} + 8\hat{j} - 12\hat{k}) = 40 + 16 + 12$$

$$\Rightarrow \quad \vec{r} \cdot (20\hat{i} + 8\hat{j} - 12\hat{k}) = 68$$

$$\Rightarrow \quad \vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$$

The cartesian equation of the plane is given by

$$(x\hat{i} + y\hat{j} + 3\hat{k}) \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$$

$$\Rightarrow \quad 5x + 2y - 3z = 17.$$

ILLUSTRATION 2 *If from a point $P(a, b, c)$ perpendiculars PA and PB are drawn to yz and zx-planes, find the vector equation of the plane OAB.*

SOLUTION The coordinates A and B are $(0, b, c)$ and $(a, 0, c)$ respectively.

$$\therefore \quad \vec{OA} = b\hat{j} + c\hat{k} \text{ and } \vec{OB} = a\hat{i} + c\hat{k}$$

The plane OAB passes through $O\,(\vec{0})$ and is perpendicular to $\vec{n} = \vec{OA} \times \vec{OB}$.

We have,

$$\vec{n} = \vec{OA} \times \vec{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & b & c \\ a & 0 & c \end{vmatrix} = bc\hat{i} + ac\hat{j} - ab\hat{k}$$

So, the equation of plane OAB is

$$(\vec{r} - \vec{0}) \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = 0 \Rightarrow \vec{r} \cdot (bc\hat{i} + ca\hat{j} - ab\hat{k}) = 0.$$

EXERCISE 28.4

1. Find the vector equation of a plane which is at a distance of 3 units from the origin and has $\hat{k}$ as the unit vector normal to it.
2. Find the vector equation of a plane which is at a distance of 5 units from the origin and which is normal to the vector $\hat{i} - 2\hat{j} - 2\hat{k}$.

3. Reduce the equation $2x - 3y - 6z = 14$ to the normal form and hence find the length of perpendicular from the origin to the plane. Also, find the direction cosines of the normal to the plane.
4. Reduce the equation $\vec{r} \cdot (\hat{i} - 2\hat{j} + 2\hat{k}) + 6 = 0$ to normal form and hence find the length of perpendicular from the origin to the plane.
5. Write the normal form of the equation of the plane $2x - 3y + 6z + 14 = 0$.
6. The direction ratios of the perpendicular from the origin to a plane are $12, -3, 4$ and the length of the perpendicular is 5. Find the equation of the plane.
7. Find a unit normal vector to the plane $x + 2y + 3z - 6 = 0$.
8. Find the equation of a plane which is at a distance of $3\sqrt{3}$ units from the origin and the normal to which is equally inclined with the coordinate axes.
9. Find the equation of the plane passing through the point $(1, 2, 1)$ and perpendicular to the line joining the points $(1, 4, 2)$ and $(2, 3, 5)$. Find also the perpendicular distance of the origin from this plane. [CBSE 1996]
10. Find the vector equation of the plane passing through the points $(1, 1, 1)$, $(1, -1, 1)$ and $(-7, -3, -5)$.

ANSWERS

1. $\vec{r} \cdot \hat{k} = 3$
2. $\vec{r} \cdot \left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k} \right) = 5$
3. $2, \frac{2}{7}, -\frac{3}{7}, -\frac{6}{7}$
4. $\vec{r} \cdot \left(-\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k} \right) = 2; 2$
5. $-\frac{2}{7}x + \frac{3}{7}y - \frac{6}{7}z = 2$
6. $\frac{12}{13}x - \frac{3}{13}y + \frac{4}{13}z = 5$
7. $\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$
8. $x + y + z = 9$
9. $x - y + 3z - 2 = 0, \frac{2}{\sqrt{11}}$
10. $\vec{r} \cdot (3i - 4\hat{k}) + 1 = 0$

28.7 ANGLE BETWEEN TWO PLANES

DEFINITION *The angle between two planes is defined as the angle between their normals.*

VECTOR FORM

THEOREM 1 *The angle θ between the planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ is given by*

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

PROOF Let θ be the angle between the planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$. Then, θ is the angle between their normals $\vec{n_1}$ and $\vec{n_2}$.

$$\therefore \quad \cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

Condition of perpendicularity : If the planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ are perpendicular, then $\vec{n_1}$ and $\vec{n_2}$ are perpendicular.

$$\therefore \quad \vec{n_1} \cdot \vec{n_2} = 0$$

Condition of parallelism : If the planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ are parallel, then $\vec{n_1}$ and $\vec{n_2}$ are parallel. Therefore, there exists a scalar λ such that

$$\vec{n_1} = \lambda \vec{n_2}$$

CARTESIAN FORM

THEOREM 2 *The angle θ between the planes $a_1 x + b_1 y + c_1 z = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ is given by*

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

PROOF Let $\vec{n_1}$ and $\vec{n_2}$ be the vectors normal to the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. Then,

$$\vec{n_1} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \text{ and } \vec{n_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}.$$

The angle θ between $\vec{n_1}$ and $\vec{n_2}$ is given by

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\ |\vec{n_2}|} \Rightarrow \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition of perpendicularity: If the planes are perpendicular, then $\vec{n_1}$ and $\vec{n_2}$ are perpendicular.

$\therefore \quad \vec{n_1}\cdot\vec{n_2} = 0$

$\Rightarrow \quad (a_1\hat{i} + b_1\hat{j} + c_1\hat{k})\cdot(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = 0$

$\Rightarrow \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$

Thus, the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular, if

$$a_1a_2 + b_1 b_2 + c_1c_2 = 0$$

Condition of parallelism: If the planes are parallel, then $\vec{n_1}$ and $\vec{n_2}$ are parallel.

$\therefore \quad \vec{n_1} = \lambda\,\vec{n_2}$ for some scalar λ

$\Rightarrow \quad (a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) = \lambda\,(a_2\hat{i} + b_2\hat{j} + c_2\hat{k})$

$\Rightarrow \quad a_1 = \lambda\,a_2,\ b_1 = \lambda\,b_2$ and $c_1 = \lambda\,c_2$

$\Rightarrow \quad \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Thus, the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

ILLUSTRATIVE EXAMPLES

Type I **ON FINDING THE ANGLE BETWEEN TWO PLANES**

EXAMPLE 1 *Find the angle between the planes $\vec{r}\cdot(2\hat{i} - \hat{j} + \hat{k}) = 6$ and $\vec{r}\cdot(\hat{i} + \hat{j} + 2\hat{k}) = 5$.*

SOLUTION We know that the angle between the planes $\vec{r}\cdot\vec{n_1} = d_1$ and $\vec{r}\cdot\vec{n_2} = d_2$ is given by

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\ |\vec{n_2}|}$$

Here, $\quad \vec{n_1} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{n_2} = \hat{i} + \hat{j} + 2\hat{k}$.

$$\therefore \quad \cos\theta = \frac{(2\hat{i}-\hat{j}+\hat{k})\cdot(\hat{i}+\hat{j}+2\hat{k})}{|2\hat{i}-\hat{j}+\hat{k}|\ |\hat{i}+\hat{j}+2\hat{k}|} = \frac{2-1+2}{\sqrt{4+1+1}\ \sqrt{1+1+4}} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \pi/3.$$

EXAMPLE 2 *Find the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$.*

SOLUTION We know that the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Therefore, the angle between $x + y + 2z = 9$ and $2x - y + z = 15$ is given by

$$\cos\theta = \frac{(1)(2) + (1)(-1) + (2)(1)}{\sqrt{1^2 + 1^2 + 2^2}\ \sqrt{2^2 + (-1)^2 + 1^2}} = \frac{2-1+2}{\sqrt{6}\ \sqrt{6}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

EXAMPLE 3 *Show that the planes $2x + 6y + 6z = 7$ and $3x + 4y - 5z = 8$ are at right angles.*

SOLUTION We know that the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles, if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

We have, $(2)(3) + (6)(4) + (6)(-5) = 0$.

Therefore, planes $2x + 6y + 6z = 7$ and $3x + 4y - 5z = 8$ are at right angles.

EXAMPLE 4 *If the planes $\vec{r}\cdot(2\hat{i}-\hat{j}+\lambda\hat{k}) = 5$ and $\vec{r}\cdot(3\hat{i}+2\hat{j}+2\hat{k}) = 4$ are perpendicular. Find the value of λ.*

SOLUTION We know that the planes $\vec{r}\cdot\vec{n_1} = d$ and $\vec{r}\cdot\vec{n_2} = d_2$ are perpendicular, if $\vec{n_1}\cdot\vec{n_2} = 0$. Therefore, given planes will be perpendicular to each other, if

$$(2\hat{i}-\hat{j}+\lambda\hat{k})\cdot(3\hat{i}+2\hat{j}+2\hat{k}) = 0 \Rightarrow 6-2+2\lambda = 0 \Rightarrow \lambda = -2.$$

Type II ON FINDING A PLANE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO TWO GIVEN PLANES

The normal to the plane passing through a point having position vector $\vec{a}$ and perpendicular to the planes $\vec{r}\cdot\vec{n_1} = d_1$ and $\vec{r}\cdot\vec{n_2} = d_2$ is perpendicular to the vectors $\vec{n_1}$ and $\vec{n_2}$. So, it is parallel to $\vec{n_1} \times \vec{n_2}$. We may use the following algorithm to find the plane passing through a given point and perpendicular to the two planes.

ALGORITHM

<u>STEP I</u> *Obtain the position vector of the given point say, $\vec{a}$.*

<u>STEP II</u> *Obtain the normal vectors to two planes. Let the normal vectors be $\vec{n_1}$ and $\vec{n_2}$.*

<u>STEP III</u> *Compute $\vec{n} = \vec{n_1} \times \vec{n_2}$. Clearly, $\vec{n}$ is normal to the required plane.*

<u>STEP IV</u> *Write the equation of the desired plane as*

$$(\vec{r} - \vec{a})\cdot(\vec{n_1} \times \vec{n_2}) = 0$$

$$\text{or,} \quad \vec{r}\cdot(\vec{n_1} \times \vec{n_2}) = \vec{a}\cdot(\vec{n_1} \times \vec{n_2})$$

EXAMPLE 5 *Find the equation of the plane passing through the point $(1, 1, -1)$ and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.* **[CBSE 2003]**

SOLUTION The equation of any plane passing through $(1, 1, -1)$ is

$$a(x-1) + b(y-1) + c(z+1) = 0 \quad \text{...(i)}$$

If plane (i) is perpendicular to each one of the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$, then

$$a + 2b + 3c = 0 \qquad \text{...(ii)}$$

and,
$$2a - 3b + 4c = 0 \qquad \text{...(iii)}$$

On solving (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{(2)(4)-(3)(-3)} = \frac{b}{(3)(2)-(1)(4)} = \frac{c}{(1)(-3)-(2)(2)}$$

$$\Rightarrow \quad \frac{a}{17} = \frac{b}{2} = \frac{c}{-7} = \lambda \text{ (say)}$$

$$\Rightarrow \quad a = 17\lambda,\ b = 2\lambda \text{ and } c = -7\lambda$$

Putting $a = 17\lambda$, $b = 2\lambda$ and $c = -7\lambda$ in (i), we get

$$17\lambda(x-1) + 2\lambda(y-1) - 7\lambda(z+1) = 0$$

$$\Rightarrow \quad 17x + 2y - 7z = 26$$

This is the equation of the required plane.

<u>ALITER</u> The required plane passes through the point having position vector $\vec{a} = \hat{i} + \hat{j} - \hat{k}$. Let the normal vector to the required plane be $\vec{n}$. Then, $\vec{n}$ is perpendicular to the normals to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$ i.e. to the vectors $\vec{n_1} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{n_2} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$$\therefore \quad \vec{n} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix} = 17\hat{i} + 2\hat{j} - 7\hat{k}$$

So, the equation of the required plane is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (17\hat{i} + 2\hat{j} - 7\hat{k}) = (\hat{i} + \hat{j} - \hat{k}) \cdot (17\hat{i} + 2\hat{j} - 7\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (17\hat{i} + 2\hat{j} - 7\hat{k}) = 17 + 2 + 7$$

$$\Rightarrow \quad \vec{r} \cdot (17\hat{i} + 2\hat{j} - 7\hat{k}) = 26$$

Type III ON FINDING A PLANE PASSING THROUGH TWO GIVEN POINTS AND PERPENDICULAR TO A GIVEN PLANE

The normal to the plane passing through two points P and Q having their position vectors $\vec{a}$ and $\vec{b}$ respectively and perpendicular to the plane $\vec{r} \cdot \vec{n_1} = d_1$ is perpendicular to the vectors $\vec{PQ}$ and $\vec{n_1}$. So, the normal vector $\vec{n}$ to the plane is parallel to the vector $\vec{PQ} \times \vec{n_1}$. Thus, we may use the following algorithm to find a plane passing through two given points and perpendicular to a given plane.

ALGORITHM

<u>STEP I</u> *Obtain the position vectors of the given points. Let the positions vectors of the given points P and Q be $\vec{a}$ and $\vec{b}$ respectively.*

<u>STEP II</u> *Obtain the equation of the plane perpendicular to the required plane. Let its equation be $\vec{r} \cdot \vec{n_1} = d$.*

<u>STEP III</u> *Let $\vec{n}$ be the normal vector to the required plane. Then, $\vec{n}$ is perpendicular to both $\vec{n_1}$ and $\vec{PQ}$. So, compute $\vec{n} = \vec{n_1} \times \vec{PQ}$.*

<u>STEP IV</u> *Write the equation of the required plane as $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ or, $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$.*

EXAMPLE 6 *Find the equation of the plane through the points (2, 1, – 1) and (– 1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$.*

SOLUTION The equation of any plane through (2, 1, – 1) is

$$a(x-2)+b(y-1)+c(z+1) = 0 \qquad \text{...(i)}$$

If it passes through (–1, 3, 4), then

$$a(-1-2)+b(3-1)+c(4+1) = 0$$

$$\Rightarrow \quad -3a+2b+5c = 0 \qquad \text{...(ii)}$$

If plane (i) is perpendicular to the plane $x - 2y + 4z = 10$, then

$$a-2b+4c = 0 \qquad \text{...(iii)}$$

Solving (ii) and (iii) by the method of cross-multiplication, we obtain

$$\frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-2}$$

$$\Rightarrow \quad \frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda \text{ (say)}$$

$$\Rightarrow \quad a = 18\lambda,\ b = 17\lambda \text{ and } c = 4\lambda.$$

Putting $a = 18\lambda$, $b = 17\lambda$ and $c = 4\lambda$ in (i), we obtain

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1) = 0$$

$$\Rightarrow \quad 18x+17y+4z = 49$$

This is the required equation of the plane.

ALITER The required plane passes through the points $P(2, 1, -1)$ and $Q(-1, 3, 4)$. Let $\vec{a}$ and $\vec{b}$ be the position vectors of points P and Q respectively. Then, $\vec{a} = 2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = -\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{PQ} = \vec{b}-\vec{a} = -3\hat{i}+2\hat{j}+5\hat{k}$

The required plane is perpendicular to the plane $x - 2y + 4z = 10$.. Let $\vec{n_1}$ be the normal vector to this plane. Then, $\vec{n_1} = \hat{i}-2\hat{j}+4\hat{k}$.

Let $\vec{n}$ be the normal vector to the desired plane. Then,

$$\vec{n} = \vec{n_1} \times \vec{PQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ -3 & 2 & 5 \end{vmatrix} = -18\hat{i}-17\hat{j}-4\hat{k}$$

The required plane passes through a point having position vector $\vec{a} = 2\hat{i}+\hat{j}-\hat{k}$ and is normal to the vector $\vec{n} = -18\hat{i}-17\hat{j}-4\hat{k}$. So, its vector equation is

$$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$$\Rightarrow \quad \vec{r}\cdot(-18\hat{i}-17\hat{j}-4\hat{k}) = (2\hat{i}+\hat{j}-\hat{k})\cdot(-18\hat{i}-17\hat{j}-4\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(-18\hat{i}-17\hat{j}-4\hat{k}) = -36-17+4$$

$$\Rightarrow \quad \vec{r}\cdot(18\hat{i}+17\hat{j}+4\hat{k}) = 49$$

The cartesian equation of the plane is $18x + 17y + 4z = 49$.

EXAMPLE 7 *Find the equation of the plane through the points (3, 4, 2) and (7, 0, 6) and is perpendicular to the plane $2x - 5y = 15$.*

SOLUTION The equation of a plane passing through (3, 4, 2) is

$$a(x-3)+b(y-4)+c(z-2) = 0 \qquad \text{...(i)}$$

This passes through (7, 0, 6).

$\therefore \quad a(7-3)+b(0-4)+c(6-2)=0$

$\Rightarrow \quad 4a-4b+4c=0 \Rightarrow a-b+c=0 \quad \text{...(ii)}$

The plane (i) is perpendicular to the plane $2x-5y+0z=15$.

$\therefore \quad 2a+(-5)b+(0)c=0 \quad \text{...(iii)}$

Solving (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{5}=\frac{b}{2}=\frac{c}{-3}=\lambda\text{ (say)} \Rightarrow a=5\lambda,\ b=2\lambda,\ c=-3\lambda$$

Substituting the values of a, b, c in (i), we get

$$5\lambda(x-3)+2\lambda(y-4)-3\lambda(z-2)=0 \Rightarrow 5x+2y-3z-17=0$$

This is the equation of the required plane.

EXERCISE 28.5

1. Find the angle between the planes :
   (i) $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ and $\vec{r}\cdot(-\hat{i}+\hat{j})=4$
   (ii) $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=6$ and $\vec{r}\cdot(3\hat{i}+6\hat{j}-2\hat{k})=9$
   (iii) $\vec{r}\cdot(2\hat{i}+3\hat{j}-6\hat{k})=5$ and $\vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})=9$
2. Find the angle between the planes:
   (i) $2x-y+z=4$ and $x+y+2z=3$
   (ii) $x+y-2z=3$ and $2x-2y+z=5$
   (iii) $x-y+z=5$ and $x+2y+z=9$
   (iv) $2x-3y+4z=1$ and $-x+y=4$
3. Show that the following planes are at right angles:
   (i) $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=5$ and $\vec{r}\cdot(-\hat{i}-\hat{j}+\hat{k})=3$
   (ii) $x-2y+4z=10$ and $18x+17y+4z=49$
4. Determine the value of λ for which the following planes are perpendicular to each other.
   (i) $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=7$ and $\vec{r}\cdot(\lambda\hat{i}+2\hat{j}-7\hat{k})=26$
   (ii) $2x-4y+3z=5$ and $x+2y+\lambda z=5$
   (iii) $3x-6y-2z=7$ and $2x+y-\lambda z=5$
5. Find the equation of a plane passing through the point $(-1,-1,2)$ and perpendicular to the planes $3x+2y-3z=1$ and $5x-4y+z=5$. [CBSE 2004, 2008]
6. Obtain the equation of the plane passing through the point $(1,-3,-2)$ and perpendicular to the planes $x+2y+2z=5$ and $3x+3y+2z=8$. [CBSE 2009]
7. Find the equation of the plane passing through the origin and perpendicular to each of the planes, $x+2y-z=1$ and $3x-4y+z=5$.
8. Find the equation of the plane passing through the points $(1,-1,2)$ and $(2,-2,2)$ and which is perpendicular to the plane $6x-2y+2z=9$. [CBSE 2005]
9. Find the equation of the plane passing through the points $(2, 2, 1)$ and $(9, 3, 6)$ and perpendicular to the plane $2x+6y+6z=1$.
10. Find the equation of the plane passing through the points whose coordinates are $(-1, 1, 1)$ and $(1,-1,1)$ and perpendicular to the plane $x+2y+2z=5$.

ANSWERS

1. (i) $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$ (ii) $\cos^{-1}\left(\frac{-4}{21}\right)$ (iii). $\cos^{-1}\left(\frac{-16}{21}\right)$

2. (i) $\frac{\pi}{3}$ (ii) $\cos^{-1}\left(\frac{-2}{3\sqrt{6}}\right)$ (iii) $\frac{\pi}{2}$ (iv) $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$

4. (i) 17 (ii) 2 (iii) 0 5. $5x+9y+11z-8=0$

6. $2x-4y+3z-8=0$ 7. $x+2y+5z=0$ 8. $x+y-2z+4=0$

9. $3x+4y-5z=9$ 10. $2x+2y-3z+3=0$

28.8 EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND PARALLEL TO TWO GIVEN VECTORS OR LINES

PARAMETRIC FORM

THEOREM 1 *The equation of the plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$, where λ and μ are scalars.*

PROOF Let O be the origin and π be a plane passing through a point A having position vector $\vec{a}$. Let the plane π be parallel to vectors $\vec{b}$ and $\vec{c}$ respectively. Through the point A draw two vectors $\vec{AB}$ and $\vec{AC}$ such that $\vec{AB}=\vec{b}$ and $\vec{AC}=\vec{c}$. Let P be an arbitrary point on the plane. Let the position vector of P be $\vec{r}$. Complete the parallelogram $ALPM$. Since $\vec{AL}$ and $\vec{AM}$ are parallel to $\vec{b}$ and $\vec{c}$ respectively. Therefore,

$$\vec{AL}=\lambda\vec{b} \text{ and } \vec{AM}=\mu\vec{c} \text{ for some scalars } \lambda \text{ and } \mu.$$

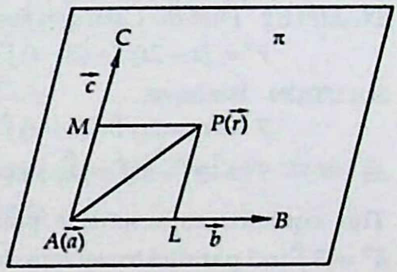

Fig. 28.5

In ΔAPL, we have

$$\vec{AP}=\vec{AL}+\vec{LP}$$

$$\Rightarrow \quad \vec{r}-\vec{a}=\lambda\vec{b}+\mu\vec{c}$$

$$\Rightarrow \quad \vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$$

Since $\vec{r}$ denotes the position vector of any point P on the plane. Therefore, the equation of the plane is $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$, where λ and μ are scalars.

NOTE *In the above equation λ and μ are variable scalars, because for different points on the plane the values of λ and μ are different. That is why, it is called the parametric form of the plane passing through a given point and parallel to given vectors.*

NON-PARAMETRIC FORM

THEOREM 2 *To show that the equation of the plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$* and *$\vec{c}$ is*

$$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0 \text{ or, } \vec{r}\cdot(\vec{b}\times\vec{c})=\vec{a}\cdot(\vec{b}\times\vec{c}) \text{ or, } [\vec{r}\,\vec{b}\,\vec{c}]=[\vec{a}\,\vec{b}\,\vec{c}].$$

PROOF From Fig. 28.5, the plane is parallel to vectors $\vec{AB}=\vec{b}$ and $\vec{AC}=\vec{c}$. Therefore, it is perpendicular to the vector $\vec{n}=\vec{AB}\times\vec{AC}=\vec{b}\times\vec{c}$.

Since the plane passes through a point A having position vector $\vec{a}$. Therefore, the equation of the plane in scalar product form is

$$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0 \qquad [\text{Using}: (\vec{r}-\vec{a})\cdot\vec{n}=0]$$

$$\Rightarrow \quad \vec{r}\cdot(\vec{b}\times\vec{c})-\vec{a}\cdot(\vec{b}\times\vec{c})=0$$

$\Rightarrow \quad \vec{r}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(\vec{b}\times\vec{c})$

$\Rightarrow \quad [\vec{r}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the vector equation of the following plane in scalar product form:*

$$\vec{r} = (\hat{i}-\hat{j}) + \lambda(\hat{i}+\hat{j}+\hat{k}) + \mu(\hat{i}-2\hat{j}+3\hat{k}).$$

SOLUTION We know that the equation $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$. Here,

$$\vec{a} = \hat{i}-\hat{j},\ \vec{b} = \hat{i}+\hat{j}+\hat{k} \text{ and } \vec{c} = \hat{i}-2\hat{j}+3\hat{k}.$$

The given plane is perpendicular to the vector

$$\vec{n} = \vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\hat{i}-2\hat{j}-3\hat{k}$$

So, the vector equation of the plane in scalar product form is

$$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$\Rightarrow \quad \vec{r}\cdot(5\hat{i}-2\hat{j}-3\hat{k}) = (\hat{i}-\hat{j})\cdot(5\hat{i}-2\hat{j}-3\hat{k})$

$\Rightarrow \quad \vec{r}\cdot(5\hat{i}-2\hat{j}-3\hat{k}) = 5+2+0$ or $\vec{r}\cdot(5\hat{i}-2\hat{j}-3\hat{k}) = 7.$

EXAMPLE 2 *Find the Cartesian form of the equation of the plane*

$$\vec{r} = (s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}.$$

SOLUTION We have,

$$\vec{r} = (s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}$$

$\Rightarrow \quad \vec{r} = 3\hat{j} + s(\hat{i}+2\hat{k}) + t(-2\hat{i}-\hat{j}+\hat{k})$

This equation represents a plane passing through a point having position vector $\vec{a} = 3\hat{j}$ and parallel to vectors $\vec{b} = \hat{i}+2\hat{k}$ and $\vec{c} = -2\hat{i}-\hat{j}+\hat{k}$.

So, it is perpendicular to

$$\vec{n} = \vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ -2 & -1 & 1 \end{vmatrix} = 2\hat{i}-5\hat{j}-\hat{k}$$

The equation of the plane in scalar product form is

$$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$\Rightarrow \quad \vec{r}\cdot(2\hat{i}-5\hat{j}-\hat{k}) = (3\hat{j})\cdot(2\hat{i}-5\hat{j}-\hat{k})$

$\Rightarrow \quad \vec{r}\cdot(2\hat{i}-5\hat{j}-\hat{k}) = -15$

The cartesian form of the equation of this plane is given by

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}-5\hat{j}-\hat{k}) = -15 \Rightarrow 2x-5y-z = -15$$

EXAMPLE 3 *Find the vector equation of the plane*

$$\vec{r} = (1+s-t)\hat{i} + (2-s)\hat{j} + (3-2s+2t)\hat{k}$$

in non-parametric form.

SOLUTION We have,

$$\vec{r} = (1+s-t)\hat{i} + (2-s)\hat{j} + (3-2s+2t)\hat{k}$$

$\Rightarrow \quad \vec{r} = (\hat{i}+2\hat{j}+3\hat{k}) + s(\hat{i}-\hat{j}-2\hat{k}) + t(-\hat{i}+2\hat{k})$

This is the vector equation of the plane passing through the point $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and parallel to vectors $\vec{b} = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{k}$. So, it is perpendicular to the vector

$$\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -2 \\ -1 & 0 & 2 \end{vmatrix} = -2\hat{i} + 0\hat{j} - \hat{k}$$

The vector equation of the plane in non-parametric form is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\Rightarrow \quad \vec{r} \cdot (-2\hat{i} + 0\hat{j} - \hat{k}) = (\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (-2\hat{i} + 0\hat{j} - \hat{k})$

$\Rightarrow \quad \vec{r} \cdot (-2\hat{i} + 0\hat{j} - \hat{k}) = -2 + 0 - 3$

$\Rightarrow \quad \vec{r} \cdot (2\hat{i} + 0\hat{j} + \hat{k}) = 5$

EXAMPLE 4 *Find the vector equation of the plane passing through the points* $\hat{i} + \hat{j} - 2\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$.

SOLUTION Let A, B, C be the points with position vectors $\hat{i} + \hat{j} - 2\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ respectively.

Then, $\quad \vec{AB} = \text{P.V. of } B - \text{P.V. of } A = (2\hat{i} - \hat{j} + \hat{k}) - (\hat{i} + \hat{j} - 2\hat{k}) = \hat{i} - 2\hat{j} + 3\hat{k}$

and, $\quad \vec{BC} = \text{P.V. of } C - \text{P.V. of } B = (\hat{i} + 2\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} + 3\hat{j} + 0\hat{k}$

A vector normal to the plane containing points A, B, and C is

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ -1 & 3 & 0 \end{vmatrix} = -9\hat{i} - 3\hat{j} + \hat{k}$$

The required plane passes through the point having position vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and is normal to the vector $-9\hat{i} - 3\hat{j} + \hat{k}$. So, its vector equation is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\Rightarrow \quad \vec{r} \cdot (-9\hat{i} - 3\hat{j} + \hat{k}) = (\hat{i} + \hat{j} - 2\hat{k}) \cdot (-9\hat{i} - 3\hat{j} + \hat{k})$

$\Rightarrow \quad \vec{r} \cdot (-9\hat{i} - 3\hat{j} + \hat{k}) = -9 - 3 - 2$

$\Rightarrow \quad \vec{r} \cdot (9\hat{i} + 3\hat{j} - \hat{k}) = 14.$

EXERCISE 28.6

1. Find the vector equation of the following planes in scalar product form $(\vec{r} \cdot \vec{n} = d)$:
   (i) $\vec{r} = (2\hat{i} - \hat{k}) + \lambda\hat{i} + \mu(\hat{i} - 2\hat{j} - \hat{k})$
   (ii) $\vec{r} = (1 + s - t)\hat{i} + (2 - s)\hat{j} + (3 - 2s + 2t)\hat{k}$
   (iii) $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k}) + \mu(-\hat{i} + \hat{j} - 2\hat{k})$
   (iv) $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(4\hat{i} - 2\hat{j} + 3\hat{k})$
2. Find the cartesian form of the equation of the following planes:
   (i) $\vec{r} = (\hat{i} - \hat{j}) + s(-\hat{i} + \hat{j} + 2\hat{k}) + t(\hat{i} + 2\hat{j} + \hat{k})$
   (ii) $\vec{r} = (1 + s + t)\hat{i} + (2 - s + t)\hat{j} + (3 - 2s + 2t)\hat{k}$
3. Find the vector equation of the following planes in non-parametric form:
   (i) $\vec{r} = (\lambda - 2\mu)\hat{i} + (3 - \mu)\hat{j} + (2\lambda + \mu)\hat{k}$
   (ii) $\vec{r} = (2\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k}) + \mu(5\hat{i} - 2\hat{j} + 7\hat{k})$

4. Find the vector equation of the plane passing through the points $3\hat{i}+4\hat{j}+2\hat{k}$, $2\hat{i}-2\hat{j}-\hat{k}$ and $7\hat{i}+6\hat{k}$.

ANSWERS

1. (i) $\vec{r}\cdot(\hat{j}-2\hat{k}) = 2$ (ii) $\vec{r}\cdot(2\hat{i}+\hat{k})=5$ (iii) $\vec{r}\cdot(-\hat{i}+\hat{j}+\hat{k})=0$
2. (i) $x-y+z=2$ (ii) $2y-z=1$
3. (i) $\vec{r}\cdot(2\hat{i}-5\hat{j}-\hat{k})+15=0$ (ii) $\vec{r}\cdot(5\hat{i}+2\hat{j}-3\hat{k})=17$
4. $\vec{r}\cdot(9\hat{i}+2\hat{j}-7\hat{k}) = 21$

EQUATION OF A PLANE PARALLEL TO A GIVEN PLANE

OR FORM Since parallel planes have the common normal, therefore, equation of a parallel to the plane $\vec{r}\cdot\vec{n}= d_1$ is $\vec{r}\cdot\vec{n} = d_2$, where d_2 is a constant determined given condition.

TRATION 1 *Find the equation of plane passing through the point $\hat{i}+\hat{j}+\hat{k}$ and parallel to lane $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=5$.*

LUTION The equation of a plane parallel to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=5$ is

$$\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k}) = d \quad \text{...(i)}$$

Since it passes through $\hat{i}+\hat{j}+\hat{k}$. Therefore,

$$(\hat{i}+\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+2\hat{k}) = d \Rightarrow 2-1+2 = d \Rightarrow d = 3.$$

Putting $d=3$ in (i), we obtain $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=3$.

This is the equation of the required plane.

ALITER The required planes passes through the point $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and is parallel to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=5$. So, it is normal to the vector $\vec{n}=2\hat{i}-\hat{j}+2\hat{k}$. Hence, its equation is

$$(\vec{r}-\vec{a})\cdot\vec{n} = 0$$

$$\Rightarrow \quad \vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$$\Rightarrow \quad \vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k}) = (\hat{i}+\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+2\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k}) = 2-1+2$$

$$\Rightarrow \quad \vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k}) = 3.$$

CARTESIAN FORM *Let $ax+by+cz+d=0$ be a plane. Then, direction ratios of its normal are proportional to a, b, c. Since parallel planes have common normal. Therefore, the direction ratios of the normal to the parallel plane are also proportional to a, b, c. Thus, the equation of a plane parallel to the plane $ax+by+cz+d=0$ is $ax+by+cz+k=0$, where k is an arbitrary constant and is determined by the given condition.*

ILLUSTRATION 2 *Find the equation of the plane through the point $(1, 4, -2)$ and parallel to the plane $-2x+y-3z=7$.*

SOLUTION Let the equation of a plane parallel to the plane $-2x+y-3z=7$ be

$$-2x+y-3z+k = 0 \quad \text{...(i)}$$

This passes through $(1, 4, -2)$. Therefore,

$$(-2)(1)+4-3(-2)+k = 0$$

$$\Rightarrow \quad -2+4+6+k = 0 \Rightarrow k = -8.$$

Putting $k = -8$ in (i), we obtain

$$-2x + y - 3z - 8 = 0 \Rightarrow 2x - y + 3z + 8 = 0$$

This is the equation of the required plane.

28.10 EQUATION OF A PLANE PASSING THROUGH THE INTERSECTION OF TWO PLANES

The intersection of two planes is always a straight line. For example, *xy*-plane and *xz*-plane intersect to form *x*-axis. The plane containing the line of intersection of two given planes is known as the plane passing through the intersection of two given planes. In the following discussion we will obtain the equation of family of planes passing through the intersection of two given planes.

VECTOR FORM

THEOREM 1 *The equation of a plane passing through the intersection of the planes* $\vec{r} \cdot \vec{n_1} = d_1$ *and* $\vec{r} \cdot \vec{n_2} = d_2$ *is given by*

$$(\vec{r} \cdot \vec{n_1} - d_1) + \lambda (\vec{r} \cdot \vec{n_2} - d_2) = 0$$

or, $\quad \vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$, *where* λ *is an arbitrary constant.*

PROOF The equation $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$ is of the form $\vec{r} \cdot \vec{n} = d$. So, it represents a plane.

In order to prove that it represents a plane, it is sufficient to show that every point on the line of intersection of the plane $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ lies on $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$.

Let $\vec{r_1}$ be the position vector of any point on the line of intersection of $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$. Then,

$$\vec{r_1} \cdot \vec{n_1} = d_1 \text{ and } \vec{r_1} \cdot \vec{n_2} = d_2$$

$$\Rightarrow \quad \vec{r_1} \cdot \vec{n_1} - d_1 = 0 \text{ and } \vec{r_1} \cdot \vec{n_2} - d_2 = 0$$

$$\Rightarrow \quad (\vec{r_1} \cdot \vec{n_1} - d_1) + \lambda (\vec{r_1} \cdot \vec{n_2} - d_2) = 0$$

$$\Rightarrow \quad \vec{r_1} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$$

$$\Rightarrow \quad \vec{r_1} \text{ lies on the plane } \vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2.$$

Thus, equation $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$ represents a plane passing through the intersection of $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$.

CARTESIAN FORM

THEOREM 2 *The equation of a plane passing through the intersection of*

$a_1 x + b_1 y + c_1 z + d_1 = 0$ *and* $a_2 x + b_2 y + c_2 z + d_2 = 0$ is

$(a_1 x + b_1 y + c_1 z + d_1) + \lambda (a_2 x + b_2 y + c_2 z + d_2) = 0$, *where* λ *is a constant.*

PROOF Consider the equation

$$(a_1 x + b_1 y + c_1 z + d_1) + \lambda (a_2 x + b_2 y + c_2 z + d_2) = 0 \qquad \text{...(i)}$$

or, $\quad x (a_1 + \lambda a_2) + y (b_1 + \lambda b_2) + z (c_1 + \lambda c_2) + d_1 + \lambda d_2 = 0$

This is a first degree equation in x, y, z. So, it represents a plane.

In order to prove that equation (i) represents a plane passing through the intersection of planes

$$a_1x + b_1y + c_1z + d_2 = 0 \qquad \text{...(ii)}$$

and, $$a_2x + b_2y + c_2z + d_2 = 0 \qquad \text{...(iii)}$$

It is sufficient to show that every point on the line of intersection of (ii) and (iii) is a point on plane (i).

Let (α, β, γ) be a point on the line of intersection of (ii) and (iii). Then,

$$a_1 \alpha + b_1 \beta + c_1 \gamma + d_1 = 0 \text{ and } a_2 \alpha + b_2 \beta + c_2 \gamma + d_2 = 0$$

$$\Rightarrow \quad (a_1 \alpha + b_1 \beta + c_1 \gamma + d_1) + \lambda (a_2 \alpha + b_2 \beta + c_2 \gamma + d_2) = 0 + \lambda 0 = 0$$

$\Rightarrow$ (α, β, γ) lies on plane (i).

Thus, $(a_1x + b_1y + c_1z + d_1) + \lambda (a_2x + b_2y + c_2z + d_2) = 0$ represents a plane passing through the intersection of the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

ILLUSTRATIVE EXAMPLES

Type 1 EQUATION OF A PLANE PASSING THROUGH THE INTERSECTION OF TWO GIVEN PLANES AND A GIVEN POINT

EXAMPLE 1 *Find the equation of a plane through the intersection of the planes* $\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 5$ and $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$ *and passing through the point* $(2, 1, -2)$.

SOLUTION The equation of a plane through the intersection of $\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 5$ and $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$ is

$$[\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) - 5] + \lambda [\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) - 3] = 0$$

$$\Rightarrow \quad \vec{r} \cdot [(1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (-1 + \lambda)\hat{k}] - 5 - 3\lambda = 0 \qquad \text{...(i)}$$

If plane in (i) passes through $(2, 1, -2)$, then the vector $2\hat{i} + \hat{j} - 2\hat{k}$ should satisfy it.

$$\therefore \quad (2\hat{i} + \hat{j} - 2\hat{k}) \cdot [(1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (-1 + \lambda)\hat{k}] - (5 + 3\lambda) = 0$$

$$\Rightarrow \quad 2(1 + 2\lambda) + 1(3 - \lambda) - 2(-1 + \lambda) - (5 + 3\lambda) = 0$$

$$\Rightarrow \quad -2\lambda + 2 = 0 \Rightarrow \lambda = 1$$

Putting $\lambda = 1$ in (i), we get the required equation of the plane as

$$\vec{r} \cdot (3\hat{i} + 2\hat{j} + 0\hat{k}) = 8.$$

EXAMPLE 2 *Find the equation of the plane containing the line of intersection of the plane* $x + y + z - 6 = 0$ and $2x + 3y + 4z + 5 = 0$ *and passing through the point* $(1, 1, 1)$.

SOLUTION The equation of the plane through the line of intersection of the given planes is

$$(x + y + z - 6) + \lambda (2x + 3y + 4z + 5) = 0 \qquad \text{...(i)}$$

If (i) passes through $(1, 1, 1)$, we have

$$-3 + 14\lambda = 0 \Rightarrow \lambda = 3/14$$

Putting $\lambda = \frac{3}{14}$ in (i), we obtain the equation of the required plane as

$$(x + y + z - 6) + \frac{3}{14}(2x + 3y + 4z + 5) = 0 \Rightarrow 20x + 23y + 26z - 69 = 0.$$

EXAMPLE 3 *Find the direction ratios of the normal to the plane passing through the point* $(2, 1, 3)$ *and the line of intersection of the planes* $x + 2y + z = 3$ *and* $2x - y - z = 5$.

SOLUTION The equation of the plane passing through the intersection of the planes $x+2y+z=3$ and $2x-y-z=5$ is given by

$$(x+2y+z-3)+\lambda(2x-y-z-5) = 0$$

$$\Rightarrow \quad x(2\lambda+1)+y(2-\lambda)+z(1-\lambda)-3-5\lambda = 0 \quad ...(i)$$

It passes through (2, 1, 3). Therefore,

$$2(2\lambda+1)+(2-\lambda)+3(1-\lambda)-3-5\lambda = 0$$

$$\Rightarrow \quad 4\lambda+2+2-\lambda+3-3\lambda-3-5\lambda = 0$$

$$\Rightarrow \quad 4-5\lambda = 0 \Rightarrow \lambda = \frac{4}{5}$$

Substituting $\lambda = \frac{4}{5}$ in (i), we get

$13x+6y+z-35 = 0$ as the equation of the required plane.

Clearly, direction ratios of normal to this plane are proportional to 13, 6, 1.

Type II **EQUATION OF A PLANE PASSING THROUGH THE INTERSECTION OF TWO PLANES AND PERPENDICULAR TO A GIVEN PLANE.**

EXAMPLE 4 *Find the equation of the plane which is perpendicular to the plane* $5x+3y+6z+8=0$ *and which contains the line of intersection of the planes* $x+2y+3z-4=0$ *and* $2x+y-z+5=0$. **[CBSE 2007]**

SOLUTION The equation of a plane through the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ is

$$(x+2y+3z-4)+\lambda(2x+y-z+5) = 0$$

$$\Rightarrow \quad x(1+2\lambda)+y(2+\lambda)+z(3-\lambda)-4+5\lambda = 0 \quad ...(i)$$

This is perpendicular to the plane $5x+3y+6z+8=0$, therefore,

$$5(1+2\lambda)+3(2+\lambda)+6(3-\lambda) = 0 \quad [\text{Using}: a_1a_2+b_1b_2+c_1c_2=0]$$

$$\Rightarrow \quad 7\lambda+29 = 0 \Rightarrow \lambda = -29/7$$

Putting $\lambda = -29/7$ in (i), we obtain the equation of the required plane as

$$-51x-15y+50z-173 = 0 \Rightarrow 51x+15y-50z+173 = 0.$$

EXAMPLE 5 *Find the equation of the plane through the line of intersection of* $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ *and* $\vec{r}\cdot(\hat{i}-\hat{j})+4=0$ *and perpendicular to* $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0$.

SOLUTION The equation of any plane through the line of intersection of the given planes is

$$[\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})-1]+\lambda[\vec{r}\cdot(\hat{i}-\hat{j})+4] = 0$$

$$\Rightarrow \quad \vec{r}\cdot[(2+\lambda)\hat{i}-(3+\lambda)\hat{j}+4\hat{k}] = 1-4\lambda \quad ...(i)$$

If plane (i) is perpendicular to $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0$, then

$$[(2+\lambda)\hat{i}-(3+\lambda)\hat{j}+4\hat{k}]\cdot(2\hat{i}-\hat{j}+\hat{k}) = 0 \quad [\text{Using } \vec{n_1}\cdot\vec{n_2} = 0]$$

$$\Rightarrow \quad 2(2+\lambda)+(3+\lambda)+4 = 0$$

$$\Rightarrow \quad 3\lambda+11 = 0 \Rightarrow \lambda = -\frac{11}{3}$$

Putting $\lambda = -\frac{11}{3}$ in (i), we obtain the equation of the required plane as

$$\vec{r}\cdot\left\{\left(2-\frac{11}{3}\right)\hat{i}-\left(3-\frac{11}{3}\right)\hat{j}+4\hat{k}\right\} = 1+\frac{44}{3}$$

$$\Rightarrow \quad \vec{r}\cdot(-5\hat{i}+2\hat{j}+12\hat{k}) = 47.$$

EXAMPLE 6 *Find the equation of the plane passing through the intersection of the planes $2x-3y+z-4=0$ and $x-y+z+1=0$ and perpendicular to the plane $x+2y-3z+6=0$.*

SOLUTION The equation of a plane the intersection of the planes $x-y+z+1=0$ and $2x-3y+z-4=0$ is

$$(2x-3y+z-4)+\lambda(x-y+z+1) = 0$$

$$\Rightarrow \quad x(2+\lambda)-y(3+\lambda)+z(1+\lambda)-4+\lambda = 0 \qquad \text{...(i)}$$

This is perpendicular to the plane $x+2y-3z+6=0$

$$\therefore \quad 1(2+\lambda)-2(3+\lambda)-3(1+\lambda) = 0$$

$$\Rightarrow \quad 2+\lambda-6-2\lambda-3-3\lambda = 0 \Rightarrow -4\lambda-7 = 0 \Rightarrow \lambda = -\frac{7}{4}$$

Putting $\lambda=-\frac{7}{4}$ in (i), we obtain

$$x\left(2-\frac{7}{4}\right)-y\left(3-\frac{7}{4}\right)+z\left(1-\frac{7}{4}\right)-4-\frac{7}{4} = 0 \Rightarrow x-5y-3z-23 = 0,$$

which is the equation of the required plane.

EXAMPLE 7 *Find the cartesian as well as vector equations of the planes through the intersection of the planes $\vec{r}\cdot(2\hat{i}+6\hat{j})+12=0$ and $\vec{r}\cdot(3\hat{i}-\hat{j}+4\hat{k})=0$ which are at a unit distance from the origin.* **[CBSE 2005]**

SOLUTION The equation of the planes through the intersection of the planes $\vec{r}\cdot(2\hat{i}+6\hat{j})+12=0$ and $\vec{r}\cdot(3\hat{i}-\hat{j}+4\hat{k})=0$ is

$$[\vec{r}\cdot(2\hat{i}+6\hat{j})+12]+\lambda[\vec{r}\cdot(3\hat{i}-\hat{j}+4\hat{k})] = 0$$

$$\Rightarrow \quad \vec{r}\cdot\{(2+3\lambda)\hat{i}+(6-\lambda)\hat{j}+4\lambda\hat{k}\}+12 = 0 \qquad \text{...(i)}$$

This planes is at a unit distance from the origin. Therefore,

Length of the perpendicular from the origin on (i) $=1$ unit.

$$\Rightarrow \quad \frac{12}{\sqrt{(2+3\lambda)^2+(6-\lambda)^2+16\lambda^2}} = 1$$

$$\Rightarrow \quad 144 = (2+3\lambda)^2+(6-\lambda)^2+16\lambda^2$$

$$\Rightarrow \quad 144 = 40+26\lambda^2 \Rightarrow 26\lambda^2=104 \Rightarrow \lambda^2=4 \Rightarrow \lambda=\pm 2$$

Putting the values of λ in (i), we obtain

$$\vec{r}\cdot(8\hat{i}+4\hat{j}+8\hat{k}) = 12 = 0 \text{ and } \vec{r}\cdot(-4\hat{i}+8\hat{j}-8\hat{k})+12 = 0$$

as the equations of the required planes. These equations can also be written as

$$\vec{r}\cdot(2\hat{i}+\hat{j}+2\hat{i})+3=0 \text{ and } \vec{r}\cdot(-\hat{i}+2\hat{j}-2\hat{k})+3 = 0$$

EXERCISE 28.7

1. Find the equation of the plane which is parallel to $2x-3y+z=0$ and which passes through $(1,-1,2)$.
2. Find the equation of the plane through $(3, 4, -1)$ which is parallel to the plane $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+2=0$.

3. Find the equation of the plane passing through the line of intersection of the planes $2x-7y+4z-3=0, 3x-5y+4z+11=0$ and the point $(-2,1,3)$.
4. Find the equation of the plane through the point $2\hat{i}+\hat{j}-\hat{k}$ and passing through the line of intersection of the planes $\vec{r}\cdot(\hat{i}+3\hat{j}-\hat{k})=0$ and $\vec{r}\cdot(\hat{j}+2\hat{k})=0$.
5. Find the equation of the plane passing through the line of intersection of the planes $2x-y=0$ and $3z-y=0$ and perpendicular to the plane $4x+5y-3z=8$.
6. Find the equation of the plane which contains the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ and which is perpendicular to the plane $5x+3y-6z+8=0$.
7. Find the equation of the plane through the line of intersection of the planes $x+2y+3z+4=0$ and $x-y+z+3=0$ and passing through the origin.
8. Find the vector equation (in scalar product form) of the plane containing the line of intersection of the planes $x-3y+2z-5=0$ and $2x-y+3z-1=0$ and passing through $(1,-2,3)$.
9. Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8=0$ and which contains the line of intersection of the planes $x+2y+3z-4=0, 2x+y-z+5=0$.
10. Find the equation of the plane through the line of intersection of the planes $\vec{r}\cdot(\hat{i}+3\hat{j})+6=0$ and $\vec{r}\cdot(3\hat{i}-\hat{j}-4\hat{k})=0$, which is at a unit distance from the origin. **[CBSE 2010]**
11. Find the equation of the plane passing through the intersection of the planes $2x+3y-z+1=0$ and $x+y-2z+3=0$ and perpendicular to the plane $3x-y-2z-4=0$
12. Find the equation of the plane that contains the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0$ and $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$.
13. Find the equation of the plane passing through (a,b,c) and parallel to the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$.
14. Find the equation of the plane passing through the intersection of the planes $\vec{r}\cdot(2\hat{i}+\hat{j}+3\hat{k})=7, \vec{r}\cdot(2\hat{i}+5\hat{j}+3\hat{k})=9$ and the point $(2,1,3)$. **[CBSE 2007]**

ANSWERS

1. $2x-3y+z=7$
2. $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0$
3. $15x-47y+28z=7$
4. $\vec{r}\cdot(\hat{i}+9\hat{j}+11\hat{k})=0$
5. $28x-17y+9z=0$
6. $33x+45y+50z-41=0$
7. $x-10y-5z=0$
8. $\vec{r}\cdot(\hat{i}+7\hat{j})+13=0$
9. $51x+15y-50z+173=0$
10. $\vec{r}\cdot(-2\hat{i}+4\hat{j}+4\hat{k})+6=0$ or, $\vec{r}\cdot(4\hat{i}+2\hat{j}-4\hat{k})+6=0$
11. $7x+13y+4z-9=0$
12. $33x+45y-50z-41=0$
13. $x+y+z=a+b+c$
14. $\vec{r}\cdot(2\hat{i}-13\hat{j}+3\hat{k})=0$

28.11 DISTANCE OF A POINT FROM A PLANE

In this section, we shall find the perpendicular distance of a point from a plane in both Cartesian and vector forms.

VECTOR FORM

THEOREM 1 *The length of the perpendicular from a point having position vector $\vec{a}$ to the plane $\vec{r}\cdot\vec{n}=d$ is given by*

$$p = \frac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}.$$

PROOF Let π be the given plane and $P(\vec{a})$ be the given point. Let PM be the length of the perpendicular from P on the plane π. Since line PM passes through $P(\vec{a})$ and is parallel to the vector $\vec{n}$ which is normal to the plane π. So, vector equation of line PM is

$$\vec{r} = \vec{a} + \lambda\vec{n}, \quad \text{...(i)}$$

where λ is a scalar.

Point M is the intersection of line (i) and the given plane. Therefore, for point M, we have

$$(\vec{a} + \lambda\vec{n})\cdot\vec{n} = d \quad \left[\begin{array}{l}\text{On substituting } \vec{r} = \vec{a} + \lambda\vec{n} \\ \text{in the equation of the plane}\end{array}\right]$$

$$\Rightarrow \quad \vec{a}\cdot\vec{n} + \lambda\,\vec{n}\cdot\vec{n} = d$$

$$\Rightarrow \quad \lambda = \frac{d - (\vec{a}\cdot\vec{n})}{\vec{n}\cdot\vec{n}}$$

$$\Rightarrow \quad \lambda = \frac{d - (\vec{a}\cdot\vec{n})}{|\vec{n}|^2}$$

Putting this value of λ in (i), we obtain the position vector of M given by

$$\vec{r} = \vec{a} + \left(\frac{d - (\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right)\vec{n}$$

$$\Rightarrow \quad \vec{PM} = \text{P.V. of } M - \text{P.V. of } P$$

$$\Rightarrow \quad \vec{PM} = \vec{a} + \left\{\frac{d - (\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right\}\vec{n} - \vec{a}$$

$$\Rightarrow \quad \vec{PM} = \frac{(d - (\vec{a}\cdot\vec{n}))\,\vec{n}}{|\vec{n}|^2}$$

$$\Rightarrow \quad PM = |\vec{PM}| = \left|\frac{[d - (\vec{a}\cdot\vec{n})]\,\vec{n}}{|\vec{n}|^2}\right|$$

$$\Rightarrow \quad PM = \frac{|\vec{d} - (\vec{a}\cdot\vec{n})|\;|\vec{n}|}{|\vec{n}|^2}$$

$$\Rightarrow \quad PM = \frac{|(\vec{a}\cdot\vec{n}) - d|}{|\vec{n}|}$$

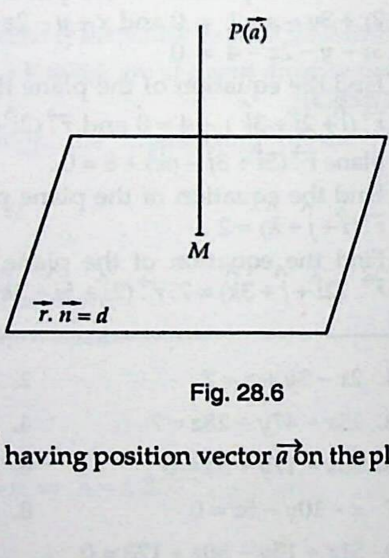

Fig. 28.6

Thus, the length of the perpendicular from a point having position vector $\vec{a}$ on the plane $\vec{r}\cdot\vec{n} = d$ is given by

$$\frac{|(\vec{a}\cdot\vec{n}) - d|}{|\vec{n}|}$$

CARTESIAN FORM

THEOREM 2 *The length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by*

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

PROOF Let M be the foot of the perpendicular from $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$. Then, the equation of PM is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \quad \text{...(i)}$$

The coordinates of any point on this line are (x_1+ar, y_1+br, z_1+cr), where r is a real number. This point coincides with M iff it lies on the plane i.e., iff

$$a(x_1+ar)+b(y_1+br)+c(z_1+cr)+d=0$$

$$\Rightarrow \quad r=-\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2} \quad \text{...(ii)}$$

Now,

$$PM=\sqrt{(x_1+ar-x_1)^2+(y_1+br-y_1)^2+(z_1+cr-z_1)^2}$$

$$\Rightarrow PM=\sqrt{a^2+b^2+c^2}\,|r|$$

$$\Rightarrow PM=\sqrt{a^2+b^2+c^2}\left|\frac{-(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}\right| \quad \text{[From (ii)]}$$

$$\Rightarrow PM=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}.$$

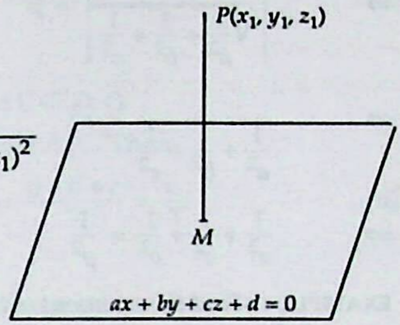

Fig. 28.7

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the distance of the point* $2\hat{i}+\hat{j}-\hat{k}$ *from the plane* $\vec{r}\cdot(\hat{i}-2\hat{j}+4\hat{k})=9$.

SOLUTION We know that the perpendicular distance of a point P with position vector $\vec{a}$ from the plane $\vec{r}\cdot\vec{n}=d$ is given by

$$\frac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}.$$

Here, $\quad \vec{a}=2\hat{i}+\hat{j}-\hat{k}$, $\vec{n}=\hat{i}-2\hat{j}+4\hat{k}$ and $d=9$.

So, required distance $=\dfrac{|(2\hat{i}+\hat{j}-\hat{k})\cdot(\hat{i}-2\hat{j}+4\hat{k})-9|}{\sqrt{1+4+16}}=\dfrac{|2-2-4-9|}{\sqrt{21}}=\dfrac{13}{\sqrt{21}}$

EXAMPLE 2 *Find the distance of the point (2, 1, 0) from the plane* $2x+y+2z+5=0$.

SOLUTION We know that the distance of the point (x_1, y_1, z_1) from the plane $ax+by+cz+d=0$ is given by

$$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

So, required distance $=\dfrac{|2\times2+1+2\times0+5|}{\sqrt{2^2+1^2+2^2}}=\dfrac{10}{3}.$

EXAMPLE 3 *Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then* $\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}=\dfrac{1}{p^2}$.

SOLUTION The equation of the plane having intercepts a, b and c on the coordinate axes is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

It is given that this plane is at a distance of p units from the origin.

$$\therefore \quad \left| \frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \right| = p$$

$$\Rightarrow \quad \left| \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \right| = p$$

$$\Rightarrow \quad \frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} = p^2$$

$$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{1}{p^2}$$

EXAMPLE 4 *Find the equations of the planes parallel to the plane $x - 2y + 2z - 3 = 0$ which is at a unit distance from the point (1, 2, 3).*

SOLUTION The equation of a plane parallel to the plane $x - 2y + 2z - 3 = 0$ is

$$x - 2y + 2z + \lambda = 0 \qquad \text{...(i)}$$

Distance of plane (i) from point (1, 2, 3) is given by $\left| \frac{1 - 2 \times 2 + 2 \times 3 + \lambda}{\sqrt{1^2 + (-2)^2 + 2^2}} \right|$

But, this distance is also given equal to 1.

$$\therefore \quad \left| \frac{1 - 4 + 6 + \lambda}{3} \right| = 1$$

$$\Rightarrow \quad |\lambda + 3| = 3$$

$$\Rightarrow \quad \lambda + 3 = \pm 3 \Rightarrow \lambda = 0 \text{ or, } \lambda = -6$$

Putting the values of λ in (i), we obtain

$$x - 2y + 2z = 0 \text{ and } x - 2y + 2z - 6 = 0$$

as the equations of the required planes.

EXAMPLE 5 *If the points $(1, 1, \lambda)$ and $(-3, 0, 1)$ be equidistant from the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$, find the value of λ.*

SOLUTION It is given that the points $(1, 1, \lambda)$ and $(-3, 0, 1)$ are equidistant from the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$

$$\therefore \quad \left| \frac{(\hat{i} + \hat{j} + \lambda \hat{k}) \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13}{\sqrt{9 + 16 + 144}} \right| = \left| \frac{(-3\hat{i} + 0\hat{j} + \hat{k}) \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13}{\sqrt{9 + 16 + 144}} \right|$$

$$\Rightarrow \quad \left| \frac{3 + 4 - 12\lambda + 13}{13} \right| = \left| \frac{-9 + 0 - 12 + 13}{13} \right|$$

$$\Rightarrow \quad |20 - 12\lambda| = 8$$

$$\Rightarrow \quad 20 - 12\lambda = \pm 8$$

$$\Rightarrow \quad 20 - 12\lambda = 8 \text{ or, } 20 - 12\lambda = -8$$

$$\Rightarrow \quad 12\lambda = 12 \text{ or, } 12\lambda = 28 \quad \Rightarrow \lambda = 1 \text{ or, } \lambda = \frac{7}{3}.$$

EXAMPLE 6 *A variable plane which remains at a constant distance 3p from the origin cut the coordinate axes at A, B, C. Show that the locus of the centroid of triangle ABC is* $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

SOLUTION Let the equation of the plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{...(i)}$$

where a, b, c are variables.

This meets X, Y and Z axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Let (α, β, γ) be the coordinates of the centroid of triangle ABC. Then,

$$\alpha=\frac{a+0+0}{3}=\frac{a}{3},\ \beta=\frac{0+b+0}{3}=\frac{b}{3},\ \gamma=\frac{0+0+c}{3}=\frac{c}{3}. \quad \text{...(ii)}$$

The plane (i) is at a distance $3p$ from the origin.

$\therefore$ $3p$ = Length of perpendicular from (0, 0, 0) to the plane (i)

$$\Rightarrow \quad 3p=\frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$

$$\Rightarrow \quad 3p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \Rightarrow \frac{1}{9p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \quad \text{...(iii)}$$

From (ii), we have

$$a=3\alpha, b=3\beta \text{ and } c=3\gamma.$$

Substituting the values of a, b, c in (iii), we obtain

$$\frac{1}{9p^2}=\frac{1}{9\alpha^2}+\frac{1}{9\beta^2}+\frac{1}{9\gamma^2}$$

$$\Rightarrow \quad \frac{1}{p^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$$

So, the locus of (α, β, γ) is

$$\frac{1}{p^2}=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} \Rightarrow x^{-2}+y^{-2}+z^{-2}=p^{-2}.$$

EXAMPLE 7 *A variable plane is at a constant distance p from the origin and meets the coordinate axes in A, B, C. Show that the locus of the centroid of the tetrahedron OABC is* $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$.

SOLUTION Let the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{...(i)}$$

This meets the coordinate axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ respectively.

Let (α, β, γ) be the coordinates of the centroid of tetrahedron $OABC$. Then,

$$\alpha=\frac{0+a+0+0}{4}=\frac{a}{4},\ \beta=\frac{0+0+b+0}{4}=\frac{b}{4},\ \gamma=\frac{0+0+0+c}{4}=\frac{c}{4} \quad \text{...(ii)}$$

The plane in (i) is at a constant distance p from the origin. Therefore,

p = Length perpendicular from (0, 0, 0) to plane (i)

$$\Rightarrow \quad p = \frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$

$$\Rightarrow \quad p = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \qquad \text{...(iii)}$$

Eliminating variables a, b, c from (ii) and (iii), we obtain

$$\frac{1}{p^2} = \frac{1}{16\,\alpha^2}+\frac{1}{16\,\beta^2}+\frac{1}{16\,\gamma^2} \Rightarrow 16\,p^{-2} = \alpha^{-2}+\beta^{-2}+\gamma^{-2}$$

Hence, the locus of (α, β, γ) is $16p^{-2} = x^{-2}+y^{-2}+z^{-2}$.

EXAMPLE 8 *If a variable plane at a constant distance p from the origin meets the coordinate axes in points A, B and C respectively. Through these points, planes are drawn parallel to the coordinate planes. Show that the locus of the point of intersection is*

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = \frac{1}{p^2}$$

SOLUTION Let the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$$

It cuts the coordinate axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ and it is at a distance p from the origin.

$$\therefore \quad \left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| = p$$

$$\Rightarrow \quad \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = p$$

$$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{1}{p^2} \qquad \text{...(i)}$$

Let $P(\alpha, \beta, \gamma)$ be the point of intersection of planes through $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ and parallel to yz, zx and xy-planes respectively.
The equations of planes passing through $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ and parallel to respectively yz, zx and xy-planes are

$x = a$, $y = b$ and $z = c$ respectively.

These three planes intersect at (a, b, c).

$$\therefore \quad \alpha = a, \beta = b \text{ and } \gamma = c \qquad \text{...(ii)}$$

Eliminating a, b, c from (i) and (ii), we get

$$\alpha^{-2}+\beta^{-2}+\gamma^{-2} = p^{-2}$$

Hence, the locus of $P(\alpha, \beta, \gamma)$ is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

EXAMPLE 9 *Find the distance between the point* $P(6, 5, 9)$ *and the plane determined by the points* $A(3, -1, 2)$, $B(5, 2, 4)$ *and* $C(-1, -1, 6)$. **[NCERT, CBSE 2010]**

SOLUTION The equation of a plane passing through $A(3, -1, 2)$ is

$$a(x-3)+b(y+1)+c(z-2) = 0 \quad \text{...(i)}$$

If this plane passes through $B(5, 2, 4)$ and $C(-1, -1, 6)$. Then,

$$2a+3b+2c = 0$$

$$-4a+0b+4c = 0$$

$$\therefore \quad \frac{a}{12} = \frac{b}{-16} = \frac{c}{12}$$

or, $$\frac{a}{3} = \frac{b}{-4} = \frac{c}{3}$$

Substituting the values of a, b, c in (i), we obtain

$$3(x-3)-4(y+1)+3(z-2) = 0$$

or, $3x-4y+3z = 19$ as the equation of the plane passing through A, B and C.

The distance of $P(6, 5, 9)$ from this plane is given by

$$\left|\frac{18-20+27-19}{\sqrt{9+16+9}}\right| = \frac{6}{\sqrt{34}}$$

ALITER Let D be the foot of the perpendicular drawn from P to the plane passing through the points A, B and C. Then, PD is the required distance.

Also, PD = Projection of $\vec{AP}$ on $\vec{AB}\times\vec{AC} = \left|\dfrac{\vec{AP}\cdot(\vec{AB}\times\vec{AC})}{|\vec{AB}\times\vec{AC}|}\right|$

Now, $$\vec{AB}\times\vec{AC} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 2\\ -4 & 0 & 4\end{vmatrix} = 12\hat{i}-16\hat{j}+12\hat{k} \text{ and, } \vec{AP} = 3\hat{i}+6\hat{j}+7\hat{k}$$

$$\therefore \quad \vec{AP}\cdot(\vec{AB}\times\vec{AC}) = (3\hat{i}+6\hat{j}+7\hat{k})\cdot(12\hat{i}-16\hat{j}+12\hat{k}) = 36-96+84 = 24$$

and, $$\vec{AB}\times\vec{AC} = \sqrt{144+256+144} = \sqrt{544} = 4\sqrt{34}$$

$$\therefore \quad PD = \frac{24}{4\sqrt{34}} = \frac{6}{\sqrt{34}}$$

28.11.1 DISTANCE BETWEEN THE PARALLEL PLANES

Let $a_1x+b_1y+c_1z+d_1=0$ and $a_1x+b_1y+c_1z+d_2=0$ be two parallel planes. Then, the distance between them can be obtained by taking any point $P(x_1, y_1, z_1)$ on any one of the given planes and finding the perpendicular distance from $P(x_1, y_1, z_1)$ on the other. The procedure is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the distance between the parallel planes* $x+y-z+4=0$ *and* $x+y-z+5=0$.

SOLUTION Let $P(x_1, y_1, z_1)$ be any point on $2x - y + 2z + 3 = 0$. Then,

$$2x_1 - y_1 + 2z_1 + 3 = 0 \qquad \text{...(i)}$$

The length of the perpendicular from $P(x_1, y_1, z_1)$ to $x + y - z + 5 = 0$ is

$$\left|\frac{x_1 + y_1 - z_1 + 5}{\sqrt{1^2 + 1^2 + (-1)^2}}\right| = \frac{|-4+5|}{\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \text{[Using (i)]}$$

Therefore, the distance between the two given parallel planes is $\frac{1}{\sqrt{3}}$.

ALITER Putting $x = 0, y = 0$ in $x + y - z + 4 = 0$, we get $z = 4$.

So, the coordinates of a point on the plane $x + y - z + 4 = 0$ are $(0, 0, 4)$.

$\therefore$ Required distance = Length of the perpendicular from $(0, 0, 4)$ on the plane $x + y - z + 5 = 0$

$\Rightarrow$ Required distance $= \left|\frac{0 + 0 - 4 + 5}{\sqrt{1^2 + 1^2 + (-1)^2}}\right| = \frac{1}{\sqrt{3}}$

EXAMPLE 2 *Find the distance between the parallel planes* $2x - y + 2z + 3 = 0$ *and* $4x - 2y + 4z + 5 = 0$.

SOLUTION Let $P(x_1, y_1, z_1)$ be any point on $2x - y + 2z + 3 = 0$. Then,

$$2x_1 - y_1 + 2z_1 + 3 = 0 \qquad \text{...(i)}$$

The length of the perpendicular from $P(x_1, y_1, z_1)$ to $4x - 2y + 4z + 5 = 0$ is

$$\left|\frac{4x_1 - 2y_1 + 4z_1 + 5}{\sqrt{4^2 + (-2)^2 + 4^2}}\right| = \left|\frac{2(2x_1 - y_1 + 2z_1) + 5}{\sqrt{36}}\right| = \frac{|2(-3) + 5|}{6} = \frac{1}{6} \text{[Using (i)]}$$

Therefore, the distance between the two given parallel planes is $\frac{1}{6}$.

ALITER Putting $x = 0, z = 0$ in $2x - y + 2z + 3 = 0$, we get $y = 3$.

So, the coordinates of a point on the plane $2x - y + 2z + 3 = 0$ are $(0, 3, 0)$.

$\therefore$ Required distance = Length of perpendicular from $(0, 3, 0)$ on the plane $4x - 2y + 4z + 5 = 0$

$\Rightarrow$ Required distance $= \left|\frac{4 \times 0 - 2 \times 3 + 4 \times 0 + 5}{\sqrt{4^2 + (-2)^2 + 4^2}}\right| = \frac{1}{6}$.

EXAMPLE 3 *Find the distance between the parallel planes,* $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$ *and* $\vec{r} \cdot (6\hat{i} - 9\hat{j} + 18\hat{k}) + 20 = 0$.

SOLUTION Let the position vector of any point P on $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$ be $\vec{a}$. Then,

$$\vec{a} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5 \qquad \text{...(i)}$$

$\therefore$ Required distance = Length of the perpendicular from $P(\vec{a})$ to

$$\vec{r} \cdot (6\hat{i} - 9\hat{j} + 18\hat{k}) + 20 = 0$$

$\Rightarrow$ Required distance $= \left|\frac{\vec{a} \cdot (6\hat{i} - 9\hat{j} + 18\hat{k}) + 20}{\sqrt{6^2 + (-9)^2 + (18)^2}}\right|$

$\Rightarrow$ Required distance $= \left|\frac{3[\vec{a} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k})] + 20}{\sqrt{36 + 81 + 324}}\right|$

Required distance $= \left|\frac{3(5) + 20}{\sqrt{441}}\right| = \frac{35}{21} = \frac{5}{3}$.

EXERCISE 28.8

1. Find the distance of the point $2\hat{i}-\hat{j}-4\hat{k}$ from the plane $\vec{r}\cdot(3\hat{i}-4\hat{j}+12\hat{k})-9=0$.
2. Show that the points $\hat{i}-\hat{j}+3\hat{k}$ and $3\hat{i}+3\hat{j}+3\hat{k}$ are equidistant from the plane $\vec{r}\cdot(5\hat{i}+2\hat{j}-7\hat{k})+9=0$.
3. Find the distance of the point (2, 3, –5) from the plane $x+2y-2z-9=0$.
4. Find the equations of the planes parallel to the plane $x+2y-2z+8=0$ which are at distance of 2 units from the point (2, 1, 1).
5. A plane is at a constant distance p from the origin and meets the coordinate axes in A, B, C. Show that the locus of the centroid of triangle ABC is
$$x^{-2}+y^{-2}+z^{-2}=9p^{-2}.$$
6. Show that the points (1, 1, 1) and (– 3, 0, 1) are equidistant from the plane $3x+4y-.12z+13=0$.
7. Find the distance between the parallel planes, $2x-y+3z-4=0$ and $6x-3y+9z+13=0$.
8. Find the equation of the plane which passes through the point (3, 4, –1) and is parallel to the plane $2x-3y+5z+7=0$. Also find the distance between the two planes.
9. Find the equations of the planes parallel to the plane $x-2y+2z-3=0$ and which are at a unit distance from the point (1, 1, 1).
10. Find the equation of the plane mid-parallel to the planes
$$2x-2y+z+3=0 \text{ and } 2x-2y+z+9=0.$$
11. Find the distance of the point (2, 3, 5) from the xy-plane.
12. Find the distance of the point (3, 3, 3) from the plane $\vec{r}\cdot(5i+2j-7k)+9=0$
13. Find the distance between the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})+7=0$ and $\vec{r}\cdot(2\hat{i}+4\hat{j}+6\hat{k})+7=0$.
14. If the product of distances of the point (1, 1, 1) from the origin and the plane $x-y+z+\lambda=0$ be 5, find the value of λ.

ANSWERS

1. $\frac{47}{13}$ 3. 3 4. $x+2y-2z+4=0$ or $x+2y-2z-8=0$ 7. $\frac{25}{3\sqrt{14}}$

8. $2x-3y+5z+11=0$ 9. $x-2y+2z+2=0$, $x-2y+2z-4=0$

10. $2x-2y+z+6=0$ 11. 5 12. $\frac{9}{\sqrt{78}}$ 13. $\frac{7}{\sqrt{56}}$ 14. 4 15. $\frac{6}{\sqrt{34}}$

HINTS TO SELECTED PROBLEM

10. Let the equation of the plane be $2x-2y+z+k=0$. This plane is equidistant from the given planes. So,
$$|-k+3|=|-k+9| \Rightarrow -k+3=-(-k+9) \Rightarrow 2k=12 \Rightarrow k=6.$$

28.12 LINE AND A PLANE

28.12.1 UNSYMMETRICAL FORM OF A LINE

From elementary solid geometry we know that two non-parallel planes intersect in a straight line. We have seen that every first degree equation in x, y, z represents a plane, therefore a line in space can be represented by two equations of first degree in x, y and

z. Thus, if $u = a_1x + b_1y + c_1z + d_1 = 0$ and $v = a_2x + b_2y + c_2z + d_2 = 0$ are equations of two non-parallel planes, then these two equations taken together represent a line because any point on the line will lie on the two planes and conversely any point which lies on two planes will also lie on the straight line which can be written as $u = 0 = v$. This form is called *unsymmetrical* form of a line.

For example, x-axis is the intersection of zx-plane i.e., $y = 0$ and xy-plane *i.e.*, $z = 0$. So, the equations of x-axis are $y = 0 = z$. Similarly, the equations of y-axis are $x = 0 = z$ and the equations of z-axis are $x = 0 = y$.

REDUCTION OF UNSYMMETRICAL FORM TO SYMMETRICAL FORM

Let the unsymmetrical form of a line be

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0 \end{aligned}\right\} \qquad \text{...(i)}$$

where $a_1 : b_1 : c_1 \neq a_2 : b_2 : c_2$.

To transform the equations to symmetrical form, we require

(i) direction ratios of the line, and

(ii) coordinates of a point on the line.

(i) Let a, b, c be direction ratios of the line (i). Then, as this line lies in both the planes, it must be perpendicular to normals to these planes. We, therefore, have

$$aa_1 + bb_1 + cc_1 = 0 \qquad \text{...(ii)}$$

and,

$$aa_2 + bb_2 + cc_2 = 0 \qquad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we obtain

$$\frac{a}{b_1c_2 - b_2c_1} = \frac{b}{c_1a_2 - c_2a_1} = \frac{c}{a_1b_2 - a_2b_1}$$

So, direction ratios of the line (i) are proportional to

$$b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1.$$

(ii) There are infinitely many points on the line (i) from which we have to choose a point. For this, we put $z = 0$ (we may put $y = 0$ or $x = 0$) in the equations of the planes to obtain:

$$a_1x + b_1y + d_1 = 0$$
$$a_2x + b_2y + d_2 = 0$$

Solving these two equations, we obtain

$$\frac{x}{b_1d_2 - b_2d_1} = \frac{y}{d_1a_2 - d_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

So, the coordinates of a point on line (i) are

$$\left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, 0\right)$$

Thus, the symmetrical form of the line (i) is given by

$$\frac{x - \left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}\right)}{b_1c_2 - b_2c_1} = \frac{y - \left(\dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}\right)}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Reduce in symmetrical form, the equations of the line* $x - y + 2z = 5$, $3x + y + z = 6$.

SOLUTION Let a, b, c be the direction ratios of the required line. Since the required line lies in both the given planes, we must have

$$a-b+2c=0 \text{ and } 3a+b+c=0$$

Solving these two equations by cross-multiplication, we get

$$\frac{a}{-1-2}=\frac{b}{6-1}=\frac{c}{1+3} \Rightarrow \frac{a}{-3}=\frac{b}{5}=\frac{c}{4}$$

In order to find a point on the required line, we put $z=0$ in the two given equations to obtain

$$x-y=5,\ 3x+y=6$$

Solving these two equations, we obtain $x=\frac{11}{4}$, $y=-\frac{9}{4}$.

Therefore, coordinates of a point on the required line are $\left(\frac{11}{4},-\frac{9}{4},0\right)$.

Hence, the equations of the required line are

$$\frac{x-\frac{11}{4}}{-3}=\frac{y-\left(-\frac{9}{4}\right)}{5}=\frac{z-0}{4}$$

$$\Rightarrow \quad \frac{4x-11}{-12}=\frac{4y+9}{20}=\frac{z-0}{4}$$

$$\Rightarrow \quad \frac{4x-11}{-3}=\frac{4y+9}{5}=\frac{z-0}{1}.$$

EXAMPLE 2 *Reduce in symmetrical form, the equations of the line $x=ay+b$, $z=cy+d$.*

SOLUTION Let l, m, n be the direction ratios of the required line. Since the required line lies in both the given planes, we must have

$$l+m(-a)+n\,0=0 \text{ and } l\cdot 0+m(-c)+n=0$$

Solving these two equations, we obtain

$$\frac{l}{-a}=\frac{m}{-1}=\frac{n}{-c} \text{ or } \frac{l}{a}=\frac{m}{1}=\frac{n}{c}.$$

To obtain the coordinates of a point on the required line, we put $y=0$ in the two given equations to obtain

$$x=b,\ z=d.$$

So, the coordinates of a point on the required lines are $(b, 0, d)$.

Therefore its equation are

$$\frac{x-b}{a}=\frac{y-0}{1}=\frac{z-d}{c}.$$

EXAMPLE 3 *Find the angle between the lines $x-2y+z=0=x+2y-2z$ and $x+2y+z=0=3x+9y+5z$.*

SOLUTION Let a_1, b_1, c_1 be the direction ratios of the line $x-2y+z=0$ and $x+2y-2z=0$. Since it lies in both the planes, therefore, it is $\perp$ to the normals to the two planes.

$$\therefore \quad a_1-2b_1+c_1=0$$

$$a_1+2b_1-2c_1=0$$

Solving these two equations by cross-multiplication, we have

$$\frac{a_1}{4-2}=\frac{b_1}{1+2}=\frac{c_1}{2+2} \Rightarrow \frac{a_1}{2}=\frac{b_1}{3}=\frac{c_1}{4}.$$

Let a_2, b_2, c_2 be the direction ratios of the line $x+2y+z=0=3x+9y+5z$. Then, as discussed above

$$a_2+2b_2+c_2=0 \quad 3a_2+9b_2+5c_2=0$$

$$\Rightarrow \quad \frac{a_2}{10-9} = \frac{b_2}{3-5} = \frac{c_2}{9-6} \Rightarrow \frac{a_2}{1} = \frac{b_2}{-2} = \frac{c_2}{3}$$

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}} = \frac{(1)(2)+(-2)(3)+(3)(4)}{\sqrt{2^2+3^2+4^2}\ \sqrt{1^2+(-2)^2+(3)^2}}$$

$$\Rightarrow \quad \cos\theta = \frac{2-6+12}{\sqrt{29}\ \sqrt{14}} = \frac{8}{\sqrt{406}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{8}{\sqrt{406}}\right).$$

28.12.2 ANGLE BETWEEN A LINE AND A PLANE

The angle between a line and a plane is the complement of the angle between the line and normal to the plane.

VECTOR FORM

THEOREM 1 *If θ is the angle between a line $\vec{r} = (a + \lambda\,\vec{b})$ and the plane $\vec{r}\cdot\vec{n} = d$, then*

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\ |\vec{n}|}$$

PROOF The line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the vector $\vec{b}$ and the plane is normal to the vector $\vec{n}$. Therefore, if θ is the angle between the given line and given plane, then the angle between $\vec{b}$ and $\vec{n}$ is $\left(\frac{\pi}{2} - \theta\right)$.

$$\therefore \quad \cos\left(\frac{\pi}{2} - \theta\right) = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\ |\vec{n}|}$$

$$\Rightarrow \quad \sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\ |\vec{n}|}$$

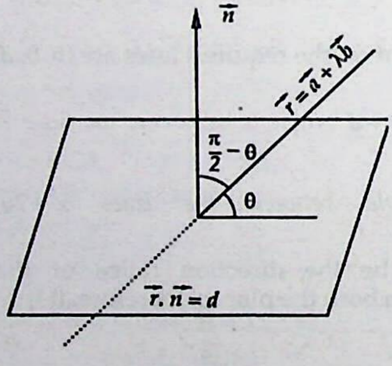

Fig. 28.8

Condition of perpendicularity: If the line is perpendicular to the plane, then it is parallel to the normal to the plane. Therefore, $\vec{b}$ and $\vec{n}$ are parallel.

So, $\vec{b} \times \vec{n} = 0$ or, $\vec{b} = \lambda\,\vec{n}$ for some scalar λ.

Condition of parallelism : If the line is parallel to the plane, then it is perpendicular to the normal to the plane. Therefore, $\vec{b}$ and $\vec{n}$ are perpendicular.

So, $\vec{b}\cdot\vec{n} = 0$.

CARTESIAN FORM

THEOREM 2 *If θ is the angle between the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$, then*

$$\sin\theta = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\ \sqrt{l^2+m^2+n^2}}$$

PROOF Clearly, the given line is parallel to the vector $\vec{b}=l\hat{i}+m\hat{j}+n\hat{k}$ and the given plane is normal to the vector $\vec{n}=a\hat{i}+b\hat{j}+c\hat{k}$. Therefore, if θ is the angle between the line and the plane, then

$$\cos\left(\frac{\pi}{2}-\theta\right) = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\ |\vec{n}|}$$

$$\Rightarrow \quad \sin\theta = \frac{(l\hat{i}+m\hat{j}+n\hat{k})\cdot(a\hat{i}+b\hat{j}+c\hat{k})}{\sqrt{l^2+m^2+n^2}\ \sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow \quad \sin\theta = \frac{al+bm+cn}{\sqrt{l^2+m^2+n^2}\ \sqrt{a^2+b^2+c^2}}$$

Condition of perpendicularity : If the line is perpendicular to the plane, then it is parallel to its normal. Therefore, $\vec{b}=l\hat{i}+m\hat{j}+n\hat{k}$ and $\vec{n}=a\hat{i}+b\hat{j}+c\hat{k}$ are parallel. Consequently, we have

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$$

Condition of parallelism : If the line is parallel to the plane, then it is perpendicular to its normal. Therefore, $\vec{b}=l\hat{i}+m\hat{j}+n\hat{k}$ and $\vec{n}=a\hat{i}+b\hat{j}+c\hat{k}$ are perpendicular. Consequently, we have

$$\vec{b}\cdot\vec{n}=0 \Rightarrow al+bm+cn=0$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the angle between the line $\vec{r}=(\hat{i}+2\hat{j}-\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=4$.*

SOLUTION We know that the angle θ between the line $\vec{r}=\vec{a}+\lambda\vec{b}$ and the plane $\vec{r}\cdot\vec{n}=d$ is given by

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\ |\vec{n}|}$$

Here, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{n}=2\hat{i}-\hat{j}+\hat{k}$.

$$\therefore \quad \sin\theta = \frac{(\hat{i}-\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+\hat{k})}{\sqrt{1^2+(-1)^2+1^2}\ \sqrt{2^2+(-1)^2+1^2}} = \frac{2+1+1}{\sqrt{3}\ \sqrt{6}} = \frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

EXAMPLE 2 *Find the angle between the line $\frac{x+1}{3}=\frac{y-1}{2}=\frac{z-2}{4}$. and the plane $2x+y-3z+4=0$.*

SOLUTION The given line is parallel to the vector $\vec{b}=3\hat{i}+2\hat{j}+4\hat{k}$ and the given plane is normal to the vector $\vec{n}=2\hat{i}+\hat{j}-3\hat{k}$. Therefore, the angle θ between the given line and given plane is given by

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|} = \frac{(3\hat{i}+2\hat{j}+4\hat{k})\cdot(2\hat{i}+\hat{j}-3\hat{k})}{\sqrt{3^2+2^2+4^2}\ \sqrt{2^2+(1)^2+(-3)^2}}$$

$$\Rightarrow \quad \sin\theta = \frac{6+2-12}{\sqrt{29}\ \sqrt{14}} = \frac{-4}{\sqrt{29\times 14}} = \frac{-4}{\sqrt{406}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{-4}{\sqrt{406}}\right)$$

EXAMPLE 3 *If the line $\vec{r} = (\hat{i}-2\hat{j}+\hat{k}) + \lambda(2\hat{i}+\hat{j}+2\hat{k})$ is parallel to the plane $\vec{r}\cdot(3\hat{i}-2\hat{j}+m\hat{k}) = 14$, find the value of m.*

SOLUTION The given line is parallel to the vector $\vec{b} = 2\hat{i}+\hat{j}+2\hat{k}$ and the given plane is normal to the vector $\vec{n} = 3\hat{i}-2\hat{j}+m\hat{k}$.

If the line is parallel to the plane, then normal to the plane is perpendicular to the line.

$\therefore \quad \vec{b} \perp \vec{n}$

$\Rightarrow \quad \vec{b}\cdot\vec{n} = 0$

$\Rightarrow \quad (2\hat{i}+\hat{j}+2\hat{k})\cdot(3\hat{i}-2\hat{j}+m\hat{k}) = 0$

$\Rightarrow \quad 6-2+2m = 0 \Rightarrow m = -2$

EXAMPLE 4 *Show that the line whose vector equation is $\vec{r} = (2\hat{i}-2\hat{j}+3\hat{k}) + \lambda(\hat{i}-\hat{j}+4\hat{k})$ is parallel to the plane whose vector equation is $\vec{r}\cdot(\hat{i}+5\hat{j}+\hat{k}) = 5$. Also, find the distance between them.* **[CBSE 2001C, 2004]**

SOLUTION The given line passes through the point having position vector $\vec{a} = 2\hat{i}-2\hat{j}+3\hat{k}$ and is parallel to the vector $\vec{b} = \hat{i}-\hat{j}+4\hat{k}$. The given plane is normal to the vector $\vec{n} = \hat{i}+5\hat{j}+\hat{k}$.

We have, $\vec{b}\cdot\vec{n} = (\hat{i}-\hat{j}+4\hat{k})\cdot(\hat{i}+5\hat{j}+\hat{k}) = 1-5+4 = 0$

So, $\vec{b}$ perpendicular to $\vec{n}$.

Hence, the given line is parallel to the given plane. The distance between the line and the parallel plane is the distance between any point on the line and the given plane. Since the line passes through the point $\vec{a} = 2\hat{i}-2\hat{j}+3\hat{k}$. Therefore,

Distance between the line and the plane

$$= \text{Length of perpendicular from } \vec{a} = 2\hat{i}-2\hat{j}+3\hat{k} \text{ to the given plane}$$

$$= \frac{|(2\hat{i}-2\hat{j}+3\hat{k})\cdot(\hat{i}+5\hat{j}+\hat{k})-5|}{\sqrt{1^2+5^2+1^2}} = \frac{|(2-10+3)-5|}{\sqrt{27}} = \frac{10}{\sqrt{27}}$$

EXAMPLE 5 *Find the vector equation of the line passing through the point with position vector $2\hat{i}-3\hat{j}-5\hat{k}$ and perpendicular to the plane $\vec{r}\cdot(6\hat{i}-3\hat{j}+5\hat{k})+2 = 0$.*

SOLUTION The required line is perpendicular to the plane $\vec{r}\cdot(6\hat{i}-3\hat{j}+5\hat{k})+2=0$. Therefore, it is parallel to the normal $\vec{n} = (6\hat{i}-3\hat{j}+5\hat{k})$. Thus, the required line passes through the point with position vector $\vec{a} = 2\hat{i}-3\hat{j}-5\hat{k}$ and is parallel to the vector $\vec{n} = 6\hat{i}-3\hat{j}+5\hat{k}$.

So, its vector equation is

$$\vec{r} = \vec{a} + \lambda\vec{n} \text{ or, } \vec{r} = (2\hat{i}-3\hat{j}-5\hat{k}) + \lambda(6\hat{i}-3\hat{j}+5\hat{k})$$

EXAMPLE 6 *Find the equations of the line passing through the point (3, 0, 1) and parallel to the planes $x+2y=0$ and $3y-z=0$.*

SOLUTION Let a, b, c be the direction ratios of the required line. Then, its equation is

$$\frac{x-3}{a} = \frac{y-0}{b} = \frac{z-1}{c} \qquad \text{...(i)}$$

Since (i) is parallel to the planes $x + 2y + 0z = 0$ and $0x + 3y - z = 0$, therefore

$$a(1) + b(2) + c(0) = 0 \text{ and } a(0) + b(3) + c(-1) = 0$$

Solving these two equations by cross-multiplication, we obtain

$$\frac{a}{(2)(-1)-(0)(3)} = \frac{b}{(0)(0)-(1)(-1)} = \frac{c}{(1)(3)-(0)(2)}$$

$$\Rightarrow \quad \frac{a}{-2} = \frac{b}{1} = \frac{c}{3} = \lambda \text{ (say)}$$

$$\Rightarrow \quad a = -2\lambda,\ b = \lambda,\ c = 3\lambda$$

Substituting the values of a, b, c in (i), we obtain the equation of the required line as

$$\frac{x-3}{-2} = \frac{y-0}{1} = \frac{z-1}{3}$$

ALITER The required line passes through the point having its position vector $\vec{a} = 3\hat{i} + \hat{k}$ and is parallel to the planes $x + 2y = 0$ and $3y - z = 0$. So, it is perpendicular to their normals $\vec{n_1} = \hat{i} + 2\hat{j}$ and $\vec{n_2} = 3\hat{j} - \hat{k}$ respectively.

Consequently, the required line is parallel to the vector

$$\vec{b} = \vec{n_1} \times \vec{n_2} = -2\hat{i} + \hat{j} + 3\hat{k}.$$

Hence, the equation of the required line is

$$\vec{r} = \vec{a} + \lambda \vec{b} \text{ or, } \vec{r} = (3\hat{i} + \hat{k}) + \lambda(-2\hat{i} + \hat{j} + 3\hat{k})$$

EXAMPLE 7 *Find the equation of the plane passing through the line of intersection of the planes $2x + y - z = 3$, $5x - 3y + 4z + 9 = 0$ and parallel to the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$.*

SOLUTION The equation of the plane passing through the line of intersection of the planes $2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ is

$$(2x + y - z - 3) + \lambda(5x - 3y + 4z + 9) = 0$$

$$\Rightarrow \quad x(2 + 5\lambda) + y(1 - 3\lambda) + z(4\lambda - 1) + 9\lambda - 3 = 0 \qquad \text{...(i)}$$

The plane in (i) is parallel to the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$.

$$\therefore \quad 2(2 + 5\lambda) + 4(1 - 3\lambda) + 5(4\lambda - 1) = 0 \Rightarrow 18\lambda + 3 = 0 \Rightarrow \lambda = -\frac{1}{6}$$

Putting the value of λ in (i), we obtain

$$x\left(2 - \frac{5}{6}\right) + y\left(1 + \frac{3}{6}\right) + z\left(-\frac{4}{6} - 1\right) - \frac{9}{6} - 3 = 0 \Rightarrow 7x + 9y - 10z - 27 = 0$$

This is the equation of the required plane.

EXAMPLE 8 *Find the equation of the plane passing through the intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to x-axis.*

SOLUTION The equation of a plane passing through the intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ is

$$\{\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) - 1\} + \lambda\{\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4\} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \{(2\lambda + 1)\hat{i} + (3\lambda + 1)\hat{j} + (1 - \lambda)\hat{k}\} + (4\lambda - 1) = 0 \qquad \text{...(i)}$$

It is parallel to x-axis i.e. the vector $\hat{i}$.

$\therefore \quad \{(2\lambda+1)\hat{i}+(3\lambda+1)\hat{j}+(1-\lambda)\hat{k}\}\cdot\hat{i} = 0$

$\Rightarrow \quad 2\lambda+1 = 0 \Rightarrow \lambda = -\frac{1}{2}$

Putting $\lambda = -\frac{1}{2}$ in (i), we get

$$\vec{r}\cdot\left(-\frac{1}{2}\hat{j}+\frac{3}{2}\hat{k}\right)-3 = 0 \text{ or, } \vec{r}\cdot(-\hat{j}+3\hat{k}) = 6$$

This is the equation of the required plane.

EXAMPLE 9 *Find the equation of the plane through the points (1, 0, – 1), (3, 2, 2) and parallel to the line* $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$.

SOLUTION The equation of a plane passing through (1, 0, – 1) is

$$a(x-1)+b(y-0)+c(z+1) = 0 \qquad \text{...(i)}$$

This passes through (3, 2, 2). So,

$$a(3-1)+b(2-0)+c(2+1) = 0 \Rightarrow 2a+2b+3c = 0 \qquad \text{...(ii)}$$

The plane in (i) is parallel to the line $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$.

Therefore, normal to the plane is perpendicular to the line.

$$\therefore \quad a(1)+b(-2)+c(3) = 0 \qquad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{(2)(3)-(3)(-2)} = \frac{b}{(1)(3)-(2)(3)} = \frac{c}{(2)(-2)-(2)(1)}$$

$$\Rightarrow \quad \frac{a}{12}=\frac{b}{-3}=\frac{c}{-6} \Rightarrow \frac{a}{4}=\frac{b}{-1}=\frac{c}{-2}=\lambda \text{ (say)}$$

$$\Rightarrow \quad a=4\lambda,\ b=-\lambda,\ c=-2\lambda$$

Substituting the values of a, b, c in (i), we obtain $4x-y-2z-6=0$

This is the equation of required plane.

ALITER The required plane passes through the point $A(1, 0, -1)$ and $B(3, 2, 2)$. Let $\vec{a}$ and $\vec{b}$ be the position vectors of A and B respectively. Let $\vec{n}$ be the normal to the required plane. Then, $\vec{n}$ is perpendicular to $\vec{AB}$. Also, the required is parallel to the line plane $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$.

So, its normal vector $\vec{n}$ is perpendicular to the vector $\vec{b_1}=\hat{i}-2\hat{j}+3\hat{k}$ which is parallel to the given line.

Thus, $\vec{n}$ is perpendicular to both $\vec{AB}$ and $\vec{b_1}=\hat{i}-2\hat{j}+3\hat{k}$. Therefore, $\vec{n}$ is parallel to the vector $\vec{AB}\times\vec{b_1}$.

Now, $\quad \vec{n_1} = \vec{AB}\times\vec{b_1} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 3 \\ 1 & -2 & 3 \end{vmatrix} = 12\hat{i}-3\hat{j}-6\hat{k}$

Hence, vector equation of the required plane is

$$(\vec{r}-\vec{a})\cdot\vec{n_1} = 0$$

$$\Rightarrow \quad \vec{r}\cdot\vec{n_1} = \vec{a}\cdot\vec{n_1}$$

$$\Rightarrow \quad \vec{r}\cdot(12\hat{i}-3\hat{j}-6\hat{k}) = (\hat{i}-\hat{k})\cdot(12\hat{i}-3\hat{j}-6\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(12\hat{i}-3\hat{j}-6\hat{k}) = 18$$

$$\Rightarrow \quad \vec{r}\cdot(4\hat{i}-\hat{j}-2\hat{k}) = 6$$

EXAMPLE 10 *State when the line $\vec{r}=\vec{a}+\lambda\vec{b}$ is parallel to the plane $\vec{r}\cdot\vec{n}=d$. Show that the line $\vec{r}=\hat{i}+\hat{j}+\lambda(2\hat{i}+\hat{j}+4\hat{k})$ is parallel to the plane $\vec{r}\cdot(-2\hat{i}+\hat{k})=5$. Also, find the distance between the line and the plane.*

SOLUTION The line $\vec{r}=\vec{a}+\lambda\vec{b}$ is parallel to the plane $\vec{r}\cdot\vec{n}=d$, if the normal to the plane is perpendicular to the line i.e.,

$$\vec{b}\perp\vec{n} \Rightarrow \vec{b}\cdot\vec{n}=0.$$

Here, $\vec{a}=\hat{i}+\hat{j}$, $\vec{b}=2\hat{i}+\hat{j}+4\hat{k}$ and $\vec{n}=-2\hat{i}+\hat{k}$.

We have,

$$\vec{b}\cdot\vec{n} = (2\hat{i}+\hat{j}+4\hat{k})\cdot(-2\hat{i}+0\hat{j}+\hat{k}) = -4+0+4 = 0$$

$$\Rightarrow \quad \vec{b}\perp\vec{n}$$

$\Rightarrow$ Given line is parallel to the given plane.

Distance between the line and the plane

= Length of the perpendicular from the point $\vec{a}=\hat{i}+\hat{j}$ on the plane $\vec{r}\cdot\vec{n}=d$.

$$= \frac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|} = \frac{|(\hat{i}+\hat{j})\cdot(-2\hat{i}+\hat{k})-5|}{\sqrt{(-2)^2+(0)^2+1^2}} = \frac{|-2+0+0-5|}{\sqrt{5}} = \frac{7}{\sqrt{5}}.$$

EXAMPLE 11 *Find the equation of the plane passing through the intersection of the planes $4x-y+z=10$ and $x+y-z=4$ and parallel to the line with direction ratios proportional to 2, 1, 1. Find also the perpendicular distance of (1, 1, 1) from this plane.*

SOLUTION The equation of a plane passing through the intersection of the given planes is

$$(4x-y+z-10)+\lambda(x+y-z-4) = 0$$

$$\Rightarrow \quad x(4+\lambda)+y(\lambda-1)+z(1-\lambda)-10-4\lambda = 0 \qquad \text{...(i)}$$

This plane is parallel to the line with direction ratios proportional to 2, 1, 1.

$$\therefore \quad 2(4+\lambda)+1(\lambda-1)+1(1-\lambda) = 0 \Rightarrow \lambda = -4$$

Putting $\lambda=-4$ in (i), we obtain

$$5y-5z-6=0 \qquad \text{...(ii)}$$

This is the equation of the required plane.

Now, Length of the perpendicular from (1, 1, 1) on (ii) is given by

$$d = \left|\frac{5\times1-5\times1-6}{\sqrt{5^2+(-5)^2}}\right| = \frac{3\sqrt{2}}{5}.$$

EXAMPLE 12 *Find the equation of the plane passing through the point A (1, 2, 1) and perndicular to the line joining the points P (1, 4, 2) and Q (2, 3, 5). Also, find the distance of this plane from the line $\frac{x+3}{2}=\frac{y-5}{-1}=\frac{z-7}{-1}$.* **[CBSE 2010]**

SOLUTION The direction ratios of PQ are proportional to

$$2-1, 3-4, 5-2 \text{ i.e. } 1, -1, 3$$

So, the equation of the plane passing through A (1, 2, 1) and perpendicular to PQ is

$$1 \times (x-1) + (-1)(y-2) + 3 \times (z-1) = 0 \text{ or, } x - y + 3z = 2 \quad \text{...(i)}$$

The given line is parallel to the vector $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and the plane (i) is normal to the vector $\vec{n} = \hat{i} - \hat{j} + 3\hat{k}$ such that $\vec{b} \cdot \vec{n} = 0$. So, given line is parallel to the plane (i).

The distance between the plane (i) and the given line is the distance of any point on the line from the plane (i). The line passes through the point $(-3, 5, 7)$.

So, required distance = Length of perpendicular from $(-3, 5, 7)$ on plane (i)

$$= \left| \frac{-3-5+21-2}{\sqrt{1^2 + (-1)^2 + 3^2}} \right| = \frac{11}{\sqrt{11}} = \sqrt{11}$$

EXERCISE 28.9

1. Find the angle between the line $\vec{r} = (2\hat{i} + 3\hat{j} + 9\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 5$.
2. Find the angle between the line $\dfrac{x-1}{1} = \dfrac{y-2}{-1} = \dfrac{z+1}{1}$ and the plane $2x + y - z = 4$.
3. Find the angle between the line joining the points $(3, -4, -2)$ and $(12, 2, 0)$ and the plane $3x - y + z = 1$.
4. The line $\vec{r} = \hat{i} + \lambda(2\hat{i} - m\hat{j} - 3\hat{k})$ is parallel to the plane $\vec{r} \cdot (m\hat{i} + 3\hat{j} + \hat{k}) = 4$. Find m.
5. Show that the line whose vector equation is $\vec{r} = 2\hat{i} + 5\hat{j} + 7\hat{k} + \lambda(\hat{i} + 3\hat{j} + 4\hat{k})$ is parallel to the plane whose vector equation is $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 7$. Also, find the distance between them.
6. Find the vector equation of the line through the origin which is perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 3$.
7. Find the equation of the plane through $(2, 3, -4)$ and $(1, -1, 3)$ and parallel to x-axis.
8. Find the equation of a plane passing through the points $(0, 0, 0)$ and $(3, -1, 2)$ and parallel to the line $\dfrac{x-4}{1} = \dfrac{y+3}{-4} = \dfrac{z+1}{7}$. **[CBSE 2010]**
9. Find the equation of the line passing through $(1, 2, 3)$ and parallel to the planes $x - y + 2z = 5$ and $3x + y + z = 6$.
10. Prove that the line of section of the planes $5x + 2y - 4z + 2 = 0$ and $2x + 8y + 2z - 1 = 0$ is parallel to the plane $4x - 2y - 5z - 2 = 0$.
11. Find the vector equation of the line passing through the point $(1, -1, 2)$ and perpendicular to the plane $2x - y + 3z - 5 = 0$.
12. Find the equation of the plane through the points $(2, 2, -1)$ and $(3, 4, 2)$ and parallel to the line whose direction ratios are 7, 0, 6.
13. Find the angle between the line $\dfrac{x-2}{3} = \dfrac{y+1}{-1} = \dfrac{z-3}{2}$ and the plane $3x + 4y + z + 5 = 0$.
14. Find the equation of the plane passing through the intersection of the planes $x - 2y + z = 1$ and $2x + y + z = 8$ and parallel to the line with direction ratios proportional to 1, 2, 1. Find also the perpendicular distance of $(1, 1, 1)$ from this plane. **[CBSE 2005]**
15. State when the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the plane $\vec{r} \cdot \vec{n} = d$. Show that the line $\vec{r} = \hat{i} + \hat{j} + \lambda(3\hat{i} - \hat{j} + 2\hat{k})$ is parallel to the plane $\vec{r} \cdot (2\hat{j} + \hat{k}) = 3$. Also, find the distance between the line and the plane.
16. Show that the plane whose vector equation is $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 1$ and the line whose vector equation is $\vec{r} = (-\hat{i} + \hat{j} + \hat{k}) + \lambda(2i + \hat{j} + 4\hat{k})$ are parallel. Also find the distance between them.

17. Find the equation of the plane through the intersection of the planes $3x - 4y + 5z = 10$ and $2x + 2y - 3z = 4$ and parallel to the line $x = 2y = 3z$.
18. Find the vector equation of the straight line passing through (1, 2, 3) and perpendicular to the plane $\vec{r}\cdot(\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.
19. Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}\cdot(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}\cdot(3\hat{i} + \hat{j} + \hat{k}) = 6$.
20. Find the equation of the plane passing through the ponits (3, 4, 1) and (0, 1, 0) and parallel to the line $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$. [CBSE 2008]

ANSWERS

1. $\sin^{-1}\left(\frac{3\sqrt{3}}{29}\right)$
2. 0
3. $\sin^{-1}\left(\frac{23}{11\sqrt{11}}\right)$
4. -3
5. $\frac{7}{\sqrt{3}}$
6. $\vec{r} = \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$
7. $7y + 4z - 5 = 0$
8. $x - 19y - 11z = 0$
9. $\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$
11. $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$
12. $12x + 15y - 14z - 68 = 0$
13. $\sin^{-1}\sqrt{\frac{7}{52}}$
14. $13x - 6y + 9z - 37 = 0, \frac{21}{\sqrt{286}}$
15. Distance $= \frac{1}{\sqrt{5}}$
16. $\frac{1}{\sqrt{6}}$
17. $x - 20y + 27z = 14$
18. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 5\hat{k})$
19. $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 4\hat{k})$
20. $8x - 13y + 15z + 13 = 0$

28.12.3 INTERSECTION OF A LINE AND A PLANE

Let the equation of a line be $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and that of a plane be $ax + by + cz + d = 0$.

The coordinates of any point on the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ are given by

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \text{ (say)}$$

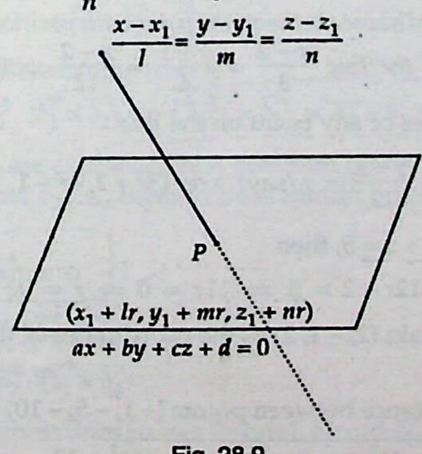

Fig. 28.9

or, $$(x_1 + lr, y_1 + mr, z_1 + nr) \quad ...(i)$$

If it lies on the plane $ax + by + cz + d = 0$, then

$$a(x_1 + lr) + b(y_1 + mr) + c(z_1 + nr) + d = 0$$

$$\Rightarrow \quad (ax_1 + by_1 + cz_1 + d) + r(al + bm + cn) = 0$$

$$\Rightarrow \quad r = -\frac{(ax_1 + by_1 + cz_1 + d)}{al + bm + cn}$$

Substituting the value of r in (i), we obtain the coordinates of the required point of intersection.

In order to find the coordinates of the point of intersection of a line and a plane, we may use the following algorithm.

ALGORITHM

STEP I *Write the coordinates of any point on the line in terms of some parameters r (say).*

STEP II *Substitute these coordinates in the equation of the plane to obtain the value of r.*

STEP III *Put the value of r in the coordinates of the point in step I.*

ILLUSTRATION 1 *Find the coordinates of the point where the line through the points A (3, 4, 1) and B (5, 1, 6) crosses the XY-plane.* [NCERT]

SOLUTION The equation of the line passing through A and B

$$\frac{x-3}{5-3} = \frac{y-4}{1-4} = \frac{z-1}{6-1} \text{ or, } \frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5}$$

The coordinates of any point on this line are given by

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5} = \lambda$$

$$\Rightarrow \quad x = 2\lambda + 3,\ y = -3\lambda + 4,\ z = 5\lambda + 1$$

So, $(2\lambda + 3, -3\lambda + 4, 5\lambda + 1)$ are coordinates of any point on the line passing through A and B. If it lies on XY-plane i.e. $z = 0$. Then,

$$5\lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{5}$$

So, the coordinates of required point are

$$\left(2 \times -\frac{1}{5} + 3, -3 \times -\frac{1}{5} + 4, 5 \times -\frac{1}{5} + 1\right) \text{ i.e., } \left(\frac{13}{5}, \frac{23}{5}, 0\right)$$

ILLUSTRATION 2 *Find the distance between the point with position vector* $-\hat{i} - 5\hat{j} - 10\hat{k}$ *and the point of intersection of the line* $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ *with the plane* $x - y + z = 5$.

SOLUTION The coordinates of any point on the line

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = r \text{ (say) are } (3r + 2, 4r - 1, 12r + 2) \quad ...(i)$$

If it lies on the plane $x - y + z = 5$, then

$$3r + 2 - 4r + 1 + 12r + 2 = 5 \Rightarrow 11r = 0 \Rightarrow r = 0.$$

Putting $r = 0$ in (i), we obtain $(2, -1, 2)$ as the coordinates of the point of intersection of the given line and plane.

Required distance = Distance between points $(-1, -5, -10)$ and $(2, -1, 2)$

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = 13.$$

28.12.4 CONDTION FOR A LINE TO LIE IN A PLANE

THEOREM 1 *(Vector form)* *If the line* $\vec{r} = \vec{a} + \lambda \vec{b}$ *lies in the plane* $\vec{r} \cdot \vec{n} = d$, *then* (i) $\vec{b} \cdot \vec{n} = 0$ *and* (ii) $\vec{a} \cdot \vec{n} = d$.

PROOF If the line $\vec{r} = \vec{a} + \lambda \vec{b}$ lies in the plane $\vec{r} \cdot \vec{n} = d$, then every point on the line lies on the plane. The position vector of any point on the line $\vec{r} = \vec{a} + \lambda \vec{b}$ is $\vec{a} + \lambda \vec{b}$. This point lies on the plane if

$$(\vec{a} + \lambda \vec{b}) \cdot \vec{n} = d \quad \text{for all values of } \lambda$$

$$\Rightarrow \quad (\vec{a} \cdot \vec{n} - d) + \lambda (\vec{b} \cdot \vec{n}) = 0 \quad \text{for all values of } \lambda.$$

$$\Rightarrow \quad \vec{a} \cdot \vec{n} - d = 0 \qquad \text{...(i)}$$

$$\text{and,} \quad \vec{b} \cdot \vec{n} = 0 \qquad \text{...(ii)}$$

The conditions (i) and (ii) also confirm the geometrical fact that a line will lie in a plane, if (i) any point lies on the plane (ii) the normal to the plane is perpendicular to the line.

THEOREM 2 *(Cartesian form)* *If the line* $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ *lies in the plane* $ax + by + cz + d = 0$, *then*

(i) $ax_1 + by_1 + cz_1 + d = 0$ *and,* (ii) $al + bm + cn = 0$.

PROOF The coordinates of any point on the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = \lambda \text{ are } (x_1 + l\lambda,\ y_1 + m\lambda,\ z_1 + n\lambda)$$

If the line lies on the plane, then every point on the line lies on the plane. Therefore, $(x_1 + l\lambda, y_1 + m\lambda, z_1 + n\lambda)$ lies on the plane $ax + by + cz + d = 0$ for all values of λ.

$$\therefore \quad a(x_1 + l\lambda) + b(y_1 + m\lambda) + c(z_1 + n\lambda) + d = 0 \quad \text{for all values } \lambda$$

$$\Rightarrow \quad (ax_1 + by_1 + cz_1 + d) + \lambda(al + bm + cn) = 0 \quad \text{for all values } \lambda$$

$$\Rightarrow \quad ax_1 + by_1 + cz_1 + d = 0 \text{ and } al + bm + cn = 0$$

28.12.5 CONDITION OF COPLANARITY OF TWO LINES AND EQUATION OF THE PLANE CONTAINING THEM

THEOREM 1 *(Vector form)* *If the lines* $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ *are coplanar, then*

$$\vec{a_1} \cdot (\vec{b_1} \times \vec{b_2}) = \vec{a_2} \cdot (\vec{b_1} \times \vec{b_2})$$

and the equation of the plane containing them is

$$\vec{r} \cdot (\vec{b_1} \times \vec{b_2}) = \vec{a_1} \cdot (\vec{b_1} \times \vec{b_2}) \text{ or } \vec{r} \cdot (\vec{b_1} \times \vec{b_2}) = \vec{a_2} \cdot (\vec{b_1} \times \vec{b_2})$$

PROOF If the given lines are coplanar, their common plane should be parallel to each of the vectors $\vec{a_1} - \vec{a_2}$, $\vec{b_1}$ and $\vec{b_2}$ i.e., these vectors should be coplanar and the condition for the same is

$$\boxed{(\vec{a_1} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2}) = 0}$$

$$\Rightarrow \quad \vec{a_1} \cdot (\vec{b_1} \times \vec{b_2}) - \vec{a_2} \cdot (\vec{b_1} \times \vec{b_2}) = 0$$

$$\Rightarrow \quad \vec{a_1} \cdot (\vec{b_1} \times \vec{b_2}) = \vec{a_2} \cdot (\vec{b_1} \times \vec{b_2})$$

The plane containing the given lines passes through $\vec{a_1}$ and $\vec{a_2}$ and is normal to the vector $\vec{b_1} \times \vec{b_2}$. So, its equation is

$$(\vec{r}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = 0 \text{ or, } (\vec{r}-\vec{a_2})\cdot(\vec{b_1}\times\vec{b_2}) = 0$$

$$\Rightarrow \quad \vec{r}\cdot(\vec{b_1}\times\vec{b_2}) = \vec{a_1}\cdot(\vec{b_1}\times\vec{b_2}) \text{ or, } \vec{r}\cdot(\vec{b_1}\times\vec{b_2}) = \vec{a_2}\cdot(\vec{b_1}\times\vec{b_2})$$

THEOREM 2 *(Cartesian form) If the line* $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ *and* $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ *are coplanar, then*

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing them is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \text{ or, } \boxed{\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0}$$

PROOF The equations of two straight lines are

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad \text{...(i)}$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \quad \text{...(ii)}$$

Let $\quad ax+by+cz+d = 0 \quad$ ...(iii)

be the plane containing the line (i). Then, (x_1, y_1, z_1) lies on the plane and the normal to the plane is perpendicular to the line.

$\therefore \quad ax_1+by_1+cz_1+d = 0 \quad$ ...(iv)

and, $\quad al_1+bm_1+cn_1 = 0 \quad$ ...(v)

Substituting the value of d obtained from (iv) in (iii), we get

$$a(x-x_1)+b(y-y_1)+c(z-z_1) = 0 \quad \text{...(vi)}$$

If the lines (i) and (ii) are coplanar then line (ii) lies in the plane (vi), i.e., the point (x_2, y_2, z_2) lies in the plane (vi) and line (ii) is perpendicular to the normal to the plane (vi).

$\therefore \quad a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1) = 0 \quad$ ...(vii)

and, $\quad al_2+bm_2+cn_2 = 0 \quad$ ...(viii)

Eliminating a, b, c from the equations (v), (vii) and (viii), we get

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

which is the required condition.

The equating of the required plane is obtained by eliminating a, b and c from the equations (vi), (vii) and (viii).

$$\therefore \quad \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \text{ is the required plane.}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Show that the lines* $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})$ *and* $\vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})$ *are coplanar. Also, find the plane containing these two lines.*

SOLUTION We know that the lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ are coplanar if

$$\vec{a_1}\cdot(\vec{b_1}\times\vec{b_2}) = \vec{a_2}\cdot(\vec{b_1}\times\vec{b_2})$$

and the equation of the plane containing them is

$$\vec{r}\cdot(\vec{b_1}\times\vec{b_2}) = \vec{a_1}\cdot(\vec{b_1}\times\vec{b_2})$$

Here, $\vec{a_1}=\hat{i}+\hat{j}-\hat{k}$, $\vec{b_1}=3\hat{i}-\hat{j}$, $\vec{a_2}=4\hat{i}+0\hat{j}-\hat{k}$ and $\vec{b_2}=2\hat{i}+0\hat{j}+3\hat{k}$.

$$\therefore \quad \vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = -3\hat{i}-9\hat{j}+2\hat{k}$$

$$\Rightarrow \quad \vec{a_1}\cdot(\vec{b_1}\times\vec{b_2}) = (\hat{i}+\hat{j}-\hat{k})\cdot(-3\hat{i}-9\hat{j}+2\hat{k}) = -3-9-2=-14$$

$$\text{and,} \quad \vec{a_2}\cdot(\vec{b_1}\times\vec{b_2}) = (4\hat{i}+0\hat{j}-\hat{k})\cdot(-3\hat{i}-9\hat{j}+2\hat{k}) = -12+0-2=-14$$

$$\Rightarrow \quad \vec{a_1}\cdot(\vec{b_1}\times\vec{b_2}) = \vec{a_2}\cdot(\vec{b_1}\times\vec{b_2})$$

So, the given lines are coplanar.

The equation of the plane containing the given lines is

$$\vec{r}\cdot(\vec{b_1}\times\vec{b_2}) = \vec{a_1}\cdot(\vec{b_1}\times\vec{b_2})$$

$$\Rightarrow \quad \vec{r}\cdot(\vec{b_1}\times\vec{b_2}) = -14$$

$$\Rightarrow \quad \vec{r}\cdot(-3\hat{i}-9\hat{j}+2\hat{k}) = -14 \qquad [\because \vec{b_1}\times\vec{b_2} = -3\hat{i}-9\hat{j}+2\hat{k}]$$

$$\Rightarrow \quad \vec{r}\cdot(3\hat{i}+9\hat{j}-2\hat{k}) = 14.$$

EXAMPLE 2 *Prove that the lines* $\dfrac{x+1}{3}=\dfrac{y+3}{5}=\dfrac{z+5}{7}$ *and* $\dfrac{x-2}{1}=\dfrac{y-4}{4}=\dfrac{z-6}{7}$ *are coplanar. Also, find the plane containing these two lines.*

SOLUTION We know that the lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad \text{and,} \quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing these two lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Here, $x_1=-1, y_1=-3, z_1=-5, x_2=2, y_2=4, z_2=6, l_1=3, m_1=5, n_1=7, l_2=1, m_2=4, n_2=7.$

$$\therefore \quad \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 3 & 7 & 11 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 21-98+77 = 0$$

So, the given lines are coplanar.

The equation of the plane containing the lines is

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

$\Rightarrow \quad (x+1)(35-28)-(y+3)(21-7)+(z+5)(12-5) = 0$

$\Rightarrow \quad x-2y+z = 0$

EXAMPLE 3 *Show that the lines*

$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta} \text{ and, } \frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$$

are coplanar. [NCERT]

SOLUTION We know that the lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and, } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

are coplanar, if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

The equations of the given lines are

$$\frac{x-(a-d)}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-(a+d)}{\alpha+\delta} \text{ and, } \frac{x-(b-c)}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-(b+c)}{\beta+\gamma}$$

Here, $x_1=a-d,\ y_1=a,\ z_1=a+d$

$x_2=b-c,\ y_2=b,\ z_2=b+c$

$l_1=\alpha-\delta,\ m_1=\alpha,\ n_1=\alpha+\delta;\ l_2=\beta-\gamma,\ m_2=\beta,\ n_2=\beta+\gamma$

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} b-c-a+d & b-a & b+c-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 2(b-a) & b-a & b+c-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$

$$\Rightarrow \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad [\because C_1 \text{ and } C_2 \text{ are proportional}]$$

Hence, given lines are coplanar.

EXAMPLE 4 *Find the vector equation of the plane that contains the lines* $\vec{r}=(\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}-\hat{k})$ *and* $\vec{r}=(\hat{i}+\hat{j})+\mu(-\hat{i}+\hat{j}-2\hat{k})$.

SOLUTION The two given lines pass through the point having position vector $\vec{a}=\hat{i}+\hat{j}$ and are parallel to the vectors $\vec{b_1}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b_2}=-\hat{i}+\hat{j}-2\hat{k}$ respectively. Therefore, the plane containing the given lines also passes through the point with position vector $\vec{a}=\hat{i}+\hat{j}$. Since the plane contains the lines which are parallel to the vectors $\vec{b_1}$ and $\vec{b_2}$ respectively. Therefore, the plane is normal to the vector

$$\vec{n}=\vec{b_1}\times\vec{b_2}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{vmatrix} = -3\hat{i}+3\hat{j}+3\hat{k}$$

Thus, the vector equation of the required planes is

$$(\vec{r}-\vec{a})\cdot\vec{n}=0 \text{ or, } \vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

$$\Rightarrow \quad \vec{r}\cdot(-3\hat{j}+3\hat{j}+3\hat{k}) = (\hat{i}+\hat{j})\cdot(-3\hat{i}+3\hat{j}+3\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot((-3\hat{i}+3\hat{j}+3\hat{k}) = -3+3$$

$$\Rightarrow \quad \vec{r}\cdot(-\hat{i}+\hat{j}+\hat{k}) = 0.$$

EXAMPLE 5 *Find the equation of the plane passing through the point (0, 7, –7) and containing the line* $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$.

SOLUTION Let the equation of a plane passing through (0, 7, – 7) be

$$a(x-0)+b(y-7)+c(z+7) = 0.\text{(i)}$$

The line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ passes through the point (– 1, 3, – 2) and has direction ratios - 3, 2, 1. If (i) contains this lines, it must pass through (– 1, 3, – 2) and must be parallel to the line. Therefore,

$$a(-1)+b(3-7)+c(-2+7) = 0$$

$$\Rightarrow \quad a(-1)+b(-4)+c(5) = 0 \quad \text{...(ii)}$$

and, $$-3a+2b+1c = 0 \quad \text{...(iii)}$$

On solving (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{-14}=\frac{b}{-14}=\frac{c}{-14} \Rightarrow \frac{a}{1}=\frac{b}{1}=\frac{c}{1}=\lambda \text{ (say)} \Rightarrow a=\lambda, b=\lambda, c=\lambda.$$

Putting the values of a, b, c in (i), we obtain

$$\lambda(x-0)+\lambda(y-7)+\lambda(z+7) = 0 \Rightarrow x+y+z = 0.$$

This is the equation of the required plane.

ALITER The required plane passes through the point A (0, 7, – 7) and contains the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$.

Clearly, the line passes through B (– 1, 3, – 2) and is parallel to the vector $\vec{b}=-3\hat{i}+2\hat{j}+\hat{k}$

Thus, the plane passes through two points A (0, 7, – 7) and B (– 1, 3, – 2) and is parallel to the vector $\vec{b}=-3\hat{i}+2\hat{j}+\hat{k}$.

Let $\vec{n}$ be the normal vector to the required plane. Then, $\vec{n}$ is perpendicular to both $\vec{b}$ and $\vec{AB}$. So, it is parallel to $\vec{AB}\times\vec{b}$.

Let $\vec{n_1} = \vec{AB}\times\vec{b}$. Then,

$$\vec{n_1} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -4 & 5 \\ -3 & 2 & 1 \end{vmatrix} = -14\hat{i}-14\hat{j}-14\hat{k}$$

Let $\vec{\alpha}$ be the position vector of A. Then, $\vec{\alpha}=7\hat{j}-7\hat{k}$.

Clearly, the required plane passes through $\vec{\alpha}=7\hat{j}-7\hat{k}$ and is perpendicular to $\vec{n_1}=-14\hat{i}-14\hat{j}-14\hat{k}$. So, its equation is

$$(\vec{r}-\vec{\alpha})\cdot\vec{n_1} = 0$$

$$\Rightarrow \quad \vec{r}\cdot n_1 = \vec{\alpha}\cdot\vec{n_1}$$

$$\Rightarrow \quad \vec{r}\cdot(-14\hat{i}-14\hat{j}-114\hat{k}) = (7\hat{j}-7\hat{k})\cdot(-14\hat{i}-14\hat{j}-14\hat{k})$$

$\Rightarrow \quad \vec{r}\cdot(-14\hat{i}-14\hat{j}-14\hat{k}) = -98+98$

$\Rightarrow \quad \vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 0$

The cartesian equation of the plane is $x+y+z = 0$.

EXAMPLE 6 *Show that the plane whose vector equation is $\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k})=3$ contains the line whose vector equation is $\vec{r}=(\hat{i}+\hat{j})+\lambda(2\hat{i}+\hat{j}+4\hat{k})$.*

SOLUTION The line $\vec{r}=(\hat{i}+\hat{j})+\lambda(2\hat{i}+\hat{j}+4\hat{k})$ passes through a point with position vector $\vec{a}=\hat{i}+\hat{j}$ and is parallel to the vector $\vec{b}=2\hat{i}+\hat{j}+4\hat{k}$. The plane $\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k})=3$ contains the given line if

(i) it passes through $\hat{i}+\hat{j}$ and (ii) it is parallel to the line.

We have, $(\hat{i}+\hat{j})\cdot(\hat{i}+2\hat{j}-\hat{k})=1+2=3$

So, the plane passes through the point $\hat{i}+\hat{j}$.

And, $(2\hat{i}+\hat{j}+4\hat{k})\cdot(\hat{i}+2\hat{j}-\hat{k})=2+2-4=0$. Therefore, the plane is parallel to the line. Hence, the given plane contains the given line.

EXAMPLE 7 *Find the vector and cartesian equation of the plane containing the two lines*

$$\vec{r}=2\hat{i}+\hat{j}-3\hat{k}+\lambda(\hat{i}+2\hat{j}+5\hat{k})$$

and, $$\vec{r}=3\hat{i}+3\hat{j}+2\hat{k}+\mu(3\hat{i}-2\hat{j}+5\hat{k})$$

SOLUTION Given lines pass through points having position vectors $\vec{a_1}=2\hat{i}+\hat{j}-3\hat{k}$ and $\vec{a_2}=3\hat{i}+3\hat{j}+2\hat{k}$ respectively and are parallel to the vectors $\vec{b_1}=\hat{i}+2\hat{j}+5\hat{k}$ and $\vec{b_2}=3\hat{i}-2\hat{j}+5\hat{k}$ respectively. Therefore, the plane containing these two lines passes through points having position vectors $\vec{a_1}$ and $\vec{a_2}$ and is perpendicular to the vector $\vec{n}=\vec{b_1}\times\vec{b_2}$.

We have,

$$\vec{n}=\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 5\\ 3 & -2 & 5\end{vmatrix}=20\hat{i}+10\hat{j}-8\hat{k}$$

So, the vector equation of the required plane is

$$(\vec{r}-\vec{a_1})\cdot\vec{n}=0$$

or, $\quad \vec{r}\cdot\vec{n}=\vec{a_1}\cdot\vec{n}$

or, $\quad \vec{r}\cdot(20\hat{i}+10\hat{j}-8\hat{k}) = (2\hat{i}+\hat{j}-3\hat{k})\cdot(20\hat{i}+10\hat{j}-8\hat{k})$

or, $\quad \vec{r}\cdot(20\hat{i}+10\hat{j}-8\hat{k}) = 40+10+24$

or, $\quad \vec{r}\cdot(10\hat{i}+5\hat{j}-4\hat{k}) = 37$

The cartesian equation is $10x+5y-4z = 37$.

EXAMPLE 8 *If $4x+4y-\lambda z=0$ is the equation of the plane through the origin that contains the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{4}$, find the value of λ.*

SOLUTION If the plane $4x+4y-\lambda z=0$ contains the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{4}$, then

$$2\times4+3\times4-4\times\lambda = 0 \Rightarrow 20-4\lambda = 0 \Rightarrow \lambda = 5.$$

EXERCISE 28.10

1. Show that the lines $\vec{r} = (2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$ and $\vec{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 4\hat{k})$ are coplanar. Also, find the equation of the plane containing them.
2. Show that the lines $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ are coplanar. Also find the equation of the plane containing them.
3. Find the equation of the plane containing the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and the point $(0, 7, -7)$ and show that the line $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ also lies in the same plane.
4. Find the equation of the plane which contains two parallel lines $\frac{x-4}{1} = \frac{y-3}{-4} = \frac{z-2}{5}$ and $\frac{x-3}{1} = \frac{y+2}{-4} = \frac{z}{5}$.
5. Show that the lines $\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2}$ and $3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$ intersect. Find the equation of the plane in which they lie and also their point of intersection.
6. Show that the plane whose vector equation is $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$ contains the line whose vector equation is $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} + \hat{j} + 4\hat{k})$.
7. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$. **[NCERT]**

ANSWERS

1. $\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) + 7 = 0$ 2. $x + y + z = 0$ 3. $x + y + z = 0$
4. $11x - y - 3z = 35$ 5. $(2, 4, -3)$, $45x - 17y + 25z + 53 = 0$ 7. 13 units

28.13 IMAGE OF A POINT IN A PLANE

DEFINITION *Let P and Q be two points and let π be a plane such that*

(i) line PQ is perpendicular to the plane π, and, (ii) mid-point of PQ lies on the plane π. Then, either of the point is the image of the other in the plane π.

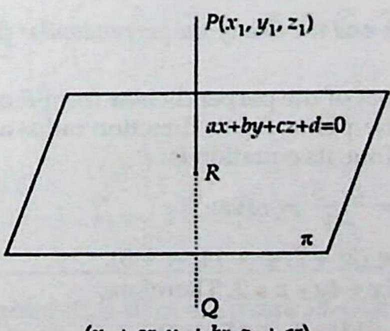

Fig. 28.10

In order to find the image of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$, we may use the following algorithm.

ALGORITHM

<u>STEP I</u> *Write the equations of the line passing through P and normal to the given plane as*
$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}.$$

<u>STEP II</u> *Write the coordinates of image Q as (x_1+ar, y_1+br, z_1+cr).*

<u>STEP III</u> *Find the coordinates of the mid-point R of PQ.*

<u>STEP IV</u> *Obtain the value of r by putting the coordinates of R in the equation of the plane.*

<u>STEP V</u> *Put the value of r in the coordinates of Q.*

The above algorithm is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the image of the point $(3,-2,1)$ in the plane $3x-y+4z=2$.*

SOLUTION Let Q be the image of the point $P(3,-2,1)$ in the plane $3x-y+4z=2$. Then, PQ is normal to the plane. Therefore, direction ratios of PQ are proportional to $3,-1,4$. Since PQ passes through $P(3,-2,1)$ and has direction ratios proportional to $3,-1,4$. Therefore, equation of PQ is

$$\frac{x-3}{3}=\frac{y+2}{-1}=\frac{z-1}{4}=r\ \text{(say)}$$

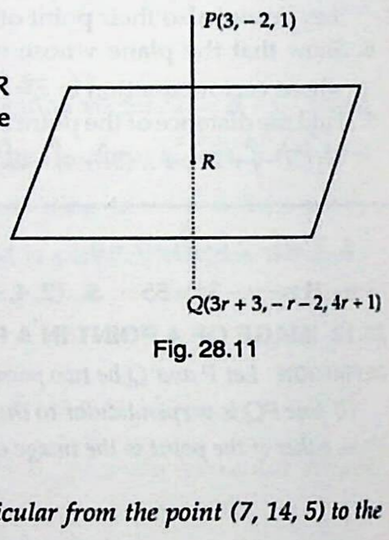

Fig. 28.11

Let the coordinates of Q be $(3r+3,-r-2,4r+1)$. Let R be the mid-point of PQ. Then, R lies on the plane $3x-y+4z=2$. The coordinates of R are

$$\left(\frac{3r+3+3}{2},\frac{-r-2-2}{2},\frac{4r+1+1}{2}\right)$$

or, $$\left(\frac{3r+6}{2},\frac{-r-4}{2},2r+1\right)$$

Since R lies on $3x-y+4z=2$.

$$3\left(\frac{3r+6}{2}\right)-\left(\frac{-r-4}{2}\right)+4(2r+1)=2$$

$$\Rightarrow\quad 13r=-13\Rightarrow r=-1$$

Hence, the coordinates of Q are $(0,-1,-3)$.

EXAMPLE 2 *Find the length and the foot of the perpendicular from the point $(7,14,5)$ to the plane $2x+4y-z=2$.*

SOLUTION Let M be the foot of the perpendicular from P on the plane $2x+4y-z=2$. Then, PM is normal to the plane. So, its direction ratios are $2,4,-1$. Since PM passes through $P(7,14,5)$. Therefore, its equation is

$$\frac{x-7}{2}=\frac{y-14}{4}=\frac{z-5}{-1}=r\ \text{(say)}$$

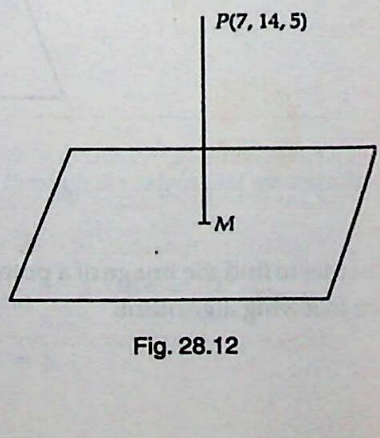

Fig. 28.12

Let the coordinates of M be $(2r+7,4r+14,-r+5)$.
Since M lies on the plane $2x+4y-z=2$. Therefore,

$$2(2r+7)+4(4r+14)-(-r+5)=2$$

$$\Rightarrow\quad 21r+63=0$$

$$\Rightarrow\quad r=-3$$

So, the coordinates of M are $(1,2,8)$.

Now,

$$PM = \text{Length of the perpendicular from } P$$

$$\Rightarrow \quad PM = \sqrt{(7-1)^2 + (14-2)^2 + (5-8)^2} = 3\sqrt{21}.$$

EXAMPLE 3 *Find the image of the point having position vector* $\hat{i} + 3\hat{j} + 4\hat{k}$ *in the plane* $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$.

SOLUTION Let Q be the image of the point $P(\hat{i} + 3\hat{j} + 4\hat{k})$ in the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$. Then, PQ is normal to the plane. Since PQ passes through P and is normal to the given plane, therefore equation of line PQ is

$$\vec{r} = (\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k}).$$

Since Q lies on line PQ, so let the position vector of Q be

$$(\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k}) = (1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (4 + \lambda)\hat{k}.$$

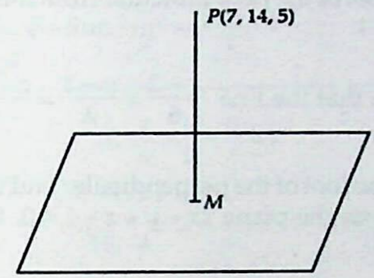

Fig. 28.13

Since R is the mid-point of PQ. Therefore, position vector of R is

$$\frac{[(1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (4 + \lambda)\hat{k}] + [\hat{i} + 3\hat{j} + 4\hat{k}]}{2}$$

$$= (\lambda + 1)\hat{i} + \left(3 - \frac{\lambda}{2}\right)\hat{j} + \left(4 + \frac{\lambda}{2}\right)\hat{k}$$

Since R lies on the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$.

$$\therefore \quad \left\{(\lambda + 1)\hat{i} + \left(3 - \frac{\lambda}{2}\right)\hat{j} + \left(4 + \frac{\lambda}{2}\right)\hat{k}\right\} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$$

$$\Rightarrow \quad 2\lambda + 2 - 3 + \frac{\lambda}{2} + 4 + \frac{\lambda}{2} + 3 = 0$$

$$\Rightarrow \quad \lambda = -2.$$

Thus, the position vector of Q is

$$(\hat{i} + 3\hat{j} + 4\hat{k}) - 2(2\hat{i} - \hat{j} + \hat{k}) = -3\hat{i} + 5\hat{j} + 2\hat{k}.$$

EXERCISE 28.11

1. Find the image of the point (0, 0, 0) in the plane $3x + 4y - 6z + 1 = 0$.
2. Find the reflection of the point (1, 2, – 1) in the plane $3x - 5y + 4z = 5$.
3. Find the coordinates of the foot of the perpendicular drawn from the point (5, 4, 2) to the line $\dfrac{x+1}{2} = \dfrac{y-3}{3} = \dfrac{z-1}{-1}$.
   Hence or otherwise deduce the length of the perpendicular.

4. Find the image of the point with position vector $3\hat{i}+\hat{j}+2\hat{k}$ in the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=4$.
5. Find the coordinates of the foot of the perpendicular from the point (1, 1, 2) to the plane $2x-2y+4z+5=0$. Also, find the length of the perpendicular.
6. Find the distance of the point $(1,-2,3)$ from the plane $x-y+z=5$ measured along a line parallel to $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$. **[CBSE 2008]**
7. Find the coordinates of the foot of the perpendicular from the point (2, 3, 7) to the plane $3x-y-z=7$. Also, find the length of the perpendicular.
8. Find the image of the point (1, 3, 4) in the plane $2x-y+z+3=0$.
9. Find the distance of the point with position vector $-\hat{i}-5\hat{j}-10\hat{k}$ from the point of intersection of the line $\vec{r}=(2\hat{i}-\hat{j}+2\hat{k})+\lambda(3\hat{i}+4\hat{j}+12\hat{k})$ with the plane $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=5$.]
10. Find the length and the foot of the perpendicular from the point (1, 1, 2) to the plane $\vec{r}\cdot(\hat{i}-2\hat{j}+4\hat{k})+5=0$. **[CBSE 2002C]**
11. Find the value of λ such that the line $\frac{x-2}{6}=\frac{y-1}{\lambda}=\frac{z+5}{-4}$ is perpendicular to the plane $3x-y-2z=7$. **[CBSE 2010]**
12. Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point $P(3,2,1)$ from the plane $2x-y+z+1=0$. Find also the image of the point in the plane. **[CBSE 2010]**

ANSWERS

1. $\left(\frac{-6}{61},\frac{-8}{61},\frac{12}{61}\right)$ 2. $\left(\frac{73}{25},\frac{-6}{5},\frac{39}{25}\right)$ 3. $(1,6,0)$, $2\sqrt{6}$ 4. $(1,2,1)$
5. $\left(-\frac{1}{12},\frac{25}{12},-\frac{1}{6}\right)$, $\frac{13}{\sqrt{24}}$ 6. 1 7. $(5,2,6)$; $\sqrt{11}$ 8. $(-3,5,2)$.
9. 13 10. $\frac{13}{12}\sqrt{6}$, $\left(-\frac{1}{12},\frac{25}{12},\frac{-2}{12}\right\}$ 11. 26 12. $(1,3,0)$, $\sqrt{6}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the equation of the plane parallel XOY-plane and passing through the point $(2,-3,5)$.
2. Write the equation of the plane parallel to YOZ-plane and passing through $(-4,1,0)$.
3. Write the equation of the plane passing through points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$.
4. Write the general equation of a plane parallel to X-axis.
5. Write the value of k for which the planes $x-2y+kz=4$ and $2x+5y-z=9$ are perpendicular.
6. Write the intercepts made by the plane $2x-3y+4z=12$ on the coordinate axes.
7. Write the ratio in which the plane $4x+5y-3z=8$ divides the line segment joining points $(-2,1,5)$ and $(3,3,2)$.

8. Write the distance between the parallel planes $2x - y + 3z = 4$ and $2x - y + 3z = 18$.
9. Write the plane $\vec{r}\cdot(2\hat{i} + 3\hat{j} - 6\hat{k}) = 14$ in normal form.
10. Write the distance of the plane $\vec{r}\cdot(2\hat{i} - \hat{j} + 2\hat{k}) = 12$ from the origin.
11. Write the equation of the plane $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ in scalar product form.
12. Write a vector normal to the plane $\vec{r} = l\vec{b} + m\vec{c}$.
13. Write the equation of the plane passing through $(2, -1, 1)$ and parallel to the plane $3x + 2y - z = 7$.
14. Write the equation of the plane containing the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$.
15. Write the position vector of the point where the line $\vec{r} = \vec{a} + \lambda\vec{b}$ meets the plane $\vec{r}\cdot\vec{n} = 0$.
16. Write the value of k for which the line $\dfrac{x-1}{2} = \dfrac{y-1}{3} = \dfrac{z-1}{k}$ is perpendicular to the normal to the plane $\vec{r}\cdot(2\hat{i} + 3\hat{j} + 4\hat{k}) = 4$.
17. Write the angle between the line $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z+3}{-2}$ and the plane $x + y + 4 = 0$.

ANSWERS

1. $z = 5$ 2. $x = -4$ 3. $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ 4. $by + cz + d = 0$

5. -8 6. $6, -4, 3$ 7. $2:1$ 8. $\sqrt{14}$

9. $\vec{r}\cdot\left(\dfrac{2}{7}\hat{i} + \dfrac{3}{7}\hat{j} - \dfrac{6}{7}\hat{k}\right) = 2$ 10. 4 11. $(\vec{r} - \vec{a})\cdot(\vec{b} \times \vec{c}) = 0$

12. $\vec{b} \times \vec{c}$ 14. $(\vec{r} - \vec{a})\cdot(\vec{b} \times \vec{c}) = 0$ 15. $\vec{a} - \left(\dfrac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\right)\vec{b}$ 16. $\dfrac{-13}{4}$ 17. $45°$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The Plane $2x - (1 + \lambda)y + 3\lambda z = 0$ passes through the intersection of the planes
   (a) $2x - y = 0$ and $y - 3z = 0$ (b) $2x + 3z = 0$ and $y = 0$
   (c) $2x - y + 3z = 0$ and $y - 3z = 0$ (d) none of these
2. The acute angle between the planes $2x - y + z = 6$ and $x + y + 2z = 3$ is
   (a) $45°$ (b) $60°$ (c) $30°$ (d) $75°$
3. The equation of the plane through the intersection of the planes $x + 2y + 3z = 4$ and $2x + y - z = -5$ and perpendicular to the plane $5x + 3y + 6z + 8 = 0$ is
   (a) $7x - 2y + 3z + 81 = 0$ (b) $23x + 14y - 9z + 48 = 0$
   (c) $51x - 15y - 50z + 173 = 0$ (d) none of these
4. The distance between the planes $2x + 2y - z + 2 = 0$ and $4x + 4y - 2z + 5 = 0$ is
   (a) $\dfrac{1}{2}$ (b) $\dfrac{1}{4}$ (c) $\dfrac{1}{6}$ (d) none of these
5. The image of the point $(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$ is
   (a) $(3, 5, 2)$ (b) $(-3, 5, 2)$ (c) $(3, 5, -2)$ (d) $(3, -5, 2)$
6. The equation of the plane containing the two lines
   $\dfrac{x-1}{2} = \dfrac{y+1}{-1} = \dfrac{z-0}{3}$ and $\dfrac{x}{-2} = \dfrac{y-2}{-3} = \dfrac{z+1}{-1}$ is
   (a) $8x + y - 5z - 7 = 0$ (b) $8x + y + 5z - 7 = 0$
   (c) $8x - y - 5z - 7 = 0$ (d) none of these

7. The equation of the plane $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$ in scalar product form is

(a) $\vec{r} \cdot (5\hat{i} - 2\hat{j} - 3\hat{k}) = 7$ (b) $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 7$

(c) $\vec{r} \cdot (5\hat{i} - 2\hat{j} + 3\hat{k}) = 7$ (d) none of these

8. The distance of the line $\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ from the plane $\vec{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$, is

(a) $\frac{5}{3\sqrt{3}}$ (b) $\frac{10}{3\sqrt{3}}$ (c) $\frac{25}{3\sqrt{3}}$ (d) none of these

9. The equation of the plane through the line $x + y + z + 3 = 0 = 2x - y + 3z + 1$ and parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is

(a) $x - 5y + 3z = 7$ (b) $x - 5y + 3z = -7$

(c) $x + 5y + 3z = 7$ (d) $x + 5y + 3z = -7$

10. The vector equation of the plane containing the line $\vec{r} = (-2\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - 2\hat{j} - \hat{k})$ and the point $\hat{i} + 2\hat{j} + 3\hat{k}$ is

(a) $\vec{r} \cdot (\hat{i} + 3\hat{k}) = 10$ (b) $\vec{r} \cdot (\hat{i} - 3\hat{k}) = 10$

(c) $\vec{r} \cdot (3\hat{i} + \hat{k}) = 10$ (d) none of these

11. A plane meets the coordinate axes at A, B, C such that the centroid of ΔABC is the point (a, b, c). If the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = k$, then $k =$

(a) 1 (b) 2 (c) 3 (d) none of these

12. The distance between the point $(3, 4, 5)$ and the point where the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ meets the plane $x + y + z = 17$, is

(a) 1 (b) 2 (c) 3 (d) none of these

13. A vector parallel to the line of intersection of the planes $\vec{r} \cdot (3\hat{i} - \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (\hat{i} + 4\hat{j} - 2\hat{k}) = 2$ is

(a) $-2\hat{i} + 7\hat{j} + 13\hat{k}$ (b) $2\hat{i} + 7\hat{j} - 13\hat{k}$

(c) $-2\hat{i} - 7\hat{j} + 13\hat{k}$ (d) $2\hat{i} + 7\hat{j} + 13\hat{k}$.

14. If a plane passes through the point (1, 1, 1) and is perpendicular to the line $\frac{x-1}{3} = \frac{y-1}{0} = \frac{z-1}{4}$ then its perpendicular distance from the origin is

(a) 3/4 (b) 4/3 (c) 7/5 (d) 1

15. The equation of the plane parallel to the lines $x - 1 = 2y - 5 = 2z$ and $3x = 4y - 11 = 3z - 4$ and passing through the point $(2, 3, 3)$ is

(a) $x - 4y + 2z + 4 = 0$ (b) $x + 4y + 2z + 4 = 0$

(c) $x - 4y + 2z - 4 = 0$ (d) none of these

16. The distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 12\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$ is

(a) 9 (b) 13 (c) 17 (d) none of these

17. The equation of the plane through the intersection of the planes $ax + by + cz + d = 0$ and $lx + my + nz + p = 0$ and parallel to the line $y = 0, z = 0$

(a) $(bl - am)\,y + (cl - an)\,z + dl - ap = 0$
(b) $(am - bl)\,x + (mc - bn)\,z + md - bp = 0$
(c) $(na - cl)\,x + (bn - cm)\,y + nd - cp = 0$
(d) none of these.

18. The equation of the plane which cuts equal intercepts of unit length on the coordinate axes is
(a) $x + y + z = 1$ (b) $x + y + z = 0$
(c) $x + y - z = 1$ (d) $x + y + z = 2$

ANSWERS

1. (a)	2. (b)	3. (d)	4. (c)	5. (b)	6. (a)	7. (a)	8. (b)
9. (a)	10. (a)	11. (c)	12. (c)	13. (a)	14. (c)	15. (a)	16. (b)
17. (a)	18. (a)						

SUMMARY

1. The general equation of first degree in x, y, z i.e., $ax + by + cz + d = 0$ always represents a plane.
2. In the equation $ax + by + cz + d = 0$, the direction ratios of normal to the plane are proportional to a, b, c.
3. A vector normal to the plane $ax + by + cz + d = 0$ is $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$.
4. If l, m, n are the direction cosines of normal to a plane which is at a distance p from the origin, then the cartesian equation of the plane is
$$lx + my + nz = p.$$
This is known as the normal form of a plane.
5. The vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to $\vec{n}$ is
$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \text{ or, } \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$
6. The cartesian equation of a plane passing through (x_1, y_1, z_1) and having direction ratios proportional to a, b, c for its normal is
$$a\,(x - x_1) + b\,(y - y_1) + c\,(z - z_1) = 0$$
7. The vector equation of a plane having $\hat{n}$ as a unit vector normal to it and at a distance 'd' from the origin is $\vec{r} \cdot \hat{n} = d$.
If l, m, n are direction cosines of the normal to the plane, then its vector equations
$$\vec{r} \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = d$$
This is the vector equation of the normal form of a plane.
8. The vector equation of a plane passing through points having position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is
$$\vec{r} \cdot (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$
9. A vector normal to the plane passing through points $A\,(\vec{a})$, $B\,(\vec{b})$ and $C\,(\vec{c})$ is
$$\vec{AB} \times \vec{AC} \text{ or, } \vec{BC} \times \vec{BA} \text{ or, } \vec{CB} \times \vec{CA} \quad \text{i.e., } \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$$
10. The cartesian equation of a plane intercepting lengths a, b and c with X, Y and Z axes respectively is
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$
11. The cartesian equation of a plane passing through points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \text{ or, } \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

12. The angle between two planes is defined as the angle between their normals.

(i) If $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ are two planes inclined at an angle θ, then

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$$

These planes are parallel, if $\vec{n_1}$ is parallel to $\vec{n_2}$.

These planes are perpendicular, if $\vec{n_1} \cdot \vec{n_2} = 0$

(ii) If $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are cartesian equations of two planes inclined at an angle θ, then

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

The planes are parallel, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

The planes are perpendicular, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

13. The vector equation of a plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$ is

$$\vec{r} = \vec{a} + m\,\vec{b} + n\,\vec{c}, \text{ where } m \text{ and } n \text{ are parameters.}$$

or, $\vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$

14. The vector equation of the plane passing through points having position vectors, $\vec{a}, \vec{b}$ and $\vec{c}$ is

$\vec{r} = (1 - m - n)\vec{a} + m\vec{b} + n\vec{c}$ [Parametric Form]

$\vec{r} \cdot (\vec{a} \times \vec{b}) + \vec{r} \cdot (\vec{b} \times \vec{c}) + \vec{r} \cdot (\vec{c} \times \vec{a}) = \vec{a} \cdot (\vec{b} \times \vec{c})$ [Non-parametric form]

15. The equation of a plane parallel to the plane

(a) $\vec{r} \cdot \vec{n} = d$ is $\vec{r} \cdot \vec{n} = d_1$

(b) $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$

16. The length of the perpendicular from the point $(x_1, y_1 z_1)$ to the plane $ax + by + cz + d = 0$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

and the coordinates (α, β, γ) of the foot of the perpendicular are given by

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = \frac{\gamma - z_1}{c} = -\left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right)$$

The coordinates (α, β, γ) of the image of the point (x_1, y_1, z_1) in the plane $ax + by + cz + d = 0$ are given by

$$\frac{x_1}{a} = \frac{\beta - y_1}{b} = \frac{\gamma - z_1}{c} = -2\left(\frac{ax_1 + by_1 + cz_1 + d}{a^2 + b^2 + c^2}\right)$$

17. The distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by

$$\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$$

18. The equation of the family of planes containing the line
$a_1 x + b_1 y + c_1 z + d_1 = 0 = a_2 x + b_2 y + c_2 z + d_2$ is
$\left(a_1 x + b_1 y + c_1 z + d_1\right) + \lambda\left(a_2 x + b_2 y + c_2 z + d_2\right) = 0$, where λ is a parameter.

19. The equations of the planes bisecting the angles between the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are given by

$$\frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{a^2+b^2+c^2}} = \pm \frac{a_2 x + b_2 y + c_2 z + d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

20. The angle θ between a line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and a plane $ax+by+cz+d=0$ is the complement of the angle between the line and normal to the plane and is given by

$$\sin\theta = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

The angle θ between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r} \cdot \vec{n} = d$ is given by

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|}$$

A line is parallel to a plane if it is perpendicular to the normal to the plane.
A line is perpendicular to a plane if it is parallel to the normal to the plane.

21. The line $\vec{r} = \vec{a} + \lambda\vec{b}$ lies in the plane $\vec{r}\cdot\vec{n} = d$, if

$$\vec{a}\cdot\vec{n} = d \text{ and } \vec{b}\cdot\vec{n} = 0$$

The line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ lies in the plane $ax+by+cz+d=0$, if

$$ax_1+by_1+cz_1+d=0 \text{ and } al+bm+cn = 0$$

22. The equation of a plane containing the line

$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$, where $al+bm+cn=0$.

23. Two lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and, $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar, if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing them is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \text{ or, } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

24. Two lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are coplanar, if

$$\vec{a_1}\cdot(\vec{b_1}\times\vec{b_2}) = \vec{a_2}\cdot(\vec{b_1}\times\vec{b_2})$$

REVISION EXERCISE ON 3-D

1. If a line makes angles of 90°, 60° and 30° with the positive direction of x, y, and z-axis respectively, find its direction cosines.
2. If a line has direction ratios 2, – 1, – 2, determine its direction cosines.
3. Find the direction cosines of the line passing through the two points (– 2, 4, – 5) and (1, 2, 3).
4. Using direction ratios show that the points $A(2, 3, -4)$, $B(1, -2, 3)$ and $C(3, 8, -11)$ are collinear.
5. Find the direction cosines of the sides of the triangle whose vertices are (3, 5, – 4), (– 1, 1, 2) and (– 5, – 5, – 2).
6. Find the vector and cartesian equations of the line through the point (5, 2, – 4) and which is parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$.
7. Find the vector equation of the line passing through the points (– 1, 0, 2) and (3, 4, 6).
8. Show that the three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$; $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$; $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.
9. Show that the line through the points (1, – 1, 2) and (3, 4, – 2) is perpendicular to the through the points (0, 3, 2) and (3, 5, 6).
10. Show that the line through the points (4, 7, 8) and (2, 3, 4) is parallel to the line through the points (– 1, – 2, 1), (1, 2, 5).
11. Find the cartesian equation of the line which passes through the point (– 2, 4, – 5) and parallel to the line given by
$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$
12. Find the distance between the lines l_1 and l_2 given by
$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \text{ and, } \vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$$
13. Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.
14. Find the shortest distance between the lines
    (i) $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and , $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$
    (ii) $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$
    (iii) $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$
    (iv) $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$
15. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, – 1) and (4, 3, – 1).
16. Find the vector equation of the line passing through the point (1, 2, – 4) and perpendicular to the lines:
$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$
17. Find the equation of a line parallel to x-axis and passing through the origin.

18. Find the vector equation of the plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and its normal vector from the origin is $2\hat{i} - 3\hat{j} + 4\hat{k}$. Also, find its cartesian form.
19. Find the direction cosines of the unit vector perpendicular to the plane $\vec{r}\cdot(6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$ passing through the origin.
20. Find the distance of the plane $2x - 3y + 4z - 6 = 0$ from the origin.
21. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x - 3y + 4z - 6 = 0$.
22. Find the vector and cartesian equations of the plane which passes through the point $(5, 2, -4)$ and perpendicular to the line with direction ratios $2, 3, -1$.
23. Find the vector equation of the plane passing through the points $P(2, 5, -3)$, $Q(-2, -3, 5)$ and $R(5, 3, -3)$.
24. Find the vector equation of the plane passing through the intersection of the planes $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 6$ and $\vec{r}\cdot(2\hat{i} + 3\hat{j} + 4\hat{k}) = -5$ and the point $(1, 1, 1)$.
25. Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} \text{ and } \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$$

are coplanar.
26. Find the angle between the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$.
27. Find the distance of a point $(2, 5, -3)$ from the plane $\vec{r}\cdot(6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$.
28. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $10x + 2y - 11z = 3$.
29. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.
30. Find the equation of the plane passing through the points $(1, 1, -1)$, $(6, 4, -5)$ and $(-4, -2, 3)$.
31. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.
32. Find the equation of the plane through the intersection of the planes $3x - y + 2z = 4$ and $x + y + z = 2$ and the point $(2, 2, 1)$.
33. Find the vector equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.
34. Find the equation of the plane that contains the point $(1, -1, 2)$ and is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$.
35. Find the vector equation of the line passing through $(1, 2, 3)$ and perpendicular to the plane $\vec{r}\cdot(\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.
36. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 2$.
37. Find the coordinates of the point where the line through $(5, 1, 6)$ and $(3, 4, 1)$ crosses the (i) yz-plane (ii) zx-plane.
38. Find the coordinates of the point where the line through $(3, -4, -5)$ and $(2, -3, 1)$ crosses the plane $2x + y + z = 7$.
39. Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

40. If the points $(1, 1, p)$ and $-3, 0, 1)$ be equidistant from the plane $\vec{r}\cdot(3\hat{i}+4\hat{j}-12\hat{k})+13=0$, then find the value of p.

41. Find the equation of the plane passing through the intersection of the planes $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})+4=0$ and parallel to x-axis.

42. If O be the origin and the coordinates of P be $(1, 2, -3)$, then find the equation of the plane passing through P and perpendicular to OP.

43. Find the vector equation of the line passing through $(1, 2, 3)$ and parallel to the planes $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=5$ and $\vec{r}\cdot(3\hat{i}+\hat{j}+\hat{k})=6$.

44. Find the equation of the plane which contains the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=4$, $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$.

ANSWERS

1. $0, \frac{1}{2}, \frac{\sqrt{3}}{2}$ **2.** $\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$ **3.** $\frac{3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{8}{\sqrt{77}}$

5. $\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}; \frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}; \frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}$

6. $\vec{r}=5\hat{i}+2\hat{j}-4\hat{k}+\lambda(3\hat{i}+2\hat{j}-8\hat{k})$; $\frac{x-5}{3}=\frac{y-2}{2}=\frac{z+4}{-8}$

7. $\vec{r}=-\hat{i}+2\hat{k}+\lambda(4\hat{i}+4\hat{j}+4\hat{k})$ **11.** $\frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}$

12. $\frac{\sqrt{293}}{7}$ units **14.** (i) $\frac{3\sqrt{2}}{2}$ (ii) $2\sqrt{29}$ (iii) $\frac{3}{\sqrt{19}}$ (iv) 9

16. $\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$ **17.** $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$

18. $\vec{r}\cdot\left(\frac{2}{\sqrt{29}}\hat{i}-\frac{3}{\sqrt{29}}\hat{j}+\frac{4}{\sqrt{29}}\hat{k}\right)=\frac{6}{\sqrt{29}}$, $2x-3y+4z=6$

19. $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$ **20.** $6\sqrt{29}$ **21.** $\left(\frac{12}{29}, \frac{-18}{29}, \frac{24}{29}\right)$

22. $2x+3y-z=20$ **23.** $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})+23=0$

24. $\vec{r}\cdot(20\hat{i}+23\hat{j}+26\hat{k})=69$ **26.** $\cos^{-1}\left(\frac{4}{21}\right)$ **27.** $\frac{13}{7}$

28. $\sin^{-1}\left(\frac{8}{21}\right)$ **29.** $\vec{r}\cdot\frac{(3\hat{i}+5\hat{j}-6\hat{k})}{\sqrt{70}}=7$

30. These points are collinar. There will be infinite number of planes passing through these points. Their equations are given $a(x-1)+b(y-1)+c(z+1)=0$, where $5a+3b-4c=0$.

31. $y=3$ **32.** $7x-5y+4z=8$ **33.** $x-z+2=0$

34. $5x-4y-z=7$ **35.** $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(\hat{i}+2\hat{j}-5\hat{k})$ **36.** $x+y+z=a+b+c$

37. (i) $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$ (ii) $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$

38. $(1, -2, 7)$ **39.** $7x-8y+3z+25=0$ **40.** $p=1, \frac{7}{3}$

41. $y-3z+6=0$ **42.** $x+2y-3z=14$

43. $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(-3\hat{i}+5\hat{j}+4\hat{k})$ **44.** $33x+45y+50z=41$

29

LINEAR PROGRAMMING

29.1 INTRODUCTION

The term 'programming' means planning and it refers to a particular plan of action from amongst several alternatives for maximizing profit or minimizing cost etc. Programming problems deal with determining optimal allocation of limited resources to meet the given objectives, such as least cost, maximum profit, highest margin or least time, when resources have alternative uses.

The term 'Linear' means that all inequations or equations used and the function to be maximized or minimized are linear. That is why linear programming deals with that class of problems for which all relations among the variables involved are linear.

Formally, linear programming deals with the optimization (maximization or minimization) of a linear function of a number of variables subject to a number of conditions on the variables, in the form of linear inequations or equations in variables involved.

In this chapter, we shall discuss mathematical formulation of linear programming problems that arise in trade, industry, commerce and military operations. We shall also discuss some elementary techniques to solve linear programming problems in two variables only.

29.2 LINEAR PROGRAMMING PROBLEMS

In this section, we shall discuss the general form of a linear programming problem. To give the general description of a linear programming problem, let us consider the following problem :

Suppose that a furniture dealer makes two products viz. chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 4 hours on machine A and 2 hours on machine B. There are 16 hours of time per day available on machine A and 20 hours on machine B. Profits gained by the manufacturer from a chair and a table are Rs 30 and Rs 50 respectively. The manufacturer is willing to know the daily product of each of the two products to maximize his profit.

The above data can be put in the following tabular form :

Item	*Chair*	*Table*	*Maximum available time*
Machine A	2 hrs	4 hrs	16 hrs
Machine B	6 hrs	2 hrs	20 hrs
Profit (in Rs)	Rs 30	Rs 50	

To maximize his profit, suppose that the manufacturer produces x chairs and y tables per day. It is given that a chair requires 2 hours on machine A and a table requires 4 hours on machine A. Hence, the total time taken by machine A to produce x chairs and y tables is $2x + 4y$. This must be less than or equal to the total hours available on machine A. Hence, $2x + 4y \le 16$. Similarly, for machine B, we have

$$6x + 2y \le 20.$$

The total profit for x chairs and y tables is $30x + 50y$. Since the number of chairs and tables is never negative. Therefore, $x \ge 0$ and $y \ge 0$.

Thus, we have to maximize

$$\text{Profit} = 30x + 50y$$

Subject to the constraints

$$2x + 4y \le 16$$
$$6x + 2y \le 20$$
$$x \ge 0,\ y \ge 0$$

Out of all the points (x, y) in the solution set of the above linear constraints, the manufacturer has to choose that point, or those points for which the profit $30x + 50y$ has the maximum value.

In the above discussion if a chair costs Rs 250 and a table costs Rs 300 then the total cost of producing x chairs and y tables is $250x + 300y$. Now, the manufacturer will be interested to choose that point, or those points, in the solution set of the above linear constraints for which the cost $250x + 300y$ has the minimum value.

The two situations discussed above give the description of a type of linear programming problems. In the above discussion, the profit function $= 30x + 50y$ or the cost function $= 250x + 300y$ is known as the objective function. The inequations $2x + 4y \le 16$, $6x + 2y \le 20$ are known as the constraints and $x \ge 0$, $y \ge 0$ are known as the non-negativity restrictions.

The general mathematical description of a linear programming problem (LPP) is given below :

$$\text{Optimize } Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \qquad \textit{(objective function)}$$

Subject to

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots + a_{1n} x_n \ (\le, =, \ge)\ b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots + a_{2n} x_n \ (\le, =, \ge)\ b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad\qquad \textit{(constraints)}$$
$$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \ldots + a_{mn} x_n \ (\le, =, \ge)\ b_m$$
$$x_1, x_2, x_3, \ldots, x_n \ge 0 \qquad \textit{(non-negativity restrictions)}$$

where all a_{ij}'s, b_i's and c_j's are constants and x_j's are variables.

The above linear programming problem may also be written in the matrix form as follows :

$$\text{Optimize (maximize or minimize) } Z = [c_1\ c_2 \ldots\ldots c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Subject to

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \{\le, =, \ge\} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1, x_2, x_3, \ldots, x_n \ge 0$$

OR

Optimize (Maximize or Minimize)

$$Z = CX$$

Subject to $AX \ (\le, =, \ge) \ B$

$$X \ge 0,$$

where $C = [c_1 \ c_2 \ldots c_n]$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

29.3 SOME DEFINITIONS

In this section, we shall formally define various terms used in a linear programming problem.

As discussed in the previous section, the general form of a linear programming problem is

Optimize (Maximize or Minimize) $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

Subject to

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \ (\le, =, \ge) \ b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \ (\le, =, \ge) \ b_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \ \{\le, =, \ge\} \ b_m$$

$$x_1, x_2, \ldots, x_n \ge 0$$

The definitions of various terms related to the LPP are as follows :

OBJECTIVE FUNCTION *If $c_1, c_2, \ldots, c_n$ are constants and $x_1, x_2, \ldots, x_n$ are variables, then the linear function $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ which is to be maximized or minimized is called the objective function.*

The objective function describes the primary purpose of the formulation of a linear programming problem and it is always non-negative. In business applications, the profit function which is to be maximized or the cost function which is to be minimized is called the objective function.

CONSTRAINTS *The inequations or equations in the variables of a LPP which describe the conditions under which the optimisation (maximization or minimization) is to be accomplished are called constraints.*

In the constraints given in the general form of a LPP there may be any one of the three signs $\le, =, \ge$.

Inequations in the form of greater than (or less than) indicate that the total use of the resources must be more than (or less than) the specified amount whereas equations in the constraints indicate that the resources described are to be fully used.

NON-NEGATIVITY RESTRICATIONS *These are the constraints which describe that the variables involved in a LPP are non-negative.*

29.4 MATHEMATICAL FORMULATION OF LINEAR PROGRAMMING PROBLEMS

In the previous section, we have introduced the general form of a linear programming problem (LPP). In this section, we shall discuss the formulation of linear programming problems. Problem formulation is the process of transforming the verbal description of a decision problem into a mathematical form. There is not any set procedure to formulate linear programming problems. In fact, one can only learn the formulation with adequate practice. However, the following algorithm will be helpful in the formulation of linear programming problems.

ALGORITHM

STEP I *In every LPP certain decisions are to be made. These decisions are represented by decision variables. These decision variables are those quantities whose values are to be determined. Identify the variables and denote them by $x_1, x_2, x_3, \ldots$*

STEP II *Identify the objective function and express it as a linear function of the variables introduced in step I.*

STEP III *In a LPP, the objective function may be in the form of maximizing profits or minimizing costs. So, after expressing the objective function as a linear function of the decision variables, we must find the type of optimization i.e. maximization or minimization. Identify the type of the objective function.*

STEP IV *Identify the set of constrainsts, stated in terms of decision variables and express them as linear inequations or equations as the case may be..*

The following examples will illustrate the formulation of linear programming problems in various situations.

ILLUSTRATIVE EXAMPLES

Type I **OPTIMAL PRODUCT LINE PROBLEMS**

EXAMPLE 1 *A factory produces two products P_1 and P_2. Each of the product P_1 requires 2 hrs for moulding, 3 hrs for grinding and 4 hrs for polishing, and each of the product P_2 requires 4 hrs for moulding, 2 hrs for grinding and 2 hrs for polishing. The factory has moulding machine available for 20 hrs, grinding machine for 24 hrs and polishing machine available for 13 hrs. The profit is Rs. 5 per unit of P_1 and Rs. 3 per unit of P_2 and the factory can sell all that it produces. Formulate the problem as a linear programing problem to maximize the profit.*

SOLUTION The given data may be put in the following tabular form :

Product / *Resources*	P_1	P_2	*Capacity*
Moulding	2	4	20
Grinding	3	2	24
Polishing	4	2	13
Profit	5	3	

Suppose x units of product P_1 and y units of product P_2 are produced to maximize the profit. Let Z denote the total profit.
Since each unit of product P_1 requires 2 hrs for moulding and each unit of product P_2 requires 4 hrs for moulding. Hence, the total hours required for moulding for x units of product P_1 and y units of product P_2 are $2x + 4y$. This must be less than or equal to the total hours available for moulding. Hence,

$$2x + 4y \leq 20$$

This is the first constraint.

The total hours required for grinding for x units of product P_1 and y units of product P_2 is $3x + 2y$. But the maximum number of hours available for grinding is 24.

$$\therefore \quad 3x + 2y \leq 24$$

This is the second constraint.

Similarly for polishing the constraint is $4x + 2y \leq 13$.

Since x and y are non-negative integers, therefore $x \geq 0,\ y \geq 0$

The total profit for x units of product P_1 and y units of product P_2 is $5x + 3y$. Since we wish to maximize the profit, therefore the objective function is

$$\text{Maximize} \quad Z = 5x + 3y$$

Hence, the linear programming problem for the given problem is as follows

$$\textit{Maximize} \quad Z = 5x + 3y$$

Subject to the constraints

$$2x + 4y \leq 20,$$

$$3x + 2y \leq 24,$$

$$4x + 2y \leq 13,$$

and, $\quad x \geq 0,\ y \geq 0$

EXAMPLE 2 *A toy company manufactures two types of doll; a basic version doll A and a deluxe version doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2,000 per day if it produces only the basic version. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes profit of Rs 3 and Rs 5 per doll respectively on doll A and doll B; how many of each should be produced per day in order to maximize profit?*

SOLUTION Let x dolls of type A and y dolls of type B be produced per day. Then,

$$\text{Total profit} = 3x + 5y.$$

Since each doll of type B takes twice as long to produce as one of type A, therefore total time taken to produce x dolls of type A and y dolls of type B is
$x + 2y$. But the company has time to make a maximum of 2000 dolls per day

$$\therefore \quad x + 2y \leq 2000$$

Since plastic is available to produce 1500 dolls only.

$$\therefore \quad x + y \leq 1500$$

Also fancy dress is available for 600 dolls per day only

$$\therefore \quad y \leq 600$$

Since the number of dolls cannot be negative. Therefore,

$$x \geq 0,\ y \geq 0$$

Hence, the linear programming problem for the given problem is as follows :

$$\textit{Maximize}\quad Z = 3x + 5y$$

Subject to the constraints

$$x + 2y \leq 2000,$$
$$x + y \leq 1500,$$
$$y \leq 600$$

and, $\quad x \geq 0,\ y \geq 0$

EXAMPLE 3 *A firm can produce three types of cloth, say C_1, C_2, C_3. Three kinds of wool are required for it, say red wool, green wool and blue wool. One unit of length C_1 needs 2 metres of red wool, 3 metres of blue wool; one unit of cloth C_2 needs 3 metres of red wool, 2 metres of green wool and 2 metres of blue wool; and one unit of cloth C_3 needs 5 metres of green wool and 4 metres of blue wool. The firm has only a stock of 16 metres of red wool, 20 metres of green wool and 30 metres of blue wool. It is assumed that the income obtained from one unit of length of cloth C_1 is Rs. 6, of cloth C_2 is Rs. 10 and of cloth C_3 is Rs. 8. Formulate the problem as a linear programming roblem to maximize the income.*

OLUTION The given information can be put in the following tabular form :

	Cloth C_1	*Cloth C_2*	*Cloth C_3*	*Total quality of wool available*
Red Wool	2	3	0	16
Green Wool	0	2	5	20
Blue Wool	3	2	4	30
Income (in Rs.)	6	10	8	

Let x_1, x_2 and x_3 be the quantity produced in metres of the cloth of type C_1, C_2 and C_3 respectively.

Since 2 metres of red wool are required for one metre of cloth C_1 and x_1 metres of cloth C_1 are produced, therefore $2x_1$ metres of red wool will be required for cloth C_1. Similarly, cloth C_2 requires $3x_2$ metres of red wool and cloth C_3 does not require red wool. Thus, the total quantity of red wool required is $2x_1 + 3x_2 + 0x_3$

But the maximum available quantity of red wool is 16 metres.

$\therefore \quad 2x_1 + 3x_2 + 0x_3 \leq 16$

Similarly, the total quantities of green and blue wool required are

$0x_1 + 2x_2 + 5x_3$ and $3x_1 + 2x_2 + 4x_3$ respectively.

But the total quantities of green and blue wool available are 20 metres and 30 metres respectively.

$\therefore \quad 0x_1 + 2x_2 + 5x_3 \leq 20$ and $3x_1 + 2x_2 + 4x_3 \leq 30$

Also, we cannot produce negative quantities, therefore

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

The total income is $\quad Z = 6x_1 + 10x_2 + 8x_3$

Hence, the linear programming problem for the given problem is

$$\textit{Maximize}\quad Z = 6x_1 + 10x_2 + 8x_3$$

Subject to the constraints

$$2x_1 + 3x_2 + 0x_3 \le 16\,,$$
$$0x_1 + 2x_2 + 5x_3 \le 20\,,$$
$$3x_1 + 2x_2 + 4x_3 \le 30\,,$$

and, $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$

EXAMPLE 4 *A furniture firm manufactures chairs and tables, each requiring the use of three machines A, B* and C. *Production of one chair requires 2 hours on machine A, 1 hour on machine B, and 1 hour on machine C. Each table requires 1 hour each on machines A and B and 3 hours on machine C. The profit realized by selling one chair is Rs 30 while for a table the figure is Rs 60. The total time available per week on machine A is 70 hours, on machine B is 40 hours, and on machine C is 90 hours. How many chairs and tables should be made per week so as to maximize profit ? Develop a mathematical formulation.*

SOLUTION The given data may be put in the following tabular form :

Machine	*Chair*	*Table*	*Available time per week (in hours)*
A	2	1	70
B	1	1	40
C	1	3	30
Profit per unit	Rs 30	Rs 60	

Let x chairs and y tables be produced per week to maximize the profit. Then, the total profit for x chairs and y tables is $30x + 60y$.

It is given that a chair requires 2 hours on machine A and a table requires 1 hour on machine A. Therefore, the total time taken by machine A to produce x chairs and y tables is $(2x + y)$ hours. This must be less than or equal to total hours available on machine A.

$\therefore \quad 2x + y \le 70$

Similarly, the total time taken by machine B to produce x chairs and y tables is $(x + y)$ hours. But the total time available per week on machine B is 40 hours.

Finally, the total time taken by machine C to produce x chairs and y tables is $x + 3y$ hours and the total time available per week on machine C is 90 hours.

$\therefore \quad x + 3y \le 90$

Since the number of chairs and tables cannot be negative.

$\therefore \quad x \ge 0$ and $y \ge 0$

Let Z denote the total profit. Then,

$$Z = 30x + 60y$$

Hence, the mathematical form of the given LPP is as follows :

Maximize $Z = 30x + 60y$

Subject to

$$2x + y \le 70$$
$$x + y \le 40$$

$$x + 3y \leq 90$$

and, $\quad x \geq 0,\ y \geq 0$

EXAMPLE 5 *A manufacturer of a line of patent medicines is preparing a production plan on medicines A and B. There are sufficient ingredients available to make 20,000 bottles of A and 40,000 bottles of B but there are only 45,000 bottles into which either of the medicines can be put. Further more, it takes 3 hours to prepare enough material to fill 1000 bottles of A, it takes one hour to prepare enough material to fill 1000 bottles of B and there are 66 hours available for this operation. The profit is Rs 8 per bottle for A and Rs 7 per bottle for B. Formulate this problem as a linear programming problem.*

SOLUTION Suppose the manufacturer produces x bottles of medicines A and y bottles of medicine B.

Since the profit is Rs 8 per bottle for A and Rs 7 per bottle for B. So, total profit in producing x bottles of medicine A and y bottles of medicine B is Rs $(8x + 7y)$.

Let Z denote the total profit. Then,

$$Z = 8x + 7y$$

Since 1000 bottles of medicine A are prepared in 3 hours. So,

Time required to prepare x bottles of medicine $A = \frac{3x}{1000}$ hours.

It is given that 1000 bottles of medicine B are prepared in 1 hour.

$\therefore$ Time required to prepare y bottles of medicine $B = \frac{y}{1000}$ hours.

Thus, total time required to prepare x bottles of medicine A and y bottles of medicine B is $\frac{3x}{1000} + \frac{y}{1000}$ hours. But, the total time available for this operation is 66 hours.

$$\therefore \quad \frac{3x}{1000} + \frac{y}{1000} \leq 66$$

$$\Rightarrow \quad 3x + y \leq 66000$$

Since there are only 45,000 bottles into which the medicines can be put.

$$\therefore \quad x + y \leq 45{,}000$$

It is given that the ingredients are available for 20,000 bottles of A and 40,000 bottles of B.

$$\therefore \quad x \leq 20{,}000 \text{ and } y \leq 40{,}000$$

Since the number of bottles can not be negative. Therefore, $x \geq 0$, $y \geq 0$.

Hence, the mathematical formulation of the given LPP is as follows :

Maximize $Z = 8x + 7y$

Subject to

$$3x + y \leq 66{,}000$$

$$x + y \leq 45{,}000$$

$$x \leq 20{,}000$$

$$y \leq 40{,}000$$

and, $\quad x > 0, y \geq 0.$

EXAMPLE 6 *A resourceful home decorator manufactures two types of lamps say A and B. Both lamps go through two technicians, first a cutter, second a finisher. Lamp A requires 2 hours of the cutter's time and 1 hour of the finisher's time. Lamp B requires 1 hour of cutter's and 2 hours of finisher's time. The cutter has 104 hours and finisher has 76 hours of time available each month. Profit on one lamp A is Rs 6.00 and on one lamp B is Rs 11.00. Assuming that he can sell all that he produces, how many of each type of lamps should he manufacture to obtain the best return.*

SOLUTION The above information can be put in the following tabular form :

Lamp	*Cutter's time*	*Finisher's time*	*Profit in Rs*
A	2	1	6
B	1	2	11
Maximum time available	104	76	

Let the decorator manufacture x lamps of type A and y lamps of type B.

$\therefore$ Total profit = Rs $(6x + 11y)$

Total time taken by the cutter in preparing x lamps of type A and y lamps of type B is $(2x + y)$ hours. But the cutter has 104 hours only for each month.

$\therefore \quad 2x + y \leq 104.$

Similarly, the total time taken by the finisher in preparing x lamps of type A and y lamps of type B is $(x + 2y)$ hours. But the cutter has 76 hours only for each month.

$\therefore \quad x + 2y \leq 76.$

Since the number of lamps cannot be negative.

$\therefore \quad x \geq 0$ and $y \geq 0$.

Let Z denote the total profit. Then, $Z = 6x + 11y$.

Since the profit is to be maximized. So, the mathematical formulation of the given LPP is as follows :

Maximize $Z = 6x + 11y$

Subject to

$$2x + y \leq 104$$
$$x + 2y \leq 76$$

and, $x \geq 0, \; y \geq 0$

EXAMPLE 7 *A company makes two kinds of leather belts, A and B. Belt A is high quality belt, and B is of lower quality. The respective profits are Rs 4 and Rs 3 per belt. Each belt of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 1000 belts per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle, and only 400 buckles per day are available. There are only 700 buckles available for belt B. What should be the daily production of each type of belt ? Formulate the problem as a LPP.*

SOLUTION Suppose the company makes per day x belts of type A and y belts of type B.

$\therefore$ Profit $= 4x + 3y$.

Let Z denote the profit. Then, $Z = 4x + 3y$ and it is to be maximized.

It is given that 1000 belts of type B can be made per day and each belt of type A requires twice as much time as a belt of type B. So, 500 belts of type A can be made in a day.

So, total time taken in preparing x belts of type A and y belts of type B is $\left(\frac{x}{500}+\frac{y}{1000}\right)$. But the company is making x belts of type A and y belts of type B in a day.

$$\therefore \quad \frac{x}{500}+\frac{y}{1000} \leq 1 \Rightarrow 2x+y \leq 1000.$$

Since the supply of leather is sufficient for only 800 belts per day.

$$\therefore \quad x+y \leq 800.$$

It is given that only 400 fancy buckles for type A and 700 buckles for type B are available per day.

$$\therefore \quad x \leq 400, \ y \leq 700.$$

Finally, the number of belts cannot be negative.

$$\therefore \quad x \geq 0 \text{ and } y \geq 0.$$

Thus, the mathematical formulation of the given LPP is as follows :

$$\text{Maximize } Z = 4x+3y$$

Subject to

$$2x+y \leq 1000$$

$$x+y \leq 800$$

$$x \leq 400$$

$$y \leq 700$$

and, $\quad x \geq 0, \ y \geq 0.$

Type II DIET PROBLEMS

EXAMPLE 8 *A dietician whishes to mix two types of food in such a way that the vitamin contents of the mixture contain at least 8 units of Vitamin A and 10 units of vitamin C. Food 'I' contains 2 units per kg of vitamin A and 1 unit per kg of vitamin C while food 'II' contains 1 unit per kg of vitamin A and 2 units per kg of vitamin C. It costs Rs 5.00 per kg to purchase food 'I' and Rs 7.00 per kg to produce food 'II'. Formulate the above linear programming problem to minimize the cost of such a mixture.*

SOLUTION The gives data may be put in the following tabular form :

Resources	*Food*		*Requirements*
	I	II	
Vitamin A	2	1	8
Vitamin C	1	2	10
Cost (in Rs)	5	7	

Let the dietician mix x kg of food 'I' and y kg of food 'II'.

Clearly, $x \geq 0, y \geq 0$.

Since one kg of food 'I' costs Rs 5 and one kg of food 'II' costs Rs 7. Therefore, total cost of x kg of food 'I' and y kg of food 'II' is Rs $(5x+7y)$.

Let Z denote the total cost. Then,

$$Z = 5x + 7y$$

Since one kg of food 'I' contains 2 units of vitamin A. Therefore, x kg of food 'I' contain $2x$ units of vitamin A. One kg of food 'II' contains one unit of vitamin A. So, y kg of food 'II' contains y units of vitamin A. Thus, x kg of food 'I' and y kg of food 'II' contain $2x + y$ units of vitamin A. But the minimum requirement of vitamin A is 8 units.

$\therefore \quad 2x + y \geq 8$

Similarly, total amount of vitamin C supplied by x units of food 'I' and y units of food 'II' is $(x + 2y)$ units and the minimum requirement of vitamin C is 10 units.

$\therefore \quad x + 2y \geq 10$

Hence, the mathematical model of the LPP is as follows :

$$\textit{Minimize } Z = 5x + 7y$$

Subject to

$$2x + y \geq 8$$

$$x + 2y \geq 10$$

and, $\quad x, y \geq 0.$

EXAMPLE 9 *A diet is to contain at least 400 units of carbohydrate, 500 units of fat, and 300 units of protein. Two foods are available : F_1, which costs Rs 2 per unit, and F_2, which costs Rs 4 per unit. A unit of food F_1 contains 10 units of carbohydrate, 20 units of fat, and 15 units of protein; a unit of food F_2 contains 25 units of carbohydrate, 10 units of fat, and 20 units of protein. Find the minimum cost for a diet that consists of a mixture of these two foods and also meets the minimum nutrition requirements. Formulate the problem as a linear programming problem.*

SOLUTION The given data may be put in the following tabular form :

Food	*Carbohydrate*	*Fat*	*Protein*	*Cost per unit*
F_1	10	20	15	Rs 2
F_2	25	10	20	Rs 4
Minimum requirement	400	500	300	

Suppose the diet contains x units of food F_1 and y units of food F_2.

Since one unit of food F_1 costs Rs 2 and one unit of food F_2 costs Rs 4. Therefore, total cost of x units of food F_1 and y units of food F_2 is Rs $(2x + 4y)$.

Let Z denote the total cost. Then,

$$Z = 2x + 4y.$$

Since each unit of food F_1 contains 10 units of carbohydrate. Therefore, x units of food F_1 contain $10x$ units of carbohydrate. A unit of food F_2 contains 25 units of carbohydrate. So, y units of food F_2 contain $25y$ units of carbohydrate.

Thus, x units of food F_1 and y units of food F_2 contain $10x + 25y$ units of carbohydrate. But, the minimum requirement of carbohydrate is 400 units.

$\therefore \quad 10x + 25y \geq 400.$

Similarly, the total amount of fat supplied by x units of Food F_1 and y units of food F_2 is $20x + 10y$ and the minimum requirement is of 500 units.

$\therefore \quad 20x + 10y \geq 500.$

Finally, the total amount of protein supplied by x units of food F_1 and y units of food F_2 is $15x + 20y$. But the minimum requirement of protein is of 300 units.

$\therefore \quad 15x + 20y \geq 300.$

Clearly, $x \geq 0$ and $y \geq 0$.

Since we have to minimize the total cost $Z = 2x + 4y$.

Thus, the mathematical form of the given LPP is as follows :

$$\textit{Minimize } Z = 2x + 4y$$

Subject to

$$10x + 25y \geq 400$$
$$20x + 10y \geq 500$$
$$15x + 20y \geq 300$$
$$x, y \geq 0.$$

EXAMPLE 10 *The objective of a diet problem is to ascertain the quantities of certain foods that should be eaten to meet certain nutritional requirement at minimum cost. The consideration is limited to milk, beaf and eggs, and to vitamins A, B, C. The number of milligrams of each of these vitamins contained within a unit of each food is given below :*

Vitamin	*Litre of milk*	*Kg of beaf*	*Dozen of eggs*	*Minimum daily requirements*
A	1	1	10	1 mg
B	100	10	10	50 mg
C	10	100	10	10 mg
Cost	Rs 1.00	Rs 1.10	Re 0.50	

What is the linear programming formulation for this problem ?

SOLUTION Let the daily diet consists of x litres of milk, y kgs of beaf and z dozens of eggs. Then,

Total cost per day = Rs $(x + 1.10y + 0.50z)$.

Let Z denote the total cost in Rs. Then,

$$Z = x + 1.10y + 0.50z$$

Total amount of vitamin A in the daily diet is

$$(x + y + 10z) \text{ mg}$$

But the minimum requirement is 1 mg of vitamin A.

$\therefore \quad x + y + 10z \geq 1$

Similarly, total amounts of vitamins B and C in the daily diet are $(100x + 10y + 10z)$ mg and $(10x + 100y + 10z)$ mg respectively and their minimum requirements are of 50 mg and 10 mg respectively.

$\therefore \quad 100x + 10y + 10z \geq 50$

and, $\quad 10x + 100y + 10z \geq 10$

Finally, the quantity of milk, kgs of beaf and dozens of eggs cannot assume negative values.

$\therefore \quad x \geq 0,\ y \geq 0,\ z \geq 0$

Hence, the mathematical formulation of the given LPP is

$$\text{Minimize } Z = x + 1.10y + 0.50z$$

Subject to

$$x + y + 10z \geq 1$$
$$100x + 10y + 10z \geq 50$$
$$10x + 100y + 10z \geq 10$$

and, $\quad x \geq 0,\ y \geq 0,\ z \geq 0.$

Type III TRANSPORTATION PROBLEMS

EXAMPLE 11 *There is a factory located at each of the two places P and Q. From these locations, a certain commodity is delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are 8 and 6 units respectively. The cost of transportation per unit is given below.*

From \ To	Cost (in Rs)		
	A	B	C
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. Formulate the above as a linear programming problem.

SOLUTION The given information can be exhibited diagramatically as follows :

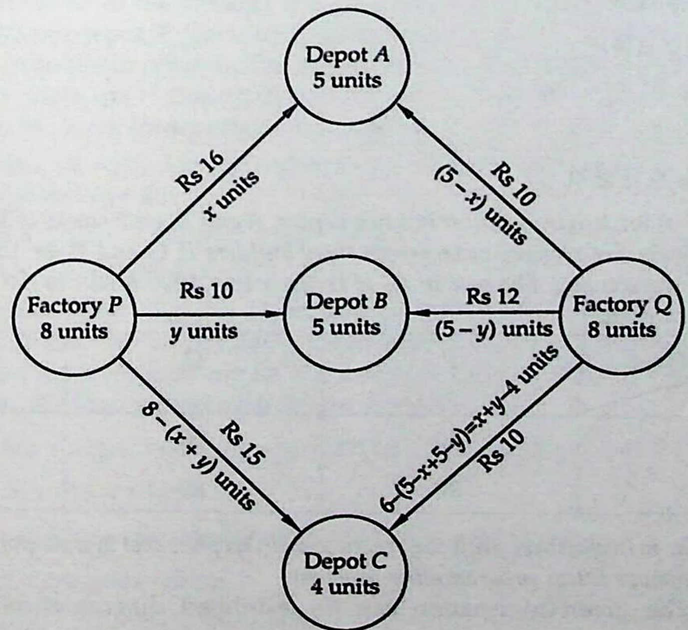

Fig. 29.1

Let the factory at P transports x units of commodity to depot at A and y units to depot at B. Since the factory at P has the capacity of 8 units of the commodity. Therefore, the left out $(8-x-y)$ units will be transported to depot at C.

Since the requirements are always non-negative quantities. Therefore,

$$x \geq 0, y \geq 0 \text{ and } 8-x-y \geq 0 \Rightarrow x \geq 0, y \geq 0 \text{ and } x+y \leq 8$$

Since the weekly requirement of the depot at A is 5 units of the commodity and x units are transported from the factory at P. Therefore, the remaining $(5-x)$ units are to be transported from the factory at Q. Similarly, $5-y$ units of the commodity will be transported from the factory at Q to the depot at B. But the factory at Q has the capacity of 6 units only, therefore the remaining $6-(5-x+5-y)=x+y-4$ units will be transported to the depot at C. As the requirements at the depots at A, B and C are always non-negative.

$\therefore \qquad 5-x \geq 0, 5-y \geq 0 \text{ and } x+y-4 \geq 0$

$\Rightarrow \qquad x \leq 5, y \leq 5 \text{ and } x+y \geq 4.$

The transportation cost from the factory at P to the depots at A, B and C are respectively Rs $16x$, $10y$ and $15(8-x-y)$. Similarly, the transportation cost from the factory at Q to the depots at A, B and C are respectively Rs $10(5-x)$, $12(5-y)$ and $10(x+y-4)$. Therefore, the total transportation cost Z is given by

$$\begin{aligned} Z &= 16x+10y+15(8-x-y)+10(5-x)+12(5-y)+10(x+y-4) \\ &= x-7y+190 \end{aligned}$$

Hence, the above LPP can be stated mathematically as follows :

Find x and y which

Minimize $Z = x-7y+190$

Subject to

$$x+y \leq 8$$
$$x+y \geq 4$$
$$x \leq 5$$
$$y \leq 5$$

and, $\quad x \geq 0, y \geq 0$

EXAMPLE 12 *A brick manufacturer has two depots, A and B, with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three builders P, Q and R for 15,000, 20,000 and 15,000 bricks respectively. The cost in Rs of transporting 1000 bricks to the builders from the depots are given below :*

To \ From	P	Q	R
A	40	20	30
B	20	60	40

How should the manufacturer fulfil the orders so as to keep the cost of transportation minimum? Formulate the above linear programming problem.

SOLUTION The given information can be exhibited diagramatically as shown in Fig. 29.2.

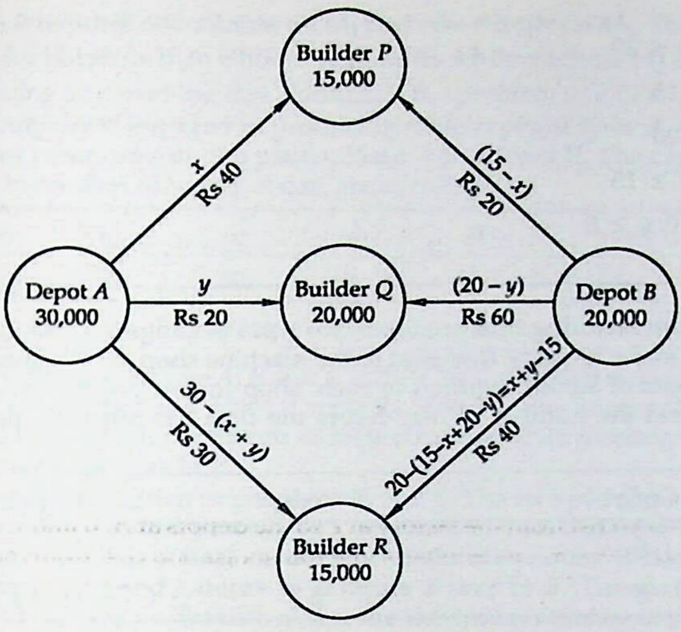

Fig. 29.2

Let the depot A transport x thousands bricks to builders P, y thousands to builder Q. Since the depot A has stock of 30,000 bricks. Thereofre, the remaining bricks i.e. $30-(x+y)$ thousands bricks will be transported to the builder R.

Since the number of bricks is always a non-negative real number.

Therefore,

$$x \geq 0, y \geq 0 \text{ and } 30-(x+y) \geq 0 \Rightarrow x \geq 0, y \geq 0 \text{ and } x+y \leq 30.$$

Now, the requirement of the builder P is of 15000 bricks and x thousand bricks are transported from the depot A. Therefore, the remaining $(15-x)$ thousands bricks are to be transported from the depot at B. The requirement of the builder Q is of 20,000 bricks and y thousand bricks are transported from depot A. Therefore, the remaining $(20-y)$ thousand bricks are to be transported from depot B.

Now, depot B has $20-(15-x+20-y)=x+y-15$ thousand bricks which are to be transported to the builder R.

Also, $\quad 15-x \geq 0,\ 20-y \geq 0$ and $x+y-15 \geq 0$

$\Rightarrow \quad x \leq 15,\ y \leq 20$ and $x+y \geq 15$

The transportation cost from the depot A to the builders P, Q and R are respectively Rs $40x$, $20y$ and $30(30-x-y)$. Similarly, the transportation cost from the depot B to the builders P, Q and R are respectively Rs $20(15-x)$, $60(20-y)$ and $40(x+y-15)$ respectively. Therefore, the total transportation cost Z is given by

$$Z = 40x+20y+30(30-x-y)+20(15-x)+60(20-y)+40(x+y-15)$$

$$\Rightarrow \quad Z = 30x-30y+1800$$

Hence, the above LPP can be stated mathematically as follows :

Find x and y in thousands which

Minimize $Z = 30x-30y+1800$

Subject to

$$x + y \leq 30$$
$$x \leq 15$$
$$y \leq 20$$
$$x + y \geq 15$$
and, $$x \geq 0, y \geq 0$$

EXERCISE 29.1

1. A small manufacturing firm produces two types of gadgets A and B, which are first processed in the foundry, then sent to the machine shop for finishing. The number of man-hours of labour required in each shop for the production of each unit of A and B, and the number of man-hours the firm has available per week are as follows :

Gadget	*Foundry*	*Machine-shop*
A	10	5
B	6	4
Firm's capacity per week	1000	600

The profit on the sale of A is Rs 30 per unit as compared with Rs 20 per unit of B. The problem is to determine the weekly production of gadgets A and B, so that the total profit is maximized. Formulate this problem as a LPP.

2. A company is making two products A and B. The cost of producing one unit of products A and B are Rs 60 and Rs 80 respectively. As per the agreement, the company has to supply at least 200 units of product B to its regular customers. One unit of product A requires one machine hour whereas product B has machine hours available abundantly within the company. Total machine hours available for product A are 400 hours. One unit of each product A and B requires one labour hour each and total of 500 labour hours are available. The company wants to minimize the cost of production by satisfying the given requirements. Formulate the problem as a LPP.

3. A firm manufactures 3 products A, B and C. The profits are Rs 3, Rs 2 and Rs 4 respectively. The firm has 2 machines and below is the required processing time in minutes for each machine on each product :

Machine	*Products*		
	A	B	C
M_1	4	3	5
M_2	2	2	4

Machines M_1 and M_2 have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but not more than 150 A's. Set up a LPP to maximize the profit.

4. A firm manufactures two types of products A and B and sells them at a profit of Rs 2 on type A and Rs 3 on type B. Each product is processed on two machines M_1 and M_2. Type A requires one minute of processing time on M_1 and two minutes

of M_2; type B requires one minute on M_1 and one minute on M_2. The machine M_1 is available for not more than 6 hours 40 minutes while machine M_2 is available for 10 hours during any working day. Formulate the problem as a LPP.

5. A rubber company is engaged in producing three types of tyres A, B and C. Each type requires processing in two plants, Plant I and Plant II. The capacities of the two plants, in number of tyres per day, are as follows :

Plant	*A*	*B*	C
I	50	100	100
II	60	60	200

The monthly demand for tyre A, B and C is 2500, 3000 and 7000 respectively. If plant I costs Rs 2500 per day, and plant II costs Rs 3500 per day to operate, how many days should each be run per month to minimize cost while meeting the demand ? Formulate the problem as LPP.

6. A company sells two different products A and B. The two products are produced in a common production process and are sold in two different markets. The production process has a total capacity of 45000 man-hours. It takes 5 hours to produce a unit of A and 3 hours to produce a unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 7000 and that of B is 10,000. If the profit is Rs 60 per unit for the product A and Rs 40 per unit for the product B, how many units of each product should be sold to maximize profit ? Formulate the problem as LPP.

7. To maintain his health a person must fulfil certain minimum daily requirements for several kinds of nutrients. Assuming that there are only three kinds of nutrients – calcium, protein and calories and the person's diet consists of only two food items, I and II, whose price and nutrient contents are shown in the table below :

	Food I (per lb)	*Food II (per lb)*	*Minimum daily requirement for the nutrient*
Calcium	10	4	20
Protein	5	5	20
Calories	2	6	13
Price (Rs)	0.60	1.00	

What combination of two food items will satisfy the daily requirement and entail the least cost ? Formulate this as a LPP.

8. A manufacturer can produce two products, A and B, during a given time period. Each of these products requires four different manufacturing operations : grinding, turning, assembling and testing. The manufacturing requirements in hours per unit of products A and B are given below.

	A	*B*
Grinding	1	2
Turning	3	1
Assembling	6	3
Testing	5	4

The available capacities of these operations in hours for the given time period are: grinding 30; turning 60, assembling 200; testing 200. The contribution to profit is Rs 2 for each unit of A and Rs 3 for each unit of B. The firm can sell all that it produces at the prevailing market price. Determine the optimum amount of A and B to produce during the given time period. Formulate this as a LPP.

9. Vitamins A and B are found in two different foods F_1 and F_2. One unit of food F_1 contains 2 units of vitamin A and 3 units of vitamin B. One unit of food F_2 contains 4 units of vitamin A and 2 units of vitamin B. One unit of food F_1 and F_2 cost Rs 5 and 2.5 respectively. The minimum daily requirements for a person of vitamin A and B is 40 and 50 units respectively. Assuming that any thing in excess of daily minimum requirement of vitamin A and B is not harmful, find out the optimum mixture of food F_1 and F_2 at the minimum cost which meets the daily minimum requirement of vitamin A and B. Formulate this as a LPP.

10. An automobile manufacturer makes automobiles and trucks in a factory that is divided into two shops. Shop A, which performs the basic assembly operation, must work 5 man-days on each truck but only 2 man-days on each automobile. Shop B, which performs finishing operations, must work 3 man-days for each automobile or truck that it produces. Because of men and machine limitations, shop A has 180 man- days per week available while shop B has 135 man-days per week. If the manufacturer makes a profit of Rs 30000 on each truck and Rs 2000 on each automobile, how many of each should he produce to maximize his profit ? Formulate this as a LPP.

11. A firm manufactures two products, each of which must be processed through two departments, 1 and 2. The hourly requirements per unit for each product in each department, the weekly capacities in each department, selling price per unit, labour cost per unit, and raw material cost per unit are summarized as follows :

	Product A	*Product B*	*Weekly capacity*
Department 1	3	2	130
Department 2	4	6	260
Selling price per unit	Rs 25	Rs 30	
Labour cost per unit	Rs 16	Rs 20	
Raw material cost per unit	Rs 4	Rs 4	

The problem is to determine the number of units to produce each product so as to maximize total contribution to profit. Formulate this as a LPP.

12. An airline agrees to charter planes for a group. The group needs at least 160 first class seats and at least 300 tourist class seats. The airline must use at least two of its model 314 planes which have 20 first class and 30 tourist class seats. The airline will also use some of its model 535 planes which have 20 first class seats and 60 tourist class seats. Each flight of a model 314 plane costs the company Rs 1 lakh, and each flight of a model 535 plane costs Rs 1.5 lakh. How many of each type of plane should be used to minimize the flight cost ? Formulate this as a LPP.

13. Amit's mathematics teacher has given him three very long lists of problems with the instruction to submit not more than 100 of them (correctly solved) for credit. The problem in the first set are worth 5 points each, those in the second set are worth 4 points each, and those in the third set are worth 6 points each. Amit knows from experience that he requires on the average 3 minutes to solve a 5 point problem, 2 minutes to solve a 4 point problem, and 4 minutes to solve a 6 point problem. Because he has other subjects to worry about, he can not afford to devote more than

$3\frac{1}{2}$ hours altogether to his mathematics assignment. Moreover, the first two sets of problems involve numerical calculations and he knows that he cannot stand more than $2\frac{1}{2}$ hours work on this type of problem. Under these circumstances, how many problems in each of these categories shall he do in order to get maximum possible credit for his efforts ? Formulate this as a LPP.

14. A farmer has a 100 acre farm. He can sell the tomatoes, lettuce, or radishes he can raise. The price he can obtain is Re 1 per kilogram for tomateoes, Rs 0.75 a head for lettuce and Rs 2 per kilogram for radishes. The average yield per acre is 2000 kgs for radishes, 3000 heads of lettuce and 1000 kilograms of radishes. Fertilizer is available at Re 0.50 per kg and the amount required per acre is 100 kgs each for tomatoes and lettuce and 50 kilograms for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes and 6 man-days for lettuce. A total of 400 man-days of labour are available at Rs 20 per man-day. Formulate this problem as a LPP to maximize the farmer's total profit.

15. Two tailors A and B earn Rs 150 and Rs 200 per day respectively. A can stitch 6 shirts and 4 pants per day while B can stitch 10 shirts and 4 pants per day. Form a linear programming problem to minimize the labour cost to produce at least 60 shirts and 32 pants. **[CBSE 2005]**

ANSWERS

1. *Max.* $Z = 30x + 20y$
Subject to
$10x + 6y \le 1000$
$5x + 4y \le 600$
$x, y \ge 0.$

2. *Mini.* $Z = 60x + 80y$
Subject to
$x + y \le 500$
$x \le 400$
$y \ge 200$
$x \ge 0, y \ge 0$

3. *Max.* $Z = 3x + 2y + 4z$
Subject to
$4x + 3y + 5z \le 2000$
$2x + 2y + 4z \le 2500$
$100 \le x \le 150$
$y \ge 200$
$z \ge 50$
$x \ge 0, y \ge 0, z \ge 0.$

4. Max. $Z = 2x + 3y$
Subject to
$x + y \le 400$
$2x + y \le 600$
$x \ge 0, y \ge 0$

5. *Mini.* $Z = 2500x + 3500y$
Subject to
$50x + 60y \ge 2500$
$100x + 60y \ge 3000$
$100x + 200y \ge 7000$
$x, y \ge 0.$

6. *Max* $Z = 60x + 40y$
Subject to
$5x + 3y \le 45000$
$x \le 7000$
$y \le 10{,}000$
$x, y \ge 0$

7. *Mini* $Z = 0.60x + 1.00y$
Subject to
$10x + 5y \ge 20$
$5x + 4y \ge 20$
$2x + 6y \ge 13$
$x, y \ge 0.$

8. *Max.* $Z = 2x + 3y$
Subject to
$x + 2y \le 30$
$3x + y \le 60$
$6x + 3y \le 200$
$5x + 4y \le 200$
$x, y \ge 0$

9. *Mini.* $Z = 3x + 2.5y$
Subject to
$2x + 4y \ge 40$

10. *Max.* $Z = 30000x + 2000y$
Subject to
$5x + 2y \le 180$

$$3x + 2y \geq 50$$
$$x \geq 0, y \geq 0$$

$$3x + 3y \leq 135$$
$$x \geq 0, y \geq 0$$

11. *Max.* $Z = 5x + 6y$

Subject to

$$3x + 2y \leq 130$$
$$4x + 6y \leq 260$$
$$x \geq 0, y \geq 0.$$
$$x, y \geq 0$$

12. *Mini.* $Z = x + 1.5y$

Subject to

$$20x + 20y \geq 160$$
$$30x + 60y \geq 300$$
$$x \geq 2$$

13. *Max.* $Z = 5x_1 + 4x_2 + 6x_3$

Subject to

$$x_1 + x_2 + x_3 \leq 100$$
$$3x_1 + 2x_2 + 4x_3 \leq 210$$
$$3x_1 + 2x_2 \leq 150.$$
$$x_1, x_2, x_3 \geq 0$$

14. *Max* $Z = 1850x + 2080y + 1875z$

Subject to

$$x + y + z \leq 100$$
$$5x + 6y + 5z \leq 400$$
$$x, y, z \geq 0$$

15. *Mini.* $Z = 150x + 200y$

Subject to

$$6x + 10y \leq 60$$
$$4x + 4y \leq 32$$
$$x, y \geq 0$$

HINTS TO SELECTED PROBLEMS

2. Let x units of product A and y units of product B be manufactured. Then, the mathematical formulation of the LPP is

Minimize $Z = 60x + 80y$

Subject to

$$x + y \leq 500 \qquad \text{(Labour hours constraint)}$$
$$x \leq 400 \qquad \text{(Machine hours constraint)}$$
$$y \geq 200 \qquad \text{(Agreement constraint)}$$
$$x \geq 0,\ y \geq 0$$

11. Suppose x units of product A and y units of product B are produced to maximize the profit. Then,

$$\text{Profit} = (25 - 16 - 4)\,x + (30 - 20 - 4)\,y \text{ or Profit} = 5x + 6y.$$

$$3x + 2y \leq 130 \qquad \text{(Capacity constraint of Department 1)}$$
$$4x + 6y \leq 260 \qquad \text{(Capacity constraint of Department 2)}$$

and $\quad x \geq 0,\ y \geq 0$

29.5 SOME DEFINITIONS AND RESULTS

In this section, we shall discuss some definitions related to the solution of linear programming problems.

The general form of a LPP is as given below :

Maximize (or *minimize*) $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ (objective function)

Subject to

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \{\leq, =, \geq\} = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \{\leq, =, \geq\}\, b_2 \qquad \text{(constraints)}$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \{\leq, =, \geq\} = b_m$$

$$x_1, x_2, \ldots, x_n \geq 0 \qquad \text{(Non-negativity restrictions)}$$

The following are some definitions related to a LPP.

SOLUTION *A set of values of variables $x_1, x_2, \ldots, x_n$ is called a solution of a LPP, if it satisfies the constraints of the LPP.*

ILLUSTRATION 1 *Consider the LPP :*

$$\text{Maximize} \quad Z = 4x + 5y$$

Subject to

$$x + 2y \leq 6$$
$$3x + y \leq 12$$
$$x \geq 0, y \geq 0.$$

Clearly, $x = 1, y = 2; x = -2, y = 3; x = -1, y = -2; x = 2, y = -3$ etc. are solutions of this LPP as they satisfy the constraints $x + 2y \leq 6$ and $3x + y \leq 12$. Note that $x = 2, y = 4$ is not a solution, because it does not satisfy $x + 2y \leq 6$.

FEASIBLE SOLUTION *A set of values of the variables $x_1, x_2, \ldots, x_n$ is called a feasible solution of a LPP, if it satisfies the constraints and non-negativity restrictions of the problem.*

In other words, a solution that also satisfies the non-negativity restrictions of a LPP, is called a feasible solution.

INFEASIBLE SOLUTION *A solution of a LPP is an infeasible solution, if the system of constraints has no point which satisfies all the constraints and non-negativity restrictions.*

ILLUSTRATION 2 *Consider a LPP*

$$\text{Maximize } Z = 6x + 8y$$

Subject to

$$3x + 2y \leq 30$$
$$x + 2y \leq 22$$
$$x, y \geq 0$$

We observe that $x = 2, y = 3; x = 5, y = 0; x = -2, y = -1; x = 0, y = -2$ etc. are solutions of this LPP. Out of these solutions $x = 2, y = 3$ and $x = 5, y = 0$ are feasible solutions. Because these solutions also satisfy non-negativity restrictions. Remaining solutions given above are infeasible solutions.

FEASIBLE REGION *The common region determined by all the constraints of a LPP is called the feasible region and every point in this region is a feasible solution of the given LPP.*

OPTIMAL FEASIBLE SOLUTION *A feasible solution of a LPP is said to be an optimal feasible solution, if it also optimizes (maximizes or minimizes) the objective function.*

Now, we shall discuss some definitions and results related to the feasible solutions of a LPP.

CONVEX SET *A set is a convex set, if every point on the line segment joining any two points in it lies in it.*

In Figs. 29.3 to 29.4 the polygons are convex sets whereas polygon in Fig. 29.5 is not a convex set.

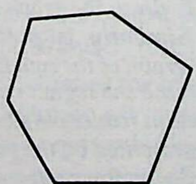

Fig. 29.3

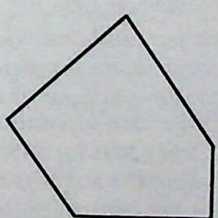

Fig. 29.4

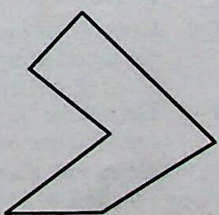

Fig. 29.5

THEOREM *The set of all feasible solutions of a LPP is a convex set.*

The proof of the above theorem is beyond the scope of the syllabus for CBSE class XII.

It follows from the above theorem that the set of all feasible solutions of a LPP is a convex polygon. When we are asked to solve a linear programming problem, it always means that we have to find its optimal solution. It is known from the general mathematical theory of linear programming that a LPP may or may not attain an optimal solution. However, if it attains an optimal solution, then one of the corner points (vertices) of the convex polygon of all feasible solutions gives the optimal solution as stated in the following theorem.

FUNDAMENTAL EXTREME POINT THEOREM *An optimal solution of a LPP, if it exists, occurs at one of the extreme (corner) points of the convex polygon of the set of all feasible solutions.*

It may happen that the two vertices of the corner polygon give the optimal value of the objective function. In such a case all points on the line segment joining these two vertices give the optimal value and the LPP is said to have infinitely many solutions. Sometimes, the convex polygon is an empty set. In such a case, we say that the LPP has no solution.

If the feasible region for a linear programming problem is bounded i.e., it can be enclosed within a circle, then the objective function has both a maximum and a minimum value and each of these values occurs at a corner point of the feasible region.

If the feasible region of a linear programming problem is unbounded i.e., it extends indefinitely in any direction, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it occurs at a corner point of the feasible region.

29.6 GRAPHICAL METHODS OF SOLVING LINEAR PROGRAMMING PROBLEMS

There are two graphical methods for the solution of linear programming problems. These methods are suitable for solving linear programming problems containing two variables only. If a LPP contains more than two variables, these graphical methods are not suitable to solve them. Such type of problems are solved by simplex method which is beyond the scope of our discussion. We shall, therefore, be concerned only with the graphical methods involving two variables x and y.

The following methods are used to solve linear programming problems graphically:

(i) Corner-Point Method

(ii) Iso-profit or iso-cost method.

We shall now apply these two methods for solving linear programming problems.

29.7 CORNER-POINT METHOD

This method is based on the Fundamental extreme point theorem which is stated in the earlier section.

Following algorithm can be used to solve a LPP in two variables graphically by using corner-point method.

ALGORITHM

STEP I *Formulate the given LPP in mathematical form if it is not given in mathematical form.*

STEP II *Convert all inequations into equations and draw their graphs. To draw the graph of a linear equation, put $y = 0$ in it and obtain a point on X-axis. Similarly, by putting $x = 0$ obtain a point on y-axis. Join these two points to obtain the graph of the equation.*

STEP III *Determine the region represented by each inequation. To determiine the region represented by an inequation replace x and y both by zero, if the inequation reduces to a valid statement, then the region containing the origin is the region represented by the given inequation. Otherwise, the region not containing the origin is the region represented by the given inequation.*

STEP IV *Obtain the region in xy-plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The polygonal region so obtained is the feasible region and is known as the convex polygon of the set of all feasible solutions of the LPP.*

STEP V *Determine the coordinates of the vertices (corner points) of the convex polygon obtained in Step II. These vertices are known as the extreme points of the set of all feasible solutions of the LPP.*

STEP VI *Obtain the values of the objective function at each of the vertices of the convex polygon. The point where the objective function attains its optimum (maximum or minimum) value is the optimal solution of the given LPP.*

REMARK 1 *If the feasible region of a LPP is bounded, i.e., it is a convex polygon. Then, the objective function $Z = ax + by$ has both a maximum value M and a minimum value m and each of these values occurs at a corner point of the convex polygon.*

REMARK 2 *Sometimes the feasible region of a LPP is not a bounded convex polygon. That is, it extends indefinitely in any direction. In such cases, we say that the feasible region is unbounded. The above algorithm is applicable when the feasible region is bounded. If the feasible region is unbounded, then we find the values of the objective function $Z = ax + by$ at each corner point of the feasible region. Let M and m respectively denote the largest and smallest values of Z at these points. In order to check whether Z has maximum and minimum values at M and m respectively, we proceed as follows:*

(i) *Draw the line $ax + by = M$ and find the open half plane $ax + by > M$. If the open half-plane represented by $ax + by > M$ has no point common with the unbounded feasible region, then M is the maximum value of Z. Otherwise Z has no maximum value.*

(ii) *Draw the line $ax + by = m$ and find the open half plane represented by $ax + by < m$. If the half-plane $ax + by < m$ has no point common with the unbounded feasible region, then m is the minimum value of Z. Otherwise, Z has no minimum value.*

Following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the following LPP graphically:*

Maximize $Z = 5x + 3y$

Subject to

$$3x + 5y \leq 15$$
$$5x + 2y \leq 10$$

and, $x, y \geq 0.$

SOLUTION Converting the given inequations into equations, we obtain the following equations:

$$3x + 5y = 15, 5x + 2y = 10, x = 0 \text{ and } y = 0$$

Region represented by $3x + 5y \leq 15$. : The line $3x + 5y = 15$ meets the coordinate axes at A_1 (5, 0) and B_1 (0, 3) respectively. Join these points to obtain the line $3x + 5y = 15$. Clearly, (0, 0) satisfies the inequation $3x + 5y \leq 15$. So, the region containing the origin represents the solution set of the inequation $3x + 5y \leq 15$.

Region Represented by $5x + 2y \leq 10$: The line $5x + 2y = 10$ meets the coordinate axes at A_2 (2, 0) and B_2 (0, 5) respectively. Join these points to obtain the line $5x + 2y = 10$. Clearly, (0, 0) satisfies the inequation $5x + 2y \leq 10$. So, the region containing the origin represents the solution set of this inequation.

Region represented by $x \geq 0$ and $y \geq 0$: Since every point in the first quadrant satisfies these inequations. So, the first quadrant is the region represented by the inequations $x \geq 0$ and $y \geq 0$.

The shaded region OA_2PB_1 in Fig. 29.6 represents the common region of the above inequations. This region is the feasible region of the given LPP.

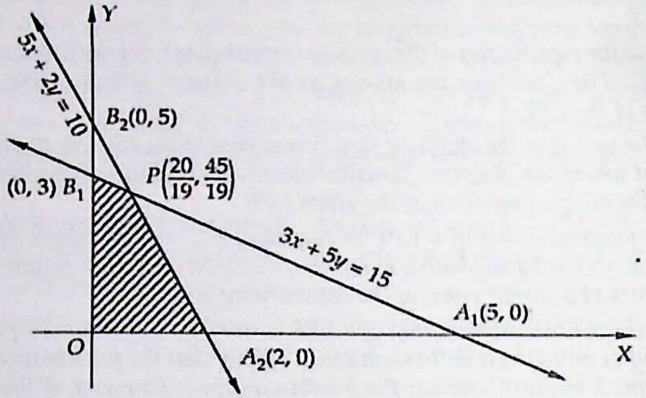

Fig. 29.6

The coordinates of the vertices (corner-points) of the shaded feasible region are $O(0, 0)$, $A_2(2, 0)$, $P\left(\frac{20}{19}, \frac{45}{19}\right)$ and $B_1(0, 3)$.

These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of the objective function* $Z = 5x + 3y$
$O(0, 0)$	$Z = 5 \times 0 + 3 \times 0 = 0$
$A_2(2, 0)$	$Z = 5 \times 2 + 3 \times 0 = 10$
$P\left(\frac{20}{19}, \frac{45}{19}\right)$	$Z = 5 \times \frac{20}{19} + 3 \times \frac{45}{19} = \frac{235}{19}$
$B_1(0, 3)$	$Z = 5 \times 0 + 3 \times 3 = 9$

Clearly, Z is maximum at $P\left(\frac{20}{19}, \frac{45}{19}\right)$. Hence, $x = \frac{20}{19}$, $y = \frac{45}{19}$ is the optimal solution of the given LPP. The optimal value of Z is $\frac{235}{19}$.

EXAMPLE 2 *Solve the following LPP by graphical method:*

Minimize $Z = 20x + 10y$

Subject to $x + 2y \leq 40$

$3x + y \geq 30$

$4x + 3y \geq 60$

and, $x, y \geq 0$

SOLUTION Converting the given inequations into equations, we obtain the following equations:

$$x + 2y = 40, 3x + y = 30, 4x + 3y = 60, x = 0 \text{ and } y = 0.$$

Region represented by $x + 2y \leq 40$: The line $x + 2y = 40$ meets the coordinate axes at $A_1(40, 0)$ and $B_1(0, 20)$ respectively. Join these points to obtain the line $x + 2y = 40$. Clearly, (0, 0) satisfies the inequation $x + 2y \leq 40$. So, the region in xy-plane that contains the origin represents the solution set of the given inequation.

Region represented by $3x+y \geq 30$: The line $3x+y=30$ meets X and Y axes at $A_2(10, 0)$ and $B_2(0, 30)$ respectively. Join these points to obtain this line. We find that $O(0, 0)$ does not satisfy the inequation $3x+y \geq 30$. So that region in xy–plane which does not contain the origin is the solution se t of this inequation.

Region represented by $4x+3y \geq 60$: The line $4x+3y=60$ meets X and Y axes at $A_3(15, 0)$ and $B_1(0, 20)$ respectively. Join these points to obtain the line $4x+3y=60$. We observe that $O(0, 0)$ does not satisfy the inequation $4x+3y \geq 60$. So, the region not containing the origin in xy–plane represents the solution set of the given inequation.

Region represented by $x \geq 0, y \geq 0$: Clearly, the region represented by $x \geq 0$ and $y \geq 0$ is the first quadrant in xy–plane.

The shaded region $A_3 A_1 QP$ in Fig. 29.7 represents the common region of the regions represented by the above inequations. This region expresents the feasible region of the given LPP.

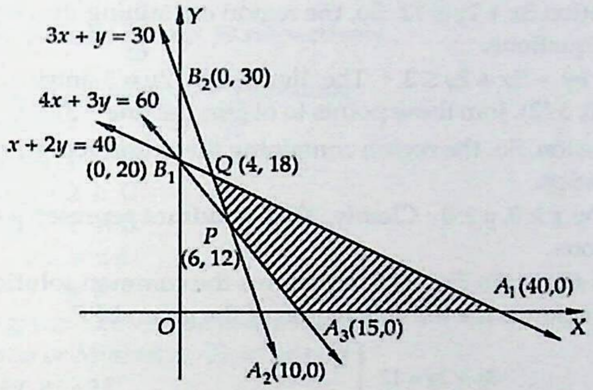

Fig. 29.7

The coordinates of the corner-points of the shaded feasible region are $A_3(15, 0)$, $A_1(40, 0)$, $Q(4, 18)$ and $P(6, 12)$. These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of the objective function* $Z = 20x + 10y$
$A_3(15, 0)$	$Z = 20 \times 15 + 10 \times 0 = 300$
$A_1(40, 0)$	$Z = 20 \times 40 + 10 \times 0 = 800$
$Q(4, 18)$	$Z = 20 \times 4 + 10 \times 18 = 260$
$P(6, 12)$	$Z = 20 \times 6 + 10 \times 12 = 240$

Clearly, Z is minimum at $P(6, 12)$. Hence, $x=6, y=12$ is the optimal solution of the given LPP. The optimal value of Z is 240.

EXAMPLE 3 *Solve the following LPP graphically:*

Minimize and Maximize $Z = 5x + 2y$

Subject to
$$-2x - 3y \leq -6$$
$$x - 2y \leq 2$$
$$3x + 2y \leq 12$$
$$-3x + 2y \leq 3$$
$$x, y \geq 0$$

SOLUTION Converting the given inequations into equations, we get

$$2x+3y=6,\ x-2y=2,\ 3x+2y=12,\ -3x+2y=3,\ x=0 \text{ and } y=0$$

Region represented by $-2x-3y \le -6$: The line $-2x-3y=-6$ or $2x+3y=6$ cuts OX and OY at A_1 (3, 0) and B_1 (0, 3) respectively. Join these points to obtain the line $2x+3y=6$.

Since O (0, 0) does not satisfy the inequation $-2x-3y \le -6$. So, the region represented by $-2x-3y \le -6$ is that part of XOY–plane which does not contain the orgin.

Region represented by $x-2y \le 2$: The line $x-2y=2$ meets the coordinate axes at A_2 (2, 0) and B_2 (0, −1). Join these points to obtain $x-2y=2$. Since (0, 0) satisfies the inequation $x-2y \le 2$, so the region containing the origin represents the solution set of this inequation.

Region represented by $3x+2y \le 12$: The line $3x+2y \le 12$ interesects OX and OY at A_3 (4, 0) and B_3 (0, 6). Join these points to obtain the line $3x+2y=12$. Clearly, (0, 0) satisfies the inequation $3x+2y \le 12$. So, the region containing the origin is the solution set of the given inequations.

Region represented by $-3x+2y \le 3$: The line $-3x+2y=3$ intersects OX and OY at A_4 (−1, 0) and B_4 (0, 3/2). Join these points to obtain the line $-3x+2y=3$. Clearly, (0,0) satisfies this inequation. So, the region containing the origin represents the solution set of the given inequation.

Region represented by $x \ge 0, y \ge 0$: Clearly, XOY quadrant represents the solution set of these two inequations.

The shaded region shown in Fig. 29.8 represents the common solution set of the above inequations. This region is the feasible region of the given LPP.

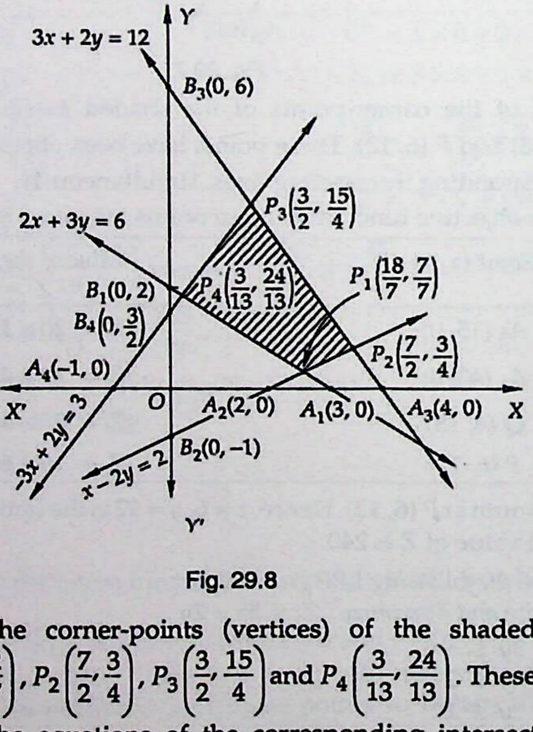

Fig. 29.8

The coordinates of the corner-points (vertices) of the shaded feasible region $P_1\,P_2\,P_3\,P_4$ are $P_1\left(\frac{18}{7}, \frac{2}{7}\right)$, $P_2\left(\frac{7}{2}, \frac{3}{4}\right)$, $P_3\left(\frac{3}{2}, \frac{15}{4}\right)$ and $P_4\left(\frac{3}{13}, \frac{24}{13}\right)$. These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.

The values of the objective function at these points are given in the following table:

Point (x, y)	Value of the objective function $Z = 5x + 2y$
$P_1\left(\frac{18}{7}, \frac{2}{7}\right)$	$Z = 5 \times \frac{18}{7} + 2 \times \frac{2}{7} = \frac{94}{7}$
$P_2\left(\frac{7}{2}, \frac{3}{4}\right)$	$Z = 5 \times \frac{7}{2} + 2 \times \frac{3}{4} = 19$
$P_3\left(\frac{3}{2}, \frac{15}{4}\right)$	$Z = 5 \times \frac{3}{2} + 2 \times \frac{15}{4} = 15$
$P_4\left(\frac{3}{13}, \frac{24}{13}\right)$	$Z = 5 \times \frac{3}{13} + 2 \times \frac{24}{13} = \frac{63}{13}$

Clearly, Z is min at $x = \frac{3}{13}, y = \frac{24}{13}$ and maximum at $x = \frac{7}{2}, y = \frac{3}{4}$. The minimum and maximum values of Z are $\frac{63}{13}$ and 19 respectively.

EXAMPLE 4 *Solve the following LPP graphically:*
Maximize and Minimize $Z = 3x + 5y$

Subject to
$$3x - 4y + 12 \geq 0$$
$$2x - y + 2 \geq 0$$
$$2x + 3y - 12 \geq 0$$
$$0 \leq x \leq 4$$
$$y \geq 2.$$

SOLUTION The given LPP can be re-written as
Maximize or Minimize $Z = 3x + 5y$

Subject to
$$3x - 4y \geq -12$$
$$2x - y \geq -2$$
$$2x + 3y \geq 12$$
$$x \leq 4$$
$$y \geq 2$$
$$x \geq 0$$

Converting the inequations into equations, we obtain the following equations $3x - 4y = -12$, $2x - y = -2$, $2x + 3y = 12$, $x = 4$, $y = 2$ and $x = 0$.

These lines are drawn on suitable scale. The shaded region $P_1 P_2 P_3 P_4 P_5$ shown in Fig. 29.9 represents the feasible region of the given LPP.

The values of the objective function at these points are given in the following table:

Point (x, y)	Value of the objective function $Z = 3x + 5y$
$P_1(3, 2)$	$Z = 3 \times 3 + 5 \times 2 = 19$
$P_2(4, 2)$	$Z = 3 \times 4 + 2 \times 5 = 22$
$P_3(4, 6)$	$Z = 3 \times 4 + 5 \times 6 = 42$
$P_4\left(\frac{4}{5}, \frac{18}{5}\right)$	$Z = 3 \times \frac{4}{5} + 5 \times \frac{18}{5} = \frac{102}{5}$
$P_5\left(\frac{3}{4}, \frac{7}{2}\right)$	$Z = 3 \times \frac{3}{4} + 5 \times \frac{7}{2} = \frac{79}{4}$

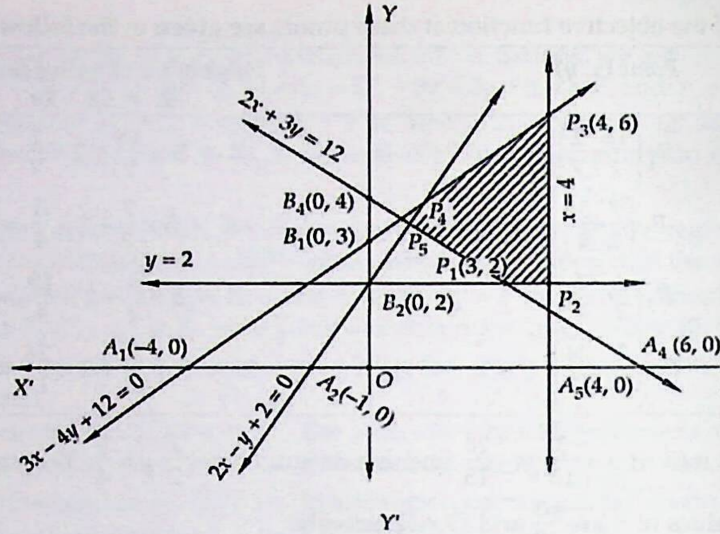

Fig. 29.9

Clearly Z assumes its minimum value 19 at $x = 3$ and $y = 2$. The maximum value of Z is 42 at $x = 4$ and $y = 6$.

EXAMPLE 5 *Determine graphically the minimum value of the objective function* $Z = -50x + 20y$

Subject to constraints:

$$2x - y \geq -5$$
$$3x + y \geq 3$$
$$2x - 3y \leq 12$$
$$x \geq 0, y \geq 0$$

SOLUTION The feasible region of the system of inequations given in constraints is shown in Fig. 29.10. We observe that the feasible region is unbounded.

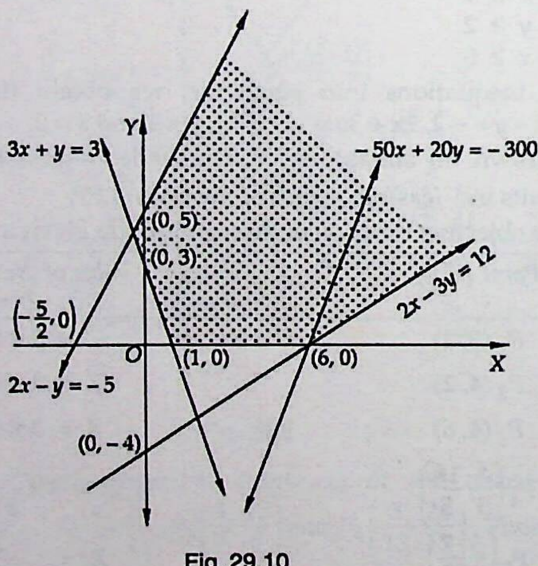

Fig. 29.10

The values of the objective function Z at the corner points are given in the following table:

Corner point (x, y)	*Value of the objective function* $Z = -50x + 20y$
$(0, 5)$	$Z = -50 \times 0 + 20 \times 5 = 100$
$(0, 3)$	$Z = -50 \times 0 + 20 \times 3 = 60$
$(1, 0)$	$Z = -50 \times 1 + 20 \times 0 = -50$
$(6, 0)$	$Z = -50 \times 6 + 20 \times 0 = -300$

Clearly, -300 is the smallest value of Z at the corner point (6, 0). Since the feasible region is unbounded. Therefore, to check whether -300 is the minimum value of Z, we draw the line $-300 = -50x + 20y$ and check whether the open half plane $-50x + 20y < -300$ has points in common with the feasible region or not. From Fig. 29.10, we find that the open half plane represented by $-50x + 20y < -300$ has points in common with the feasible region. Therefore, $Z = -50x + 20y$ has no minimum value subject to the given constraints.

29.8 ISO-PROFIT OR ISO COST METHOD

Consider the following LPP

$$\textit{Maximize} \quad Z = 10x + 6y$$

Subject to

$$3x + y \leq 12$$
$$2x + 5y \leq 34$$
$$x, y \geq 0$$

The convex set of all feasible solutions of this LPP is the set of all points in the shaded region of Fig. 29.11. Any point in this region is a feasible solution of the above LPP and only the points in this region are feasible solutions of the above LPP. In order to solve the above LPP, we have to find the point or points in the shaded region which give the largest value of the objective function. For any fixed value of Z, $Z = 10x + 6y$ or $10x + 6y = Z$ is a straight line. For example, for $Z = 5$, $10x + 6y = 5$ is a straight line. Any point on the line $Z = 10x + 6y$ will give the same value of Z. So, it is known as an *iso-profit* line.

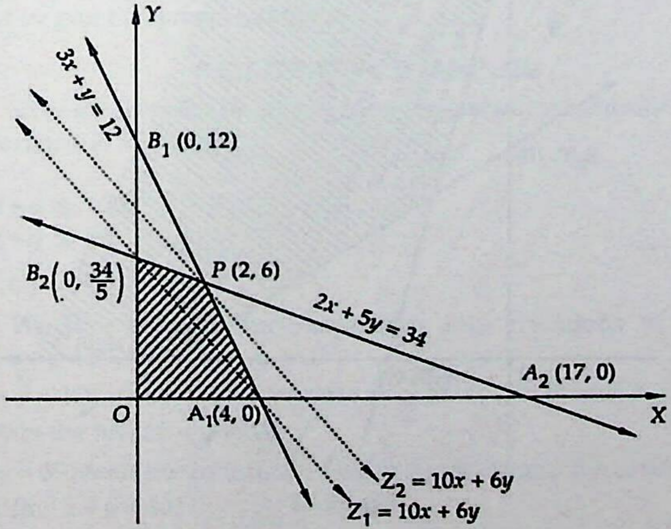

Fig. 29.11

Also, for each different value of Z, we obtain a different line. In other words, for different values of Z, equation $Z = 10x + 6y$ gives a family of parallel straight lines of slope $-\frac{10}{6} = -\frac{5}{2}$ and any point on the line $Z = 10x + 6y$, for given value of Z, gives the same value of Z. These lines are iso-profit lines. In order to maximize the objective function $Z = 10x + 6y$, we have to find the line with the largest value of Z which has at least one point in common with the shaded region. In other words, to maximize the objective function find the line parallel to $Z = 10x + 6y$ which is farthest from the origin O and which has at least one point in common with the shaded region. Clearly, $10x + 6y = Z_1$ is not farthest from the origin. However, $10x + 6y = Z_2$ is farthest from the origin and has a point $P(2, 6)$ common with the shaded region.

Thus, we see that Z_2 is the maximum value of Z, and the feasible solution which gives this value of Z is the corner $P(2, 6)$ of the shaded region. The values of the variables for the optimal solution are $x = 2, y = 6$. Substituting these values in $Z = 10x + 6y$, we get $Z = 56$ as the optimal value.

Now, consider the LPP

$$\begin{aligned} \textit{Minimize} \quad Z &= 18x + 10y \\ \textit{Subject to} \quad 4x + y &\geq 20 \\ 2x + 3y &\geq 30 \\ x, y &\geq 0 \end{aligned}$$

The convex set of all feasible solutions of this LPP is the set of all points in the shaded region of Fig. 29.12. In order to solve this LPP, we have to find the points in the shaded region which give the smallest value of the objective function. We observe that for any fixed value of Z, equation $18x + 10y = Z$ is a straight line and any point on this line gives the same value of Z. So, for some value of Z say Z_1, if the line $18x + 10y = Z_1$ has some points common with the feasible region of the LPP, then all these points give the same

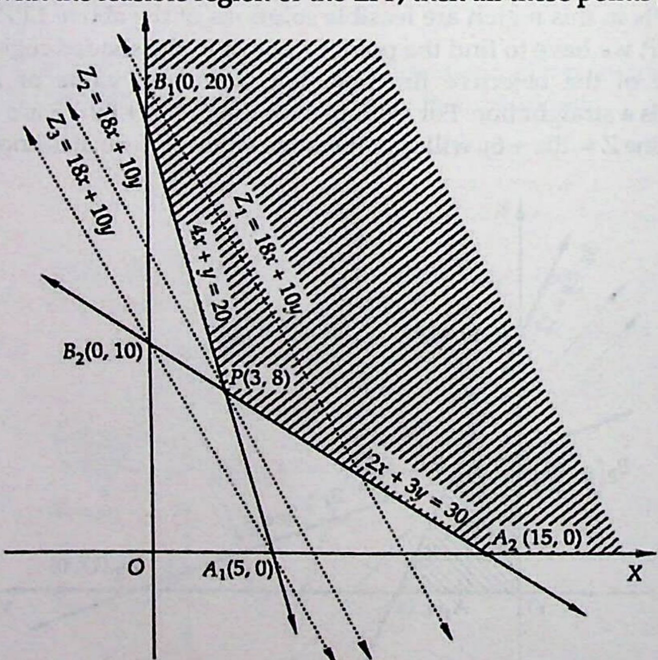

Fig. 29.12

value of Z equal to Z_1 i.e. for every point in the feasible region lying on $18x + 10y = Z_1$, we obtain the same value of Z equal to Z_1. The line $18x + 10y = Z_1$ is known as *iso-cost* line. Thus, $18x + 10y = Z$ gives a family of parallel lines of slope $-\frac{18}{10}$ in xy-plane. In order to find the minimum value of Z, we have to find the line nearest to the origin and having at least one point common with the shaded region. Clearly, $18x + 10y = Z_2$ is nearest to the origin and has a common point $P(3, 8)$ with the shaded region. The line $18x + 10y = Z_3$ is more closer to the origin than the line $18x + 10y = Z_2$, but it does not have any point common to the feasible region. Thus, Z_2 is the minimum value of Z, and the feasible solution which gives this value of Z is the corner $P(3, 8)$ of the shaded region. The values of the variables for the optimal solution are $x = 3$, $y = 8$. Substituting these values in $Z = 18x + 10y$, we get $Z = 128$ as the optimal value of Z.

The above discussion suggests the following algorithm to solve a LPP by using iso-profit (iso-cost) lines.

ALGORITHM

STEP I *Formulate the given LPP in mathematical form, if it is not given so.*

STEP II *Obtain the region in xy-plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The polygonal region so obtained is the convex set of all feasible solutions of the given LPP and it is also known as the feasible region.*

STEP III *Determine the coordinates of the vertices (Corner points) of the feasible region obtained in step II.*

STEP IV *Give some convenient value to Z and draw the line so obtained in xy-plane.*

STEP V *If the objective function is of maximization type, then draw lines parallel to the line in step IV and obtain a line which is farthest from the origin and has at least one point common to the feasible region.*
If the objective function is of minimization type, then draw lines parallel to the line in step IV and obtain a line which is nearest to the origin and has at least one point common to the feasible region.

STEP VI *Find the coordinates of the common point (s) obtained in step V. The point (s) so obtained determine the optimal solution (s) and the value (s) of the objective function at these point (s) give the optimal solution.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Solve the following linear programming problem graphically:*

$$\text{Maximize } Z = 50x + 15y$$

Subject to

$$5x + y \le 100$$
$$x + y \le 60$$
$$x, y \ge 0.$$

SOLUTION We first convert the inequations into equations to obtain the lines $5x + y = 100$, $x + y = 60$, $x = 0$ and $y = 0$.

The line $5x + y = 100$ meets the coordinate axes at $A_1(20, 0)$ and $B_1(0, 100)$. Join these points to obtain the line $5x + y = 100$.

The line $x + y = 60$ meets the coordinate axes at $A_2(60, 0)$ and $B_2(0, 60)$. Join these points to obtain the line $x + y = 60$.

Also, $x = 0$ is the y-axis and $y = 0$ is the x-axis.

The feasible region of the LPP is shaded in Fig. 29.13. The coordinates of the corner-points of the feasible region $OA_1\ PB_2$ are $O\ (0, 0)$, $A_1\ (20, 0)$, $P\ (10, 50)$ and $B_2\ (0, 60)$.

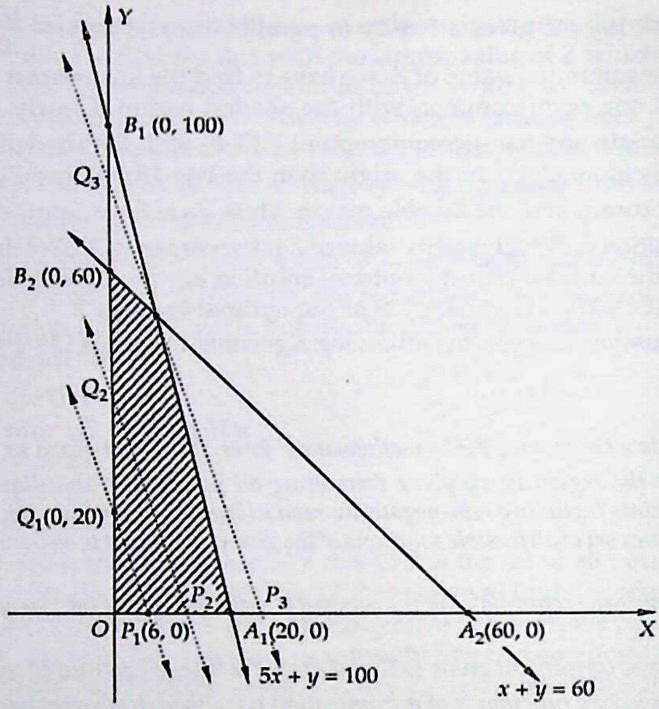

Fig. 29.13

Now, we take a constant value, say 300 (i.e. 2 times the l.c.m. of 50 and 15) for Z. Then,

$$300 = 50x + 15y$$

This line meets the coordinate axes at $P_1\ (6, 0)$ and $Q_1\ (0, 20)$. Join these points by a dotted line. Now move this line parallel to itself in the increasing direction i.e. away from the origin. $P_2\ Q_2$ and $P_3\ Q_3$ are such lines. Out of these lines locate a line which is farthest from the origin and has at least one point common to the feasible region. Clearly, $P_3\ Q_3$ is such line and it passes through the vertex $P\ (10, 15)$ the convex polygon $OA_1\ PB_2$. Hence, $x = 10$ and $y = 50$ will give the maximum value of Z. The maximum value of Z is given by

$$Z = 50 \times 10 + 15 \times 50 = 1250.$$

EXAMPLE 2 *Solve the following LPP graphically:*

$$\textit{Maximize} \quad Z = 5x + 7y$$

Subject to

$$x + y \leq 4$$
$$3x + 8y \leq 24$$
$$10x + 7y \leq 35$$
$$x, y \geq 0$$

SOLUTION Converting the inequations into equations, we obtain the following equations :

$$x + y = 4,\ 3x + 8y = 24,\ 10x + 7y = 35,\ x = 0 \text{ and } y = 0.$$

These equations represent straight lines in XOY-plane.

The line $x + y = 4$ meets the coordinate axes at A_1 (4, 0) and B_1 (0, 4). Join these points to obtain the line $x + y = 4$.

The line $3x + 8y = 24$ meets the coordinate axes at A_2 (8, 0) and B_2 (0, 3). Join these points to obtain the line $3x + 8y = 24$

The line $10x + 7y = 35$ cuts the coordinates axes at A_3 (3.5, 0) and B_3 (0, 5). These points are joined to obtain the line $10x + 7y = 35$.

Also, $x = 0$ is the y-axis and $y = 0$ is the x-axis.

The feasible region of the LPP is shaded in Fig. 29.14. The coordinates of the corner points of the feasible region $OA_3\,PQB_2$ are O (0, 0), A_3 (3.5, 0), $P\left(\frac{7}{3}, \frac{5}{3}\right)$, $Q\left(\frac{8}{5}, \frac{12}{5}\right)$ and B_2 (0, 3).

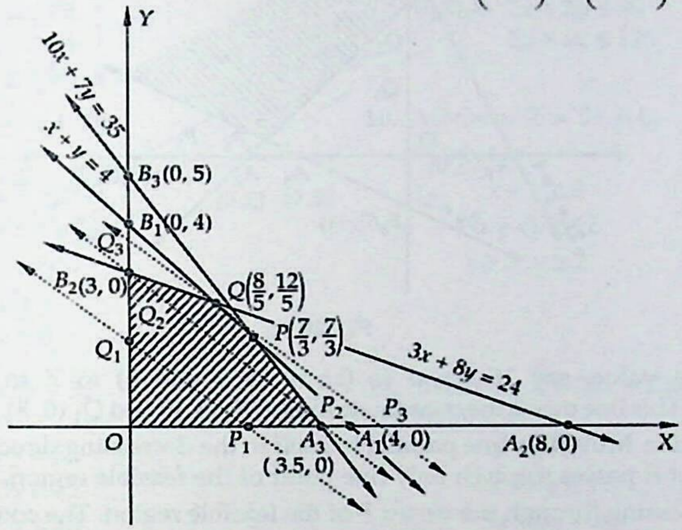

Fig. 29.14

Now, we take a constant value, say 10 for Z. Putting Z = 10 in $Z = 5x + 7y$, we obtain the line $5x + 7y = 10$. This line meets the coordinate axes at P_1 (2, 0) and $Q_1\left(0, \frac{10}{7}\right)$. Join these points by a dotted line. Now move this line parallel to itself in the increasing direction away from the origin. $P_2\,Q_2$ and $P_3\,Q_3$ are such lines. Out of these lines locate a line farthest from the origin and has at least one common point to the feasible region $OA_3\,PQB_2$. Clearly, $P_3\,Q_3$ is such line and it passes through the vertex Q (8/5, 12/5) of the feasible region. Hence $x = 8/5$ and $y = 12/5$ gives the maximum value of Z. The maximum value of Z is given by

$$Z = 5 \times \frac{8}{5} + 7 \times \frac{12}{5} = 24.8.$$

EXAMPLE 3 *Solve the following LPP graphically :*

Minimize $Z = 3x + 5y$

Subject to

$$-2x + y \leq 4$$
$$x + y \geq 3$$
$$x - 2y \leq 2$$
$$x, y \geq 0$$

SOLUTION Converting the inequations into equations, we obtain the lines $-2x + y = 4$, $x + y = 3$, $x - 2y = 2$, $x = 0$ and $y = 0$

These lines are drawn on a suitable scale and the feasible region of the LPP is shaded in Fig. 29.15.

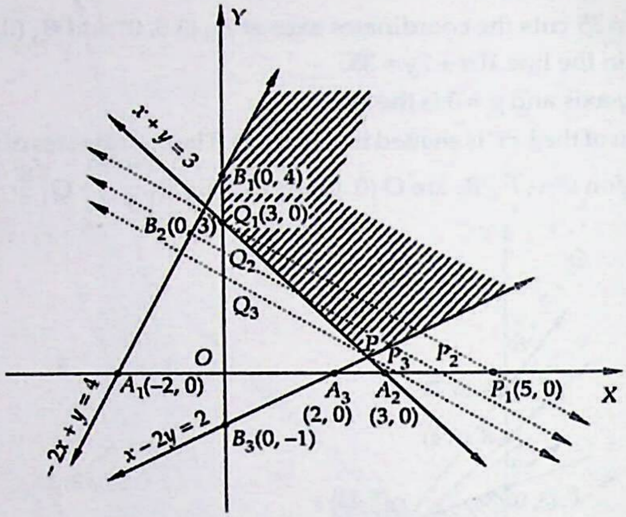

Fig. 29.15

Now, give a value, say 15 equal to (l.c.m. of 3 and 5) to Z to obtain the line $3x + 5y = 15$. This line meets the coordinate axes at P_1 (5, 0) and Q_1 (0, 3). Join these points by a dotted line. Move this line parallel to itself in the decreasing direction towards the origin so that it passes through only one point of the feasible region. Clearly, $P_3 Q_3$ is such a line passing through the vertex P of the feasible region. The coordinates of P are obtained by solving the lines $x - 2y = 2$ and $x + y = 3$. Solving these equations, we get $x = \frac{8}{3}$ and $y = \frac{1}{3}$.

Putting $x = \frac{8}{3}$ and $y = \frac{1}{3}$ in $Z = 3x + 5y$, we get

$$Z = 3 \times \frac{8}{3} + 5 \times \frac{1}{3} = \frac{29}{3}$$

Hence, the minimum value of Z is $\frac{29}{3}$ at $x = \frac{8}{3}, y = \frac{1}{3}$.

EXERCISE 29.2

Solve each of the following linear programming problems by graphical method.

1. *Maximize* $Z = 5x + 3y$
   Subject to
   $$3x + 5y \le 15$$
   $$5x + 2y \le 10$$
   $$x, y \ge 0$$

2. *Maximize* $Z = 9x + 3y$
   Subject to
   $$2x + 3y \le 13$$
   $$3x + y \le 5$$
   $$x, y \ge 0$$

3. *Minimize* $Z = 18x + 10y$
   Subject to
   $$4x + y \ge 20$$
   $$2x + 3y \ge 30$$
   $$x, y \ge 0$$

4. *Maximize* $Z = 50x + 30y$.
   Subject to
   $$2x + y \ge 18$$
   $$3x + 2y \le 34$$
   $$x, y \ge 0$$

5. *Maximize* $Z = 4x + 3y$
Subject to
$$3x + 4y \le 24$$
$$8x + 6y \le 48$$
$$x \le 5$$
$$y \le 6$$
$$x, y \ge 0$$

6. *Maximize* $Z = 15x + 10y$
Subject to
$$3x + 2y \le 80$$
$$2x + 3y \le 70$$
$$x, y \ge 0$$

7. Maximize $Z = 10x + 6y$
Subject to
$$3x + y \le 12$$
$$2x + 5y \le 34$$
$$x, y \ge 0\, x, y \ge 0$$

8. *Maximize* $Z = 3x + 4y$
Subject to
$$2x + 2y \le 80$$
$$2x + 4y \le 120$$

9. *Maximize* $Z = 7x + 10y$
Subject to
$$x + y \le 30000$$
$$y \le 12000$$
$$x \ge 6000$$
$$x \ge y$$
$$x, y \ge 0$$

10. *Minimize* $Z = 2x + 4y$
Subject to
$$x + y \ge 8$$
$$x + 4y \ge 12$$
$$x \ge 3, y \ge 2$$

11. *Minimize* $Z = 5x + 3y$
Subject to
$$2x + y \ge 10$$
$$x + 3y \ge 15$$
$$x \le 10$$
$$y \le 8$$
$$x, y \ge 0$$

12. *Minimize* $Z = 30x + 20y$
Subject to
$$x + y \le 8$$
$$x + 4y \ge 12$$
$$5x + 8y = 20$$
$$x, y \ge 0.$$

13. *Maximize* $Z = 4x + 3y$
Subject to
$$3x + 4y \le 24$$
$$8x + 6y \le 48$$
$$x \le 5$$
$$y \le 6$$
$$x, y \ge 0$$

14. *Minimize* $Z = x - 5y + 20$
Subject to
$$x - y \ge 0$$
$$-x + 2y \ge 2$$
$$x \ge 3$$
$$y \le 4$$
$$x, y \ge 0$$
[CBSE 2004]

15. *Maximize* $Z = 3x + 5y$
Subject to
$$x + 2y \le 20$$
$$x + y \le 15$$
$$y \le 5$$
$$x, y \ge 0$$

16. *Minimize* $Z = 3x_1 + 5x_2$
Subject to
$$x_1 + 3x_2 \ge 3$$
$$x_1 + x_2 \ge 2$$
$$x_1, x_2 \ge 0$$
[CBSE 2005]

17. *Maximize* $Z = 2x + 3y$
Subject to
$$x + y \ge 1$$
$$10x + y \ge 5$$
$$x + 10y \ge 1$$
$$x, y \ge 0$$

18. *Maximize* $Z = -x_1 + 2x_2$
Subject to
$$-x_1 + 3x_2 \le 10$$
$$x_1 + x_2 \le 6$$
$$x_1 - x_2 \le 2$$
$$x_1, x_2 \ge 0$$

19. *Maximize* $Z = x + y$
Subject to
$$-2x + y \le 1$$
$$x \le 2$$
$$x + y \le 3$$
$$x, y \ge 0$$

20. *Maximize* $Z = 3x_1 + 4x_2$, if possible,
Subject to the constraints
$$x_1 - x_2 \le -1$$
$$-x_1 + x_2 \le 0$$
$$x_1, x_2 \ge 0$$

21. Maximize $Z = 3x + 3y$, if possible,
Subject to the constraints
$$x - y \le 1$$
$$x + y \ge 3$$
$$x, y \ge 0$$

22. Show the solution zone of the following inequalities on a graph paper:
$$5x + y \ge 10$$
$$x + y \ge 6$$
$$x + 4y \ge 12$$
$$x \ge 0, y \ge 0.$$
Find x and y for which $3x + 2y$ is minimum subject to these inequalities. Use a graphical method.

23. Find the maximum and minimum value of $2x + y$ subject to the constraints:
$x + 3y \ge 6, x - 3y \le 3, 3x + 4y \le 24, -3x + 2y \le 6, 5x + y \ge 5, x, y \ge 0.$

24. Find the minimum value of $3x + 5y$ *subject to the constraints*
$-2x + y \le 4, x + y \ge 3, x - 2y \le 2, x, y \ge 0.$

25. Solved the following linear programming problem graphically:
Maximize $Z = 60x + 15y$
Subject to constraints
$$x + y \le 50$$
$$3x + y \le 90$$
$$x, y \ge 0$$
[CBSE 2005]

ANSWERS

1. $x = \frac{20}{19}, y = \frac{45}{19}, Z = \frac{235}{19}$ 2. $x = 2, y = 6, Z = 28$ 3. $x = 3, y = 8$, Min. $Z = 134$

4. $x = 10, y = 2, Z = 560$ 5. $x = \frac{24}{7}, y = \frac{27}{7}, Z = 24$ or $x = 5, y = \frac{4}{3}, Z = 24$ 6. $x = \frac{80}{3}, y = 0, Z = 400$ $x = 20, y = 10, Z = 400$

7. $x = 1, y = 6, Z = 56$ 8. $x = 20, y = 20, Z = 140$

9. $x = 18000, y = 12000, Z = 246000$ 10. $x = 4, y = 2, Z = 16$

11. $x = 3, y = 4, Z = 27$ 12. $x = \frac{4}{7}, y = \frac{15}{7}, Z = 60$

13. $x = \frac{24}{7}, y = \frac{24}{7}, Z = 24$ 14. $x = 4, y = 4, Z = 4; x = 5, y = \frac{4}{3}, Z = 24.$

15. $x = 10, y = 5, Z = 55$ 16. 7

17. 2 18. $\frac{20}{3}$ 19. 3 20. Does not exist

21. Max value is infinity i.e. the solution is unbounded

22. $x = 1, y = 5, Z = 13$

23. Max. $= \frac{43}{3}$ *at* $x = \frac{84}{13}, y = \frac{15}{3}$ 25. $x = 20, y = 30, Z = 1650$

29.9 DIFFERENT TYPES OF LINEAR PROGRAMMING PROBLEMS

In this section, we shall discuss formulation and solution of some important types of linear programming problems viz. Diet problems, Optimal product line problems and Transportation problems.

Type I DIET PROBLEMS

In this type of problems, we have to find the amount of different kinds of constituents/nutrients which should be included in a diet so as to minimize the cost of the desired diet such that it contains a certain minimum amount of each constituent/nutrient.

Following are some examples on this type of problems.

EXAMPLE 1 *A house wife wishes to mix together two kinds of food, X and Y, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C.*
The vitamin contents of one kg of food is given below:

	Vitamin A	*Vitamin B*	*Vitamin C*
Food X:	1	2	3
Food Y:	2	2	1

One kg of food X costs Rs 6 and one kg of food Y costs Rs 10. Find the least cost of the mixture which will produce the diet. [CBSE 2003]

SOLUTION Let x kg of food X and y kg of food Y are mixed together to make the mixture. Since one kg of food X contains one unit of vitamin A and one kg of food Y contains 2 units of vitamin A. Therefore, x kg of food X and y kg of food Y will contain $x + 2y$ units of vitamin A. But the mixture should contain at least 10 units of vitamin A. Therefore,

$$x + 2y \geq 10$$

Similarly, x kg of food X and y kg of food Y will produce $2x + 2y$ units of vitamin B and $3x + y$ units of vitamin C. But the minimum requirements of vitamins B and C are respectively of 12 and 8 units.

$$\therefore \quad 2x + 2y \geq 12 \text{ and } 3x + y \geq 8$$

Since the quantity of food X and food Y cannot be negative.

$$\therefore \quad x \geq 0, y \geq 0$$

It is given that one kg of food X costs Rs 6 and one kg of food Y costs Rs 10. So, x kg of food X and y kg of food Y will cost Rs $(6x + 10y)$.

Thus, the given linear programming problem is

$$\text{Minimize } Z = 6x + 10y$$

Subject to $x + 2y \geq 10$

$$2x + 2y \geq 12$$

$$3x + y \geq 8$$

and, $x \geq 0, y \geq 0$

To solve this LPP, we draw the lines

$$x + 2y = 10, 2x + 2y = 12 \text{ and } 3x + y = 8.$$

The feasible region of the LPP is shaded in Fig. 29.16.

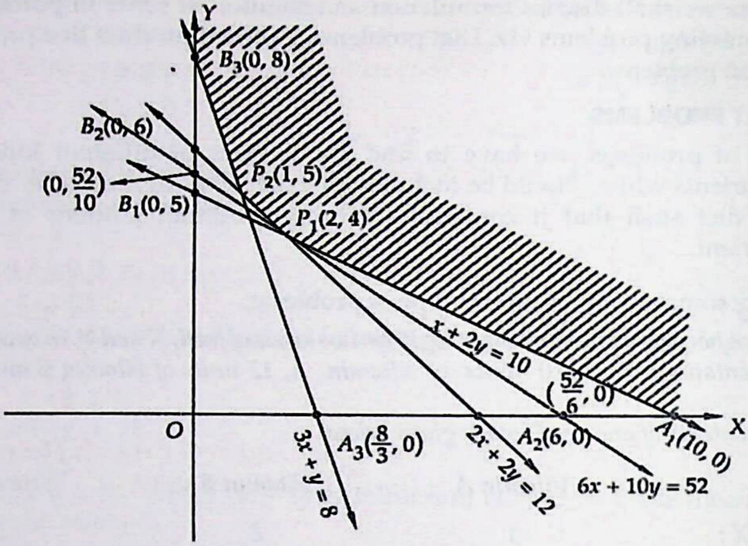

Fig. 29.16

The coordinates of the vertices (Corner-points) of shaded feasible region $A_1\ P_1\ P_2\ B_3$ are A_1 (10, 0), P_1 (2, 4), P_2 (1, 5) and B_3 (0, 8). These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of the objective function* $Z = 6x + 10y$
A_1 (10, 0)	$Z = 6 \times 10 + 10 \times 0 = 60$
A_2 (2, 4)	$Z = 6 \times 2 + 10 \times 4 = 52$
P_2 (1, 5)	$Z = 6 \times 1 + 10 \times 5 = 56$
B_3 (0, 8)	$Z = 6 \times 0 + 10 \times 8 = 80$

Clearly, Z is minimum at $x = 2$ and $y = 4$. The minimum value of Z is 52.

We observe that the open half-plane represented by $6x + 10y < 52$ does not have points in common with the feasible region. So, Z has minimum value equal to 52.

Hence, the least cost of the mixture is Rs 52.

EXAMPLE 2 *A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 10 units of vitamin C. Food 'I' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C while food 'II' contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 5.00 per kg to purchase food 'I' and Rs 7.00 per kg to produce food 'II'. Determine the minimum cost to such a mixture. Formulate the above as a LPP and solve it.*

SOLUTION Let the dietician mix x kg of food 'I' with y kg of food 'II'. Then, the mathematical model of the LPP is as follows:

Minimize $Z = 5x + 7y$

Subject to $2x + y \geq 8$

$x + 2y \geq 10$ [See Example 8, page 29.10]

and, $x, y \geq 0$

To solve this LPP graphically, we first convert the inequations into equations to obtain the following lines.

$$2x + y = 8, x + 2y = 10, x = 0, y = 0$$

The line $2x + y = 8$ meets the coordinate axes at A_1 (4, 0) and B_1 (0, 8). Join these points to obtain the line represented by $2x + y = 8$. The region not containing the origin is represented by $2x + y \geq 8$.

The line $x + 2y = 10$ meets the coordinate axes at A_2 (10, 0) and B_2 (0, 5). Join these points to obtain the line represented by $x + 2y = 10$. Clearly, O (0, 0) does not satisfy the inequation $x + 2y \geq 10$. So, the region not containing the origin is represented by this ineqution.

Clearly, $x \geq 0, y \geq 0$ represent the first quadrant.

Thus, the shaded region in Fig. 29.17 is the feasible region of the LPP. The coordinates of the corner-points of this region are A_2 (10, 0), P (2, 4) and B_1 (0, 8).

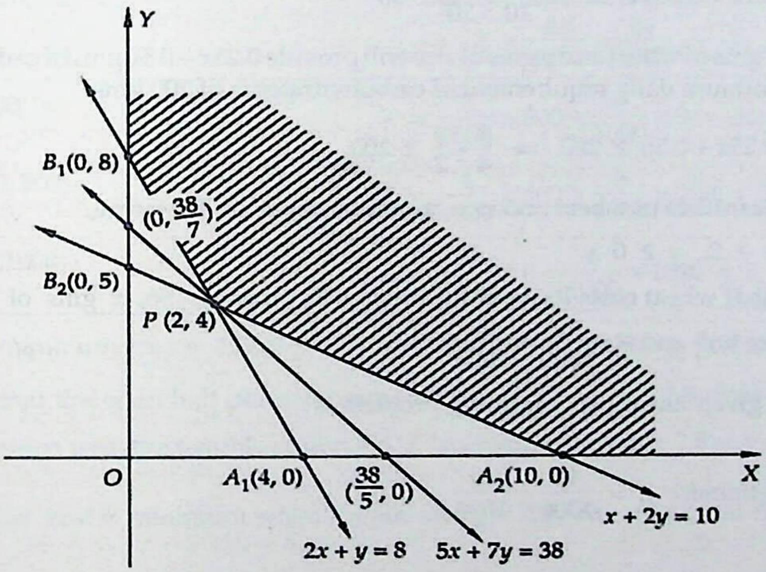

Fig. 29.17

The point P (2, 4) is obtained by solving $2x + y = 8$ and $x + 2y = 10$ simultaneously.

The values of the objective function $Z = 5x + 7y$ at the corner points of the feasible region are given in the following table :

Point (x, y)	*Value of the objective function* $Z = 5x + 7y$
A_2 (10, 0)	$Z = 5 \times 10 + 7 \times 0 = 50$
P (2, 4)	$Z = 5 \times 2 + 7 \times 4 = 38$
B_1 (0, 8)	$Z = 5 \times 0 + 7 \times 8 = 56$

Clearly, Z is minimum at $x = 2$ and $y = 4$. The minimum value of Z is 38.

We observe that open half plane represented by $5x + 7y < 38$ does not have points in common with the feasible region. So, Z has minimum value equal to 38 at $x = 2$ and $y = 4$.

Hence, the optimal mixing strategy for the dietician will be to mix 2 kg of food 'I' and 4 kg of food 'II'. In this case, his cost will be minimum and the minimum cost will be Rs 38.00.

EXAMPLE 3 *Every gram of wheat provides 0.1 gm of proteins and 0.25 gm of carbohydrates. The corresponding values of rice are 0.05 gm and 0.5 gm respectively. Wheat costs Rs. 4 per kg and rice Rs. 6. The minimum daily requirements of proteins and carbohydrates for an average child are 50 gms and 200 gms respectively. In what quantities should wheat and rice be mixed in the daily diet to provide minimum daily requirements of proteins and carbohydrates at minimum cost.*

SOLUTION Suppose x gms of wheat and y grams of rice are mixed in the daily diet.

Since every gram of wheat provides 0.1 gm of proteins and every gram of rice gives 0.05 gm of proteins. Therefore, x gms of wheat and y grams of rice will provide $0.1x + 0.05y$ gms of proteins. But the minimum daily requirement of proteins is of 50 gms.

$$\therefore \quad 0.1x + 0.05y \geq 50 \Rightarrow \frac{x}{10} + \frac{y}{20} \geq 50$$

Similarly, x gms of wheat and y gms of rice will provide $0.25x + 0.5y$ gms of carbohydrates and the minimum daily requirement of carbohydrates is of 200 gms.

$$\therefore \quad 0.25x + 0.5y \geq 200 \Rightarrow \frac{x}{4} + \frac{y}{2} \geq 200$$

Since the quantities of wheat and rice cannot be negative. Therefore,

$$x \geq 0, \; y \geq 0$$

It is given that wheat costs Rs 4 per kg and rice Rs 6 per kg. So, x gms of wheat and y gms of rice will cost Rs $\frac{4x}{1000} + \frac{6y}{1000}$

Hence, the given linear programming problem is

$$\text{Minimize } Z = \frac{4x}{1000} + \frac{6y}{1000}$$

Subject to the constraints

$$\frac{x}{10} + \frac{y}{20} \geq 50,$$

$$\frac{x}{4} + \frac{y}{2} \geq 200,$$

and, $\quad x \geq 0, \; y \geq 0$

The solution set of the linear constraints is shaded in Fig. 29.18. The vertices of the shaded region are A_2 (800,0), P (400, 200) and B_1 (0, 1000).

The values of the objective function at these points are given in the following table.

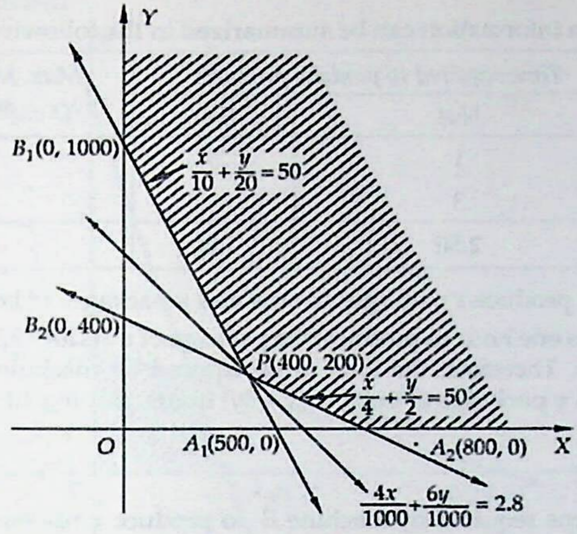

Fig. 29.18

Point (x_1, x_2)	*Value of objective function* $Z = \frac{4x}{1000} + \frac{6y}{1000}$
A_2 (800, 0)	$Z = \frac{4}{1000} \times 800 + \frac{8}{1000} \times 0 = 3.2$
P (400, 200)	$Z = \frac{4}{1000} \times 400 + \frac{6}{1000} \times 200 = 2.8$
B_1 (0, 1000)	$Z = \frac{4}{1000} \times 0 + \frac{6}{1000} \times 1000 = 6$

Clearly, Z is minimum for $x = 400$, $y = 200$ and the minimum value of Z is 2.8.

We observe that the open half plane represented by $\frac{4x}{1000} + \frac{6y}{1000} < 2.8$ does not have points is common with the feasible region. So, Z has minimum value 2.8 at $x = 400$ and $y = 200$.

Hence, the diet cost is minimum when $x = 400$ and $y = 200$. The minimum diet cost is Rs 2.8.

Type II OPTIMAL PRODUCT LINE PROBLEMS

In this type of problems, we have to determine the number of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hours per unit of the product, ware house space per unit of output, etc. in order to make maximum profit.

Following are some examples on this type of problems.

EXAMPLE 4 *A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machin B to produce a package of nuts while it takes 3 hours on macine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs 2.50 per package of nuts and Re 1.00 per package of bolts. How many packages of each should he produce each day so as to maximize his profit, if he operates his machines for at most 12 hours a day ? Formulate this mathematically and then solve it.* **[NCERT]**

SOLUTION The given information can be summarized in the following tabular form:

Machines	*Time required to produce products*		*Max. Machine hours available*
	Nut	*Bolt*	
A	1	3	12
B	3	1	12
Profit (in Rs)	2.50	1.00	

Let the manufacturer produce x packages of nuts and y packages of bolts each day. Since machine A takes one hour to produce one package of nuts and 3 hours to produce one package of bolts. Therefore, the total time required by machine A to produce x packages of nuts and y packages of bolts is $(x+3y)$ hours. But machine A operates for at most 12 hours.

$\therefore \quad x+3y \leq 12$

Similarly, the total time required by machine B to produce x packages of nuts and y packages of bolts is $(3x+y)$ hours. But machine B operates for at most 12 hours.

$\therefore \quad 3x+y \leq 12$

Since the profit on one package of nuts is Rs 2.50 and on one package of bolts the profit is Re 1. Therefore, profit on x packages of nuts and y packages of bolts is of Rs $(2.50x+y)$. Let Z denote the total profit. Then, $Z = 2.50x+y$.

Clearly, $x \geq 0$ and $y \geq 0$

Thus, the above LPP can be stated mathematically as follows:

$$\text{Maximize } Z = 2.50\,x+y$$

$$\text{Subject to } \quad x+3y \leq 12$$

$$3x+y \leq 12$$

$$\text{and,} \quad x, y \geq 0$$

To solve this LPP graphically, we first convert the inequations into equations to obtain the following equations.

$$x+3y = 12, 3x+y = 12, x = 0, y = 0$$

The line $x+3y=12$ meets the coordinate axes at $A_1\,(12, 0)$ and $B_1\,(0, 4)$. Join these two points to obtain the line represented by $x+3y=12$. The region represented by the inequation $x+3y \leq 12$ is the region containing the origin as $x=0, y=0$ satisfies the inequation $x+3y \leq 12$.

The line $3x+y=12$ meets the coordinate axes at $A_2\,(4, 0)$ and $B_2\,(0, 12)$. Join these points to obtain the line represented by $3x+y=12$. Since $x=0, y=0$ satisfies the inequation $3x+y \leq 12$. So, the region containing the origin and below the line $3x+y=12$ represents the region represented by $3x+y \leq 12$.

Clearly, $x \geq 0$ and $y \geq 0$ represent all points in first quadrant.

Thus, the shaded region OA_2PB_1 in Fig. 29.19 represents the feasible region of the given LPP.

The coordinates of the corner-points of the feasible region OA_2PB_1 are $O\,(0, 0)$, $A_2\,(4, 0)$, $P\,(3, 3)$ and $B_1\,(0, 4)$. These points are obtained by solving the corresponding intersecting lines simultaneously.

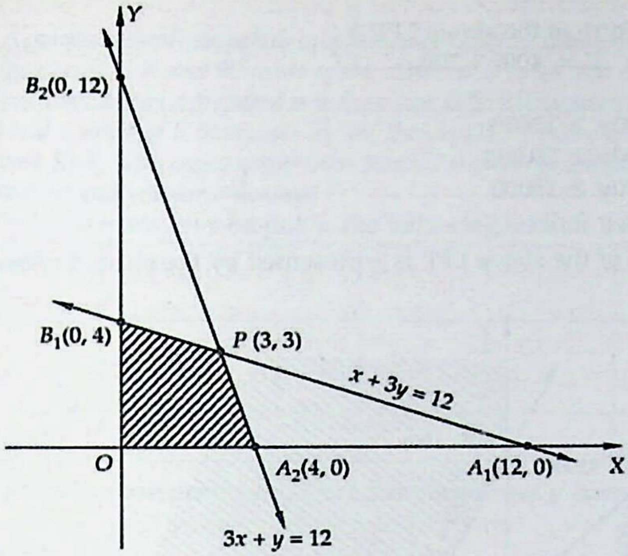

Fig. 29.19

The values of the objective function at the corner-points of the feasible region are given in the following table :

Point (x, y)	*Value of the objective function* $Z = 2.50x + y$
$O(0, 0)$	$Z = 2.50 \times 0 + 1 \times 0 = 0$
$A_2(4, 0)$	$Z = 2.50 \times 4 + 1 \times 0 = 10$
$P(3, 3)$	$Z = 2.50 \times 3 + 1 \times 3 = 10.50$
$B_1(0, 4)$	$Z = 2.50 \times 0 + 1 \times 4 = 4$

Clearly, Z is maximum at $x = 3$, $y = 3$ and the maximum value of Z is 10.50.

Hence, the optimal production strategy for the manufacturer will be to manufacture 3 packages each of nuts and bolts daily and in this case his maximum profit will be Rs 10.50.

EXAMPLE 5 *An oil company requires 12,000, 20,000 and 15,000 barrels of high-grade, medium grade and low grade oil, respectively. Refinery A produces 100, 300 and 200 barrels per day of high-grade, medium-grade and low-grade oil, respectively, while refinery B produces 200, 400 and 100 barrels per day of high-grade, medium-grade and low-grade oil, respectively. If refinery A costs Rs 400 per day and refinery B costs Rs 300 per day to operate, how many days should each be run to minimize costs while satisfying requirements.* **[CBSE 2004]**

SOLUTION The given data may be put in the following tabular form :

Refinery	*High-grade*	*Medium-grade*	*Low-grade*	*Cost per day*
A	100	300	200	Rs 400
B	200	400	100	Rs 300
Minimum Requirement	12,000	20,000	15,000	

Suppose refineries A and B should run for x and y days respectively to minimize the total cost.

The mathematical form of the above LPP is

$$\text{Minimize } Z = 400x + 300y$$

Subject to

$$100x + 200y \geq 12000$$
$$300x + 400y \geq 20{,}000$$
$$200x + 100y \geq 15000$$

and, $$x, y \geq 0$$

The feasible region of the above LPP is represented by the shaded region in Fig. 29.20.

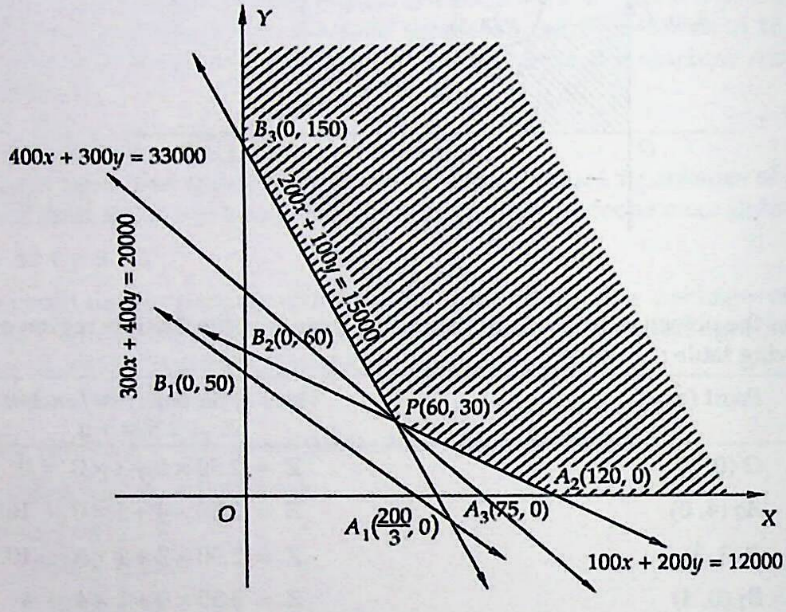

Fig. 29.20

The corner points of the feasible region are A_2 (120, 0), P (60, 30) and B_3 (0, 150). The value of the objective function at these points are given in the following table :

Point (x, y)	*Value of the objective function* $Z = 400x + 300y$
A_2 (120, 0)	$Z = 400 \times 120 + 300 \times 0 = 48000$
P (60, 30)	$Z = 400 \times 60 + 300 \times 30 = 33000$
B_3 (0, 150)	$Z = 400 \times 0 + 300 \times 150 = 45000$

Clearly, Z is minimum when $x = 60, y = 30$. The feasible region is unbounded. So, we find the half-plane represented by $400x + 300y < 33000$. Clearly, the half-plane does not have points common with the feasible region. So, Z is minimum at $x = 60,\ y = 30$.

Hence, the machine A should run for 60 days and the machine B should run for 30 days to minimize the cost while satisfying the constraints.

EXAMPLE 6 *A company produces soft drinks that has a contract which requires that a minimum of 80 units of the chemical A and 60 units of the chemical B to go into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. Supplier S has a mix of 4 units of A and 2 units of B that costs Rs 10, the supplier T has a mix of 1 unit of A and 1 unit of B that costs Rs 4. How many mixes from S and T should the company purchase to honour contract requirement and yet minimize cost ?*

SOLUTION The given data may be put in the following tabular form :

Supplier / *Chemical*	*S*	*T*	*Minimum Requirement*
A	4	1	80
B	2	1	60
Cost per unit	Rs 10	Rs 4	

Suppose x units of mix are purchased from supplier S and y units are purchased from supplier T.

Total cost $Z = 10x + 4y$.

Units of chemical A per bottle $= 4x + y$.

But the minimum requirement of chemical A per bottle = 80

$\therefore \quad 4x + y \geq 80.$

Similarly, $2x + y \geq 60$

Clearly, $x \geq 0, y \geq 0$.

Thus, the mathematical formulation of the given LPP is

Minimize $Z = 10x + 4y$

Subject to

$$4x + y \geq 80$$
$$2x + y \geq 60$$

and, $x \geq 0, y \geq 0$

Now, we find the feasible region which is the set of all points whose coordinates simultaneously satisfy all constraints including non-negativity restrictions. The shaded region in Fig. 29.21 represents the feasible region of the given LPP. The coordinates of the corner points of the feasible region are A_2 (30, 0), P (10, 40), B_1 (0, 80).

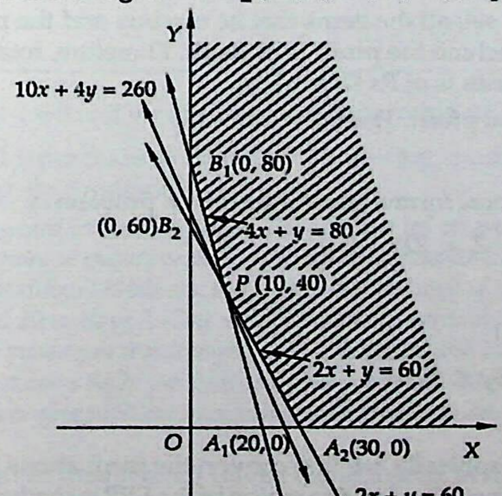

Fig. 29.21

These points are obtained by solving the equations of the corresponding intersecting lines, simultaneously.

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of objective function* $Z = 10x + 4y$
$A_2\,(30, 0)$	$Z = 10 \times 30 + 4 \times 0 = 300$
$P\,(10, 40)$	$Z = 10 \times 10 + 40 \times 4 = 260$
$B_1\,(0, 80)$	$Z = 10 \times 0 + 4 \times 80 = 320.$

Clearly, Z is minimum at (10, 40). The feasible region is unbounded and the open half plane represented by $10x + 4y < 260$ does not have points in common with the feasible region. So, Z is minimum at $x = 10, y = 40$. Hence, $x = 10, y = 40$ is the optimal solution of the given LPP.

Hence, the cost per bottle is minimum when the company purchases 10 mixes from supplier *S* and 40 m ... plier *T*.

EXAMPLE 7 *A ... a number of fans and sewing machines. He has only Rs 5760.00 to ... 20 items. A fan costs him Rs 360.00 and a sewing machine Rs 2 ... can sell a fan at a profit of Rs 22.00 and a sewing machine at a ... hat he can sell all the items that he can buy, how should he inv ... ize his profit ? Translate this problem mathematically and the ...* **[CBSE 2001C, 2002C]**

SOLUTION ... s and y sewing machines. Since the dealer has space for at mos ...

$$x + y \le 20$$

A fan costs Rs 360 and a sewing machine costs Rs 240. Therefore, total cost of x fans and y sewing machines is Rs $(360x + 240y)$. But the dealer has only Rs 5760 to invest. Therefore,

$$360x + 240y \le 5760$$

Since the dealer can sell all the items that he can buy and the profit on a fan is of Rs 22 and on a sewing machine the profit is of Rs 18. Therefore, total profit on selling x fans and y sewing machines is of Rs $(22x + 18y)$.

Let Z denote the total profit. Then, $Z = 22x + 18y$.

Clearly, $x, y \ge 0$.

Thus, the mathematical formulation of the given problem is

$$\textit{Maximize } Z = 22x + 18y$$

Subject to

$$x + y \le 20$$

$$360x + 240y \le 5760$$

and, $x \ge 0, y \ge 0$

To solve this LPP graphically, we first convert the inequations into equations and draw the corresponding lines. The feasible region of the LPP is shaded in Fig. 29.22. The corner points of the feasible region OA_2PB_1 are $O\,(0, 0)$, $A_2\,(16, 0)$, $P\,(8, 12)$ and $B_1\,(0, 20)$.

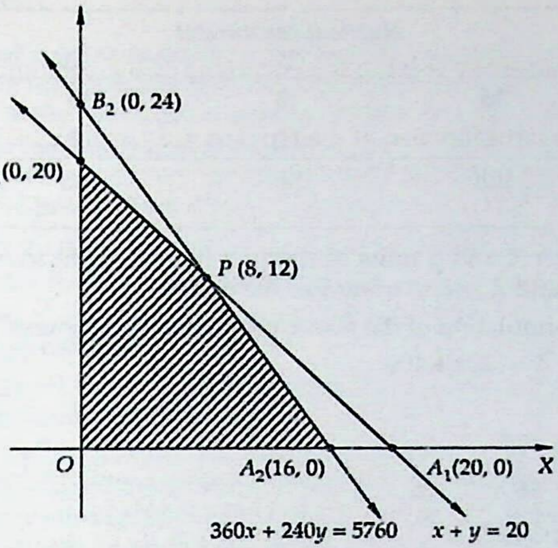

Fig. 29.22

These points have been obtained by solving the corresponding intersecting lines, simultaneously.

The values of the objective function Z at corner- points of the feasible region are given in the following table.

Point (x, y)	*Value of the objective function* $Z = 22x + 18y$
O (0 0)	$Z = 22 \times 0 + 18 \times 0 = 0$
A_2 (16, 0)	$Z = 22 \times 16 + 18 \times 0 = 352$
P (8, 12)	$Z = 22 \times 8 + 18 \times 12 = 392$
B_1 (0, 20)	$Z = 22 \times 0 + 20 \times 18 = 360$

Clearly, Z is maximum at $x = 8$ and $y = 12$. The maximum value of Z is 392.

Hence, the dealer should purchase 8 fans and 12 sewing machines to obtain the maximum profit under given conditions.

EXAMPLE 8 *A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents (call them X, Y and Z), it is necessary to buy two additional products, say, A and B. One unit of product A contains 36 units of X, 3 units of Y, and 20 units of Z. One unit of product B contains 6 units of X, 12 units of Y and 10 units of Z. The minimum requirement of X, Y and Z is 108 units, 36 units and 100 units respectively. Product A costs Rs 20 per unit and product B costs Rs 40 per unit. Formulate the above as a linear programming problem to minimize the total cost, and solve the problem by using graphical method.*

SOLUTION The data given in the problem can be summarized in the following tabular form :

Product	Nutrient constituent X	Nutrient constituent Y	Nutrient constituent Z	Cost in Rs
A	36	3	20	20
B	6	12	10	40
Minimum Requirement	108	36	100	

Let x units of product A and y units of product B are bought to fulfill the minimum requirement of X, Y and Z and to minimize the cost.

The mathematical formulation of the above problem is as follows:

$$\textit{Minimize } Z = 20x + 40y$$

Subject to

$$36x + 6y \geq 108$$
$$3x + 12y \geq 36$$
$$20x + 10y \geq 100$$

and, $\quad x, y, z \geq 0.$

The set of all feasible solutions of the above LPP is represented by the feasible region shaded darkly in Fig. 29.23. The coordinates of the corner points of the feasible region are A_2 (12, 0), P_1 (4, 2), P_2 (2, 6) and B_1 (0, 18).

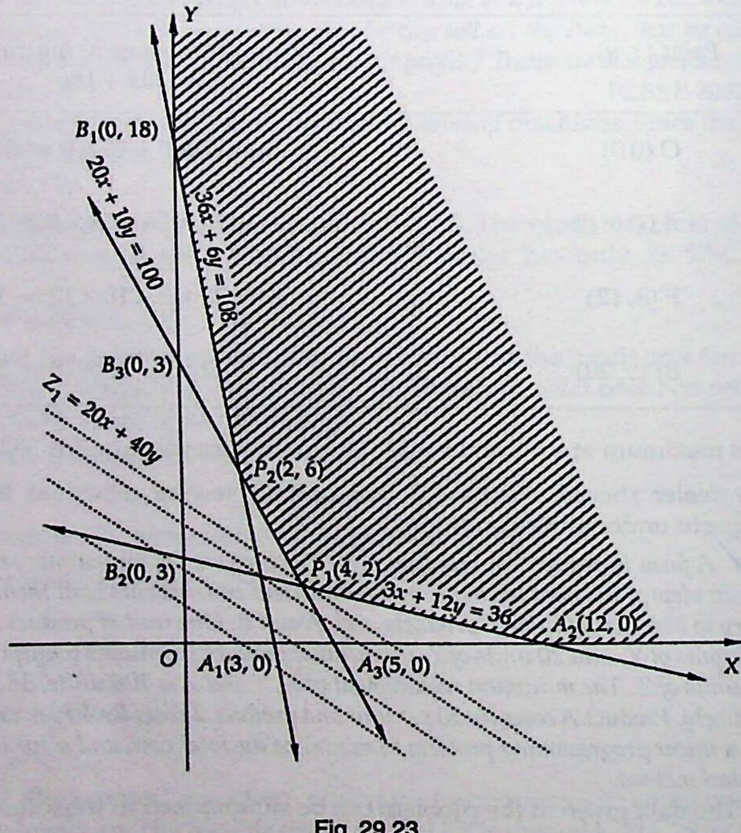

Fig. 29.23

Now, we have to find a point or points in the feasible region which give the minimum value of the objective function. For this, let us give some value to Z, say 20, and draw a dotted line $20 = 20x + 40y$. Now, draw lines parallel to this line which have at least one point common to the feasible region and locate a line which is nearest to the origin and has at least one point common to the feasible region. Clearly, such a line is $Z_1 = 20x + 40y$ and it has a point P_1 (4, 2) common with the feasible region. Thus, $Z_1 = 20x + 40y$ is the minimum value of Z, and the feasible solution which gives this value of Z is the corner P_1 (4, 2) of the shaded region. The values of the variables for the optimal solution are $x = 4, y = 2$. Substituting these values in $Z = 20x + 40y$, we get $Z = 160$ as the optimal value of Z.

Hence, 2 units of product A and 4 units of product B are sufficient to fulfill the minimum requirement at a minimum cost of Rs 160.

EXAMPLE 9 *A toy manufacturer produces two types of dolls; a basic version doll A and a deluxe version doll B. Each doll of type B takes twice as long to produce as one doll of type A. The company have time to make a maximum of 2000 dolls of type A per day, the supply of plastic is sufficient to produce 1500 dolls per day and each type requires equal amount of it. The deluxe version, i.e. type B requires a fancy dress of which there are only 600 per day available. If the company makes a profit of Rs 3 and Rs 5 per doll, respectively, on doll A and B; how many of each should be produced per day in order to maximize profit ? Solve it by graphical method.*

SOLUTION Let x dolls of type A and y dolls of type B be produced per day to maximize the profit.

The mathematical form of the given LPP is as follows:

Maximize $Z = 3x + 5y$

Subject to $x + 2y \leq 2000$

$x + y \leq 1500$ *(See Ex.2 on page 29.5)*

$y \leq 600$

and, $x, y \geq 0$.

The set of all feasible solutions of the given LPP is represented by the feasible region shaded darkly in Fig. 29.24. The coordinates of the corner points of the feasible region are O (0, 0), A_2 (1500, 0), P (1000, 500), Q (800, 600) and R (0, 600).

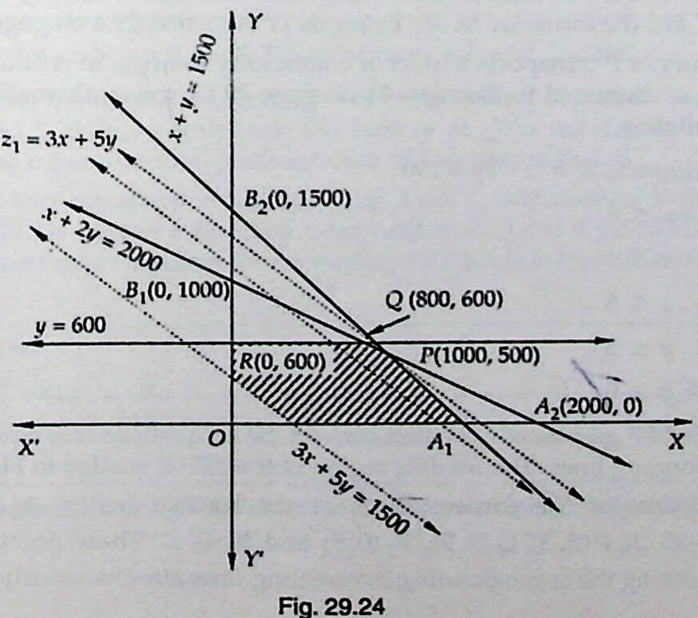

Fig. 29.24

Now, to find a point or points in the feasible region which give the maximum value of the objective function $Z = 3x + 5y$, let us give some value to Z, say 1500 and draw the dotted line $3x + 5y = 1500$ as shown in Fig. 29.23.

Now, draw lines parallel to the line $3x + 5y = 1500$ and obtain a line which is farthest from the origin and have at least one point common to the feasible region. Clearly, $Z_1 = 3x + 5y$ is such a line. This line has only one point $P(1000, 500)$ common to the feasible region. Thus, $Z = 3 \times 1000 + 5 \times 500 = 5500$ is the maximum value of Z and the optimal solution is $x = 1000$, $y = 500$.

Hence, 1000 dolls of type A and 500 dolls of type B should be produced to maximize the profit and the maximum profit is Rs 5500.

Type III TRANSPORTATION PROBLEMS

In this type of problems, we have to determine transportation schedule for a commodity from different plants or factories situated at different locations to different markets at different locations in such a way that the total cost of transportation is minimum, subject to the limitations (constraints) as regards the demand of each market and supply from each plant or factory.

Following are some examples on this type of problems:

EXAMPLE 10 *There is a factory located at each of the two places P and Q. From these locations, a certain commodity is delivered to each of these depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:*

To / From	Cost (in Rs)		
	A	B	C
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. Formulate the above LPP mathematically and then solve it.

SOLUTION For the formulation see Example 11 in section 29.4 on page 29.13.

Let the factory at P transports x units of commodity to depot at A and y units to depot at B. Then, as discussed in Example 11 on page 29.13, the mathematical model of the LPP is as follows:

$$\text{Minimize } Z = x - 7y + 190$$

$$\begin{aligned} \text{Subject to } x + y &\leq 8 \\ x + y &\geq 4 \\ x &\leq 5 \\ y &\leq 5 \\ \text{and, } \quad x \geq 0, y &\geq 0 \end{aligned}$$

To solve this LPP graphically, we first convert the inequations into equations and draw the corresponding lines. The feasible region of the LPP is shaded in Fig. 29.25.

The coordinates of the corner points of the feasible region $A_2 A_3 PQ B_3 B_2$ are $A_2(4, 0)$, $A_3(5, 0)$, $P(5, 3)$, $Q(3, 5)$, $B_3(0, 5)$ and $B_2(0, 4)$. These points have been obtained by solving the corresponding intersecting lines simultaneously.

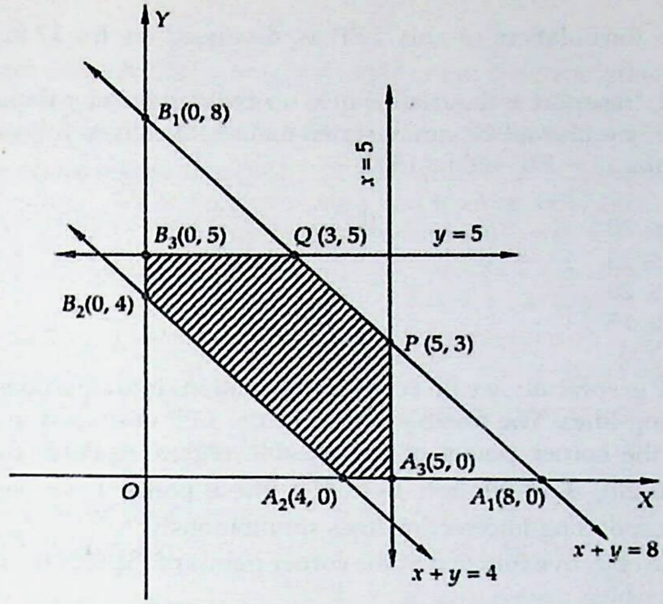

Fig. 29.25

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of the objective function* $Z = x - 7y + 190$
A_2 (4, 0)	$Z = 4 - 7 \times 0 + 190 = 194$
A_3 (5, 0)	$Z = 5 - 7 \times 0 + 190 = 195$
P (5, 3)	$Z = 5 - 7 \times 3 + 190 = 174$
Q (3, 5)	$Z = 3 - 7 \times 5 + 190 = 158$
B_3 (0, 5)	$Z = 0 - 7 \times 5 + 190 = 155$
B_2 (0, 4)	$Z = 0 - 7 \times 4 + 190 = 162$

Clearly, Z is minimum at $x = 0, y = 5$. The minimum value of Z is 155.

Thus, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 unit from the factory at Q to the depots at A, B and C respectively. The minimum transportation cost in this case is Rs 155.

EXAMPLE 11 *A brick manufacturer has two depots, A and B, with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three builders P, Q and R for 15,000, 20,000 and 15,000 bricks respectively. The cost in Rs transporting 1000 bricks to the builders from the depots are given below:*

From / *To*	P	Q	R
A	40	20	30
B	20	60	40

How should the manufacturer fulfill the orders so as to keep the cost of transportation minimum?

SOLUTION The formulation of this LPP is discussed in Ex. 12 in section 29.4 on page 29.14.

Let the depot A transport x thousand bricks to builder P and y thousand bricks to builder Q. Then, the above LPP can be stated mathematically as follows :

$$\text{Minimize } Z = 30x - 30y + 1800$$

Subject to

$$x + y \leq 30$$
$$x \leq 15$$
$$y \leq 20$$
$$x + y \geq 15$$

and, $x \geq 0, y \geq 0$

To solve this LPP graphically, we first convert inequations into equations and then draw the corresponding lines. The feasible region of the LPP is shaded in Fig. 29.26. The coordinates of the corner points of the feasible region $A_2\,PQ\,B_3\,B_2$ are $A_2\,(15, 0)$, $P\,(15, 15)$, $Q\,(10, 20)$, $B_3\,(0, 20)$ and $B_2\,(0, 15)$. These points have been obtained by solving the corresponding intersecting lines simultanously.

The values of the objective function at the corner points of the feasible region are given in the following table

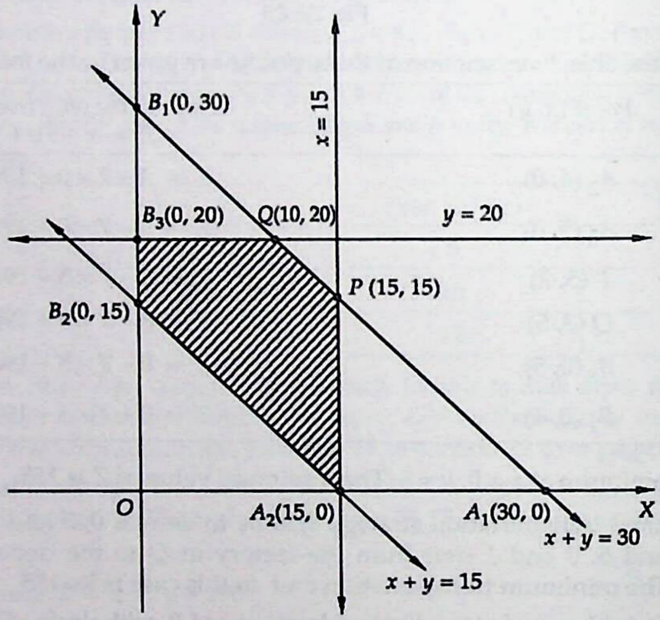

Fig. 29.26

Point (x, y)	*Value of the objective function* $Z = 30x - 30y + 1800$
$A_2\,(15, 0)$	$Z = 30 \times 15 - 30 \times 0 + 1800 = 2250$
$P\,(15, 15)$	$Z = 30 \times 15 - 30 \times 15 + 1800 = 1800$
$Q\,(10, 20)$	$Z = 30 \times 10 - 30 \times 20 + 1800 = 1500$
$B_3\,(0, 20)$	$Z = 30 \times 0 - 30 \times 20 + 1800 = 1200$
$B_2\,(0, 15)$	$Z = 30 \times 0 - 30 \times 15 + 1800 = 1350$

Clearly, Z is minimum at $x = 0, y = 20$ and the minimum value of Z is 1200.

Thus, the manufacturer should supply 0, 20 and 10 thousand bricks to builders P, Q and R from depot A and 15, 0 and 5 thousand bricks to builders P, Q and R from depot B respectively. In this case the minimum transportation cost will be Rs 1200.

29.10 SOME EXCEPTIONAL CASES

Uptill now we have been discussing linear programming problems having finite unique solutions. In this section, we shall discuss some problems which either do not have solutions or they have unbounded solutions. Consider the following linear programming problem:

$$\text{Maximize } Z = 2x + 5y$$

Subject to the constraints

$$x + y \le 4$$
$$3x + 3y \ge 18$$
$$x, y \ge 0.$$

This problem is shown graphically in Fig. 29.27. Clearly, it has no solution because the constraints are inconsistent.

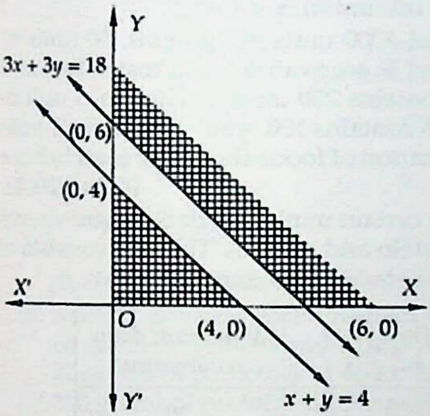

Fig. 29.27

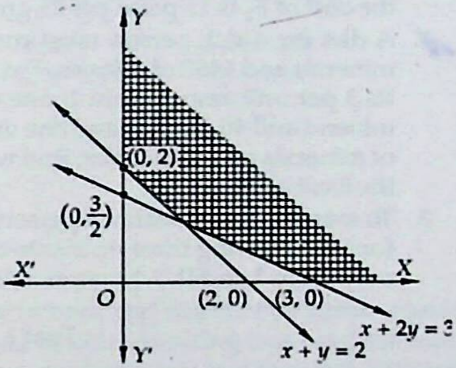

Fig. 29.28

In some linear programming problems, the common feasible region may not be bounded and the variables can take any value in the unbounded feasible region. Such type of problems are said to have unbounded solutions. Consider the following linear programming problem.

$$\text{Maximize } Z = 2x + 3y$$

Subject to the constraints

$$x + y \ge 2$$
$$x + 2y \ge 3$$
$$x, y \ge 0$$

This problem is illustrated graphically in Fig. 29.28.

The feasible region is shaded in Fig. 29.28. Clearly, x and y can take arbitrary large values. So, the objective function can be made as large as we please. Consequently, we say that the problem has unbounded solution.

EXERCISE 29.3

Type I **DIET PROBLEMS**

1. A diet of two foods F_1 and F_2 contains nutrients thiamine, phosphorous and iron. The amount of each nutrient in each of the food (in milligrams per 25 gms) is given in the following table:

Food / Nutrients	F_1	F_2
Thiamine	0.25	0.10
Phosphorous	0.75	1.50
Iron	1.60	0.80

The minimum requirement of the nutrients in the diet are 1.00 mg of thiamine, 7.50 mg of phosphorous and 10.00 mg of iron. The cost of F_1 is 20 paise per 25 gms while the cost of F_2 is 15 paise per 25 gms. Find the minimum cost of diet.

2. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 of calories. Two foods A and B, are available at a cost of Rs 4 and Rs 3 per unit respectively. If one unit of A contains 200 units of vitamin, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamin, 2 units of minerals and 40 calories, find what combination of foods should be used to have the least cost? [CBSE 2004]

3. To mantain one's health, a person must fulfil certain minimum daily requirements for the following three nutrients-calcium, protein and calories. This diet consists of only items I and II whose prices and nutrient contents are shown below:

	Food I	*Food II*	*Minimum daily requirement*
Calcium	10	4	20
Protein	5	6	20
Calories	2	6	12
Price	Re 0.60 per unit	Re 1.00 per unit	

Find the combination of food items so that the cost may be minimum.

4. A hospital dietician wishes to find the cheapest combination of two foods, A and B, that contains at least 0.5 milligram of thiamin and at least 600 calories. Each unit of A contains 0.12 milligram of thiamin and 100 calories, while each unit of B contains 0.10 milligram of thiamin and 150 calories. If each food costs 10 paise per unit, how many units of each should be combined at a minimum cost ?

5. A dietician mixes together two kinds of food in such a way that the mixture contains at least 6 units of vitamin A, 7 units of vitamin B, 11 units of vitamin C and 9 units of vitamin D. The vitamin contents of 1 kg of food X and 1 kg of food Y are given below:

	Vitamin A	*Vitamin B*	*Vitamin C*	*Vitamin D*
Food X	1	1	1	2
Food Y	2	1	3	1

One kg of food X costs Rs 5, whereas one kg of food Y costs Rs 8. Find the least cost of the mixture which will produce the desired diet.

6. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs Rs 4 per unit and F_2 costs Rs 6 per unit one unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem and find graphically the minium cost for diet that consists of mixture of these foods and also meets the mineral nutritional requirements. [CBSE 2009]

7. To maintain one's health, a person must fulfil minimum daily requirements for the following three nutrients calcium, protein and calories. His diet consists of only food items I and II whose prices and nutrient contents are shown below:

Price	*Food I* *Re. 0.60 per unit*	*Food II* *Re 1 per unit*	*Minimum* *requirement*
Calcium	10	4	20
Protein	5	5	20
Calories	2	6	12

Find the combination of food items so that the cost may be minimum.

8. Kellogg is a new cereal formed of a mixture of bran and rice that contains at least 88 grams of protein and at least 36 milligrams of iron. Knowing that bran contains 80 grams of protein and 40 milligrams of iron per kilogram, and that rice contains 100 grams of protein and 30 milligrams of iron per kilogram, find the minimum cost of producing this new cereal if bran costs Rs 5 per kg and rice costs Rs 4 per kg. [CBSE 2002]

Type II OPTIMAL PRODUCT LINE PROBLEMS

9. A manufacturer has three machines installed in his factory. Machines I and II are capable of being operated for at most 12 hours wheareas Machine III must operate at least for 5 hours a day. He produces only two items, each requiring the use of three machines. The number of hours required for producing one unit each of the items on the three machines is given in the following table :

Item	*Number of hours required by the machine*		
	I	II	III
A	1	2	1
B	2	1	5/4

He makes a profit of Rs 6.00 on item A and Rs 4.00 on item B. Assuming that he can sell all that he produces, how many of each item should he produce so as to maximize his profit? Determine his maximum profit. Formulate this LPP mathematically and then solve it.

He makes a profit of Rs 6.00 on item A and Rs 4.00 on item B. Assuming that he can sell all that he produces, how many of each item should he produce so as to maximize his profit? Determine his maximum profit. Formulate this LPP mathematically and then solve it.

10. Two tailors, A and B earn Rs 15 and Rs 20 per day respectively. A can stitch 6 shirts and 4 pants while B can stitch 10 shirts and 4 pants per day. How many days shall each work if it is desired to produce (at least) 60 shirts and 32 pants at a minimum labour cost?

11. A factory manufactures two types of screws, A and B, each type requiring the use of two machines — an automatic and a hand-operated. It takes 4 minute on the automatic and 6 minutes on the hand-operated machines to manufacture a package of screws 'A', while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a package of screws 'B'. Each machine is available for at most 4 hours on any day. The manufacturer can sell a package of screws 'A' at a profit of 70 P and screws 'B' at a profit of Re 1. Assuming that he can sell all the screws he can manufacture, how many packages of each type should the factory owner produce in a day in order to maximize his profit ? Determine the maximum profit. [NCERT]

12. A company produces two types of leather belts, say type A and B. Belt A is a superior quality and belt B is of a lower quality. Profits on each type of belt are Rs 2 and Rs 1.50 per belt, respectively. Each belt of type A requires twice as much time as required by a belt of type B. If all belts were of type B, the company could produce 1000 belts per day. But the supply of leather is sufficient only for 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 fancy buckles are available for this per day. For belt of type B, only 700 buckles are available per day.

    How should the company manufacture the two types of belts in order to have a maximum overall profit?

13. A small manufacturer has employed 5 skilled men and 10 semi-skilled men and makes an article in two qualities deluxe model and an ordinary model. The making of a deluxe model requires 2 hrs. work by a skilled man and 2 hrs. work by a semi-skilled man. The ordinary model requires 1 hr by a skilled man and 3 hrs. by a semi-skilled man. By union rules no man may work more than 8 hrs per day. The manufacturers clear profit on deluxe model is Rs 15 and on an ordinary model is Rs 10. How many of each type should be made in order to maximize his total daily profit.

14. A manufacturer makes two types A and B of tea-cups. Three machines are needed for the manufacture and the time in minutes required for each cup on the machines is given below:

	Machines		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each cup A is 75 paise and that on each cup B is 50 paise, show that 15 tea-cups of type A and 30 of type B should be manufactured in a day to get the maximum profit. [CBSE 2003, 2008]

15. A factory owner purchases two types of machines, A and B, for his factory. The requirements and limitations for the machines are as follows:

	Area occupied by the machine	*Labour force for each machine*	*Daily output in units*
Machine A	1000 sq. m	12 men	60
Machine B	1200 sq. m	8 men	40

He has an area of 7600 sq.m available and 72 skilled men who can operate the machines. How many machines of each type should he buy to maximize the daily output? **[CBSE 2003, 2008]**

16. A company produces two types of goods, A and B, that require gold and silver. Each unit of type A requires 3 gm of silver and 1 gm of gold while that of type B requires 1 gm of silver and 2 gm of gold. The company can produce 9 gm of silver and 8 gm of gold. If each unit of type A brings a profit of Rs 40 and that of type B Rs 50, find the number of units of each type that the company should produce to maximize the profit. What is the maximum profit?
17. A manufacturer of Furniture makes two products : chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hrs on machine A and 6 hrs on machine B. A table requires 4 hrs on machine A and 2 hrs on machine B. There are 16 hrs of time per day available on machine A and 30 hrs on machine B. Profit gained by the manufacturer from a chair and a table is Rs 3 and Rs 5 respectively. Find with the help of graph what should be the daily production of each of the two products so as to maximize his profit.
18. A furniture manufacturing company plans to make two products : chairs and tables. From its available resources which consists of 400 square feet of teak wood and 450 man hours. It is known that to make a chair requires 5 square feet of wood and 10 man-hours and yields a profit of Rs 45, while each table uses 20 square feet of wood and 25 man-hours and yields a profit of Rs 80.
    How many items of each product should be produced by the company so that the profit is maximum.
19. A wholesale dealer deals in two kinds, A and B (say) of mixture of nuts. Each kg. of mixture A contains 60 grams of almonds, 30 grams of cashew nuts and 30 grams of hazel nuts. Each kg. of mixture B contains 30 grams of almonds, 60 grams of cashew nuts and 180 grams of hazel nuts. The remander of both mixtures is per nuts. The dealer is contemplating to use mixtures A and B to make a bag which will contain at least 240 grams of almonds, 300 grams of cashew nuts and 540 grams of hazel nuts.
    Mixture A costs Rs 8 per kg. and mixture B costs Rs 12 per kg. Assuming that mixtures A and B are uniform, use graphical method to determine the number of kg. of each mixture which he should use to minimise the cost of the bag.
20. A firm manufactures two products A and B. Each product is processed on two machines M_1 and M_2. Product A requires 4 minutes of processing time on M_1 and 8 min. on M_2; product B requires 4 minutes on M_1 and 4 min. on M_2. The machine M_1 is available for not more than 8 hrs 20 min. while machine M_2 is available for 10 hrs. during any working day. The products A and B are sold at a profit of Rs 3 and Rs 4 respectively.
    Formulate the problem as a linear programming problem and find how many products of each type should be produced by the firm each day in order to get maximum profit.
21. A firm manufacturing two types of electric items, A and B, can make a profit of Rs 20 per unit of A and Rs 30 per unit of B. Each unit of A requires 3 motors and 4 transformers and each unit of B requires 2 motors and 4 transformers. The total supply of these per month is restricted to 210 motors and 300 transformers. Type B is an export model requiring a voltage stabilizer which has a supply restricted to 65 units per month. Formulate the linear programing problem for maximum profit and solve it graphically.

22. A factory uses three different resources for the manufacture of two different products, 20 units of the resources A, 12 units of B and 16 units of C being available. 1 unit of the first product requires 2, 2 and 4 units of the respective resources and 1 unit of the second product requires 4, 2 and 0 units of respective resources. It is known that the first product gives a profit of 2 monetary units per unit and the second 3. Formulate the linear programming problem. How many units of each product should be manufactured for maximizing the profit? Solve it graphically.
23. A publisher sells a hard cover edition of a text book for Rs 72.00 and a paperback edition of the same text for Rs 40.00. Costs to the publisher are Rs 56.00 and Rs 28.00 per book respectively in addition to weekly costs of Rs 9600.00. Both types require 5 minutes of printing time, although hardcover requires 10 minutes binding time and the paperback requires only 2 minutes. Both the printing and binding operations have 4,800 minutes available each week. How many of each type of book should be produced in order to maximize profit ?
24. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine; size B contains 1 grain of aspirin, 8 grains of bicarbonate and 66 grains of codeine. It has been found by users that it requires at least 12 grains of aspirin, 7.4 grains of bicarbonate and 24 grains of codeine for providing immediate effects. Determine graphically the least number of pills a patient should have to get immediate relief. Determine also the quantity of codeine consumed by patient.
25. A chemical company produces two compounds, A and B. The following table gives the units of ingredients, C and D per kg of compounds A and B as well as minimum requirements of C and D and costs per kg of A and B. Find the quantities of A and B which would give a supply of C and D at a minimum cost.

	Compound		*Minimum requirement*
	A	*B*	
Ingredient C	1	2	80
Ingredient D	3	1	75
Cost (in Rs) per kg	4	6	

26. A company manufactures two types of novelty Souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours available for assembling. The profit is 50 paise each for type A and 60 paise each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximize the profit? [NCERT]
27. A manufacturer makes two products A and B. Product A sells at Rs 200 each and takes $1/2$ hour to make. Product B sells at Rs 300 each and takes 1 hour to make. There is a permanent order for 14 of product A and 16 of product B. A working week consists of 40 hours of production and weekly turnover must not be less than Rs 10000. If the profit on each of product A is Rs 20 and on product B is Rs 30, then how many of each should be produced so that the profit is maximum. Also, find the maximum profit.
28. A manufacturer produces two types of steel trunks. He has two machines A and B. For completing, the first type of the trunk requires 3 hours on machine A and 3 hours on machine B, whereas the second type of the trunk requires 3 hours on machine A and 2 hours on machine B. Machines A and B can work at most for 18 hours and 15 hours per day respectively. He earns a profit of Rs 30 and Rs 25 per trunk of the first type and the second type respectively. How many trunks of each type must the make each day to make maximum profit? [CBSE 2001]

29. A manufacturer of patent medicines is preparing a production plan on medicines, A and B. There are sufficient raw materials available to make 20000 bottles of A and 40000 bottles of B, but there are only 45000 bottles into which either of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of A, it takes 1 hour to prepare enough material to fill 1000 bottles of B and there are 66 hours available for this operation. The profit is Rs 8 per bottle for A and Rs 7 per bottle for B. How should the manufacturer schedule his production in order to maximize his profit?
30. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 400 is made on each first class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats of first class. However, at least 4 times as many passengers prefer to travel by economy class to the first class. Determine how many each type of tickets must be sold in order to maximize the profit for the airline. What is the maximum profit.
31. A gardener has supply of fertilizer of type I which consists of 10% nitrogen and 6% phosphoric acid and type II fertilizer which consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, he finds that he needs at least 14 kg of nitrogen and 14 kg of phosporic acid for his crop. If the type I fertilizer costs 60 paise per kg and type II fertilizer costs 40 paise per kg, determine how many kilograms of each fertilizer should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost? [CBSE 2002, 2008]
32. Anil wants to invest at most Rs 12000 in Saving Certificates and National Saving Bonds. According to rules, he has to invest at least Rs 2000 in Saving Certificates and at least Rs 4000 in National Saving Bonds. If the rate of interest on saving certificate is 8% per annum and the rate of interest on National Saving Bond is 10% per annum, how much money should he invest to earn maximum yearly income ? Find also his maximum yearly income.
33. A man owns a field of area 1000 sq.m. He wants to plant fruit trees in it. He has a sum of Rs 1400 to purchase young trees. He has the choice of two types of trees. Type A requires 10 sq.m of ground per tree and costs Rs 20 per tree and type B requires 20 sq.m of ground per tree and costs Rs 25 per tree. When fully grown, type A produces an average of 20 kg of fruit which can be sold at a profit of Rs 2.00 per kg and type B produces an average of 40 kg of fruit which can be sold at a profit of Rs 1.50 per kg. How many of each type should be planted to achieve maximum profit when the trees are fully grown ? What is the maximum profit?
34. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of grinding/cutting machine and a sprayer. It takes 2 hours on the grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp while it takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at most 20 hours and the grinding/cutting machine for at most 12 hours. The profit from the sale of a lamp is Rs 5.00 and a shade is Rs 3.00. Assumging that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximize his profit? [NCERT]
35. A producer has 30 and 17 units of labour and capital respectively which he can use to produce two type of goods x and y. To produce one unit of x, 2 units of labour and 3 units of capital are required. Similarly, 3 units of labour and 1 unit of capital is required to produce one unit of y. If x and y are priced at Rs 100 and Rs 120 per unit respectively, how should be producer use his resources to maximize the total revenue? Solve the problem graphically. [CBSE 2000]
36. A firm manufactures two types of products A and B and sells them at a profit of Rs 5 per unit of type A and Rs 3 per unit of type B. Each product is processed on two machines M_1 and M_2. One unit of type A requires one minute of processing

time on M_1 and two minutes of processing time on M_2, whereas one unit of type B requires one minute of processing time on M_1 and one minute on M_2. Machines M_1 and M_2 are respectively available for at most 5 hours and 6 hours in a day. Find out how many units of each type of product should the firm produce a day in order to maximize the profit. Solve the problem graphically. [CBSE 2000]

37. A small firm manufacturers items A and B. The total number of items A and B that it can manufacture in a day is at the most 24. Item A takes one hour to make while item B takes only half an hour. The maximum time available per day is 16 hours. If the profit on one unit of item A be Rs 300 and one unit of item B be Rs 160, how many of each type of item be produced to maximize the profit? Solve the problem graphically. [CBSE 2001, 2004]

38. A company manufactures two types of toys A and B. Type A requires 5 minutes each for cutting and 10 minutes each for assembling. Type B requires 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours available for cutting and 4 hours available for assembling in a day. The profit is Rs 50 each on type A and Rs 60 each on type B. How many toys of each type should the company manufacture in a day to maximize the profit? [CBSE 2001]

39. A company manufactures two articles A and B. There are two departments through which these articles are processed: (i) assembly and (ii) finishing departments. The maximum capacity of the first department is 60 hours a week and that of other department is 48 hours per week. The product of each unit of article A requires 4 hours in assembly and 2 hours in finishing and that of each unit of B requires 2 hours in assembly and 4 hours in finishing. If the profit is Rs 6 for each unit of A and Rs 8 for each unit of B, find the number of units of A and B to be produced per week in order to have maximum profit. [CBSE 2003]

40. A firm makes items A and B and the total number of items it can make in a day is 24. It takes one hour to make an item of A and half an hour to make an item of B. The maximum time available per day is 16 hours. The profit on an item of A is Rs 300 and on one item of B is Rs 160. How many items of each type should be produced to maximize the profit? Solve the problem graphically. [CBSE 2004]

41. A company sells two different products, A and B. The two products are produced in a common production process, which has a total capacity of 500 man-hours. It takes 5 hours to produce a unit of A and 3 hours to produce a unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 70 and that for B is 125. If the profit is Rs 20 per unit for the product A and Rs 15 per unit for the product B, how many units of each product should be sold to maximize profit?

42. A box manufacturer makes large and small boxes from a large piece of cardboard. The large boxes require 4 sq. metre per box while the small boxes require 3 sq. metre per box. The manufacturer is required to make at least three large boxes and at least twice as many small boxes as large boxes. If 60 sq. metre of carboard is in stock, and if the profits on the large and small boxes are Rs 3 and Rs 2 per box, how many of each should be made in order to maximize the total profit?

43. A manufacturer makes two products, A and B. Product A sells at Rs 200 each and takes 1/2 hour to make. Product B sells at Rs 300 each and takes 1 hour to make. There is a permanent order for 14 units of product A and 16 units of product B. A working week consists of 40 hurs of production and the weekly turnover must not be less than Rs 10000. If the profit on each of product A is Rs 20 and an product B is Rs 30, then how many of each should be produced so that the profit is maximum? Also find the maximum profit.

44. If a young man drives his vehicle at 25 km/hr, he has to spend Rs 2 per km on petrol. If he drives it at a faster speed of 40 km/hr, the petrol cost increases to Rs 5/per km.

He has Rs 100 to spend on petrol and travel within one hour. Express this as an LPP and solve the same. **[CBSE 2007]**

45. A manufacturer produces two types of steel trunks. He has two machines A and B. The first types of the trunk requires 3 hours on machine A and 3 hours on machine B. The second type of trunk requires 3 hours on machine A and 2 hours on machine B. Machines A and B are run daily for 18 hours and 15 hours respectively. There is a profit of Rs 30 on the first type of the trunk and Rs 25 on the second type of the trunk. How many trunks of each type should be produced and sold to make maximum profit? **[CBSE 2005]**
46. A small firm manufactures gold rings and chains. The total number of rings and chains manufactured per day is atmost 24. It takes 1 hour to make a ring and 30 minutes to make a chain. The maximum number of hours available per day is 16. If the profit on a ring is Rs 300 and that on a chain is Rs 190, find the number of rings and chains that should be manufactured per day, so as to earn the maximum profit. Make it as an LPP and solve it graphically. **[CBSE 2010]**
47. A library has to accomodate two different types of books on a shelf. The books are 6 cm and 4 cm thick and weigh 1 kg and $1\frac{1}{2}$ kg each respectively. The shelf is 96 cm long and atmost can support a weight of 21 kg. How should the shelf be filled with the books of two types in order to include the greatest number of books? Make it as an LPP and solve it graphically. **[CBSE 2010]**
48. One kind of cake requires 300 gm of flour and 15 gm of fat, another kind of cake requires 150 gm of flour and 30 gm of fat. Find the maximum number of cakes which can be made from 7.5 kg of flour and 600 gm of fat, assuming that there is no shortage of the other ingradients used in making the cakes. Make it as an LPP and solve it graphically. **[CBSE 2010]**

Type III **TRANSPORTATION PROBLEMS**

49. Two godowns, A and B, have grain storage capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F, whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table: **[NCERT]**

	Transportation cost per quintal (in Rs)	
To \ *From*	*A*	*B*
D	6.00	4.00
E	3.00	2.00
F	2.50	3.00

How should the supplies be transported in order that the transportation cost is minimum ?

50. An oil company has two depots, A and B, with capacities of 7000 litres and 4000 litres respectively. The company is to supply oil to three petrol pumps, D, E, F whose requirements are 4500, 3000 and 3500 litres respectively. The distance (in km) between the depots and petrol pumps is given in the following table: **[NCERT]**

	Distance (in Km)	
To \ *From*	*A*	*B*
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost per km is Re 1.00 per litre, how should the delivery be scheduled in order that the transportation cost is minimum?

51. A medical company has factories at two places, A and B. From these places, supply is made to each of its three agencies situated at P, Q and R. The monthly requirements of the agencies are respectively 40, 40 and 50 packets of the medicines, while the production capacity of the factories, A and B, are 60 and 70 packets respectively. The transportation cost per packet from the factories to the agencies are given below:

	Transportation cost per packet (in Rs)	
From / To	*A*	*B*
P	5	4
Q	4	2
R	3	5

How many packets from each factory be transported to each agency so that the cost of transportation is minimum ? Also find the minimum cost?

ANSWERS

1. 125/2 gm of food F_1, 375/4 gm of food F_2; Min cost 425/4 Paise
2. 5 units of food A and 30 units of food B
3. Food I 3 units, Food II 1 unit, Min cost Rs 2.80
4. 1.875 units of A and 2.75 units of B
5. Rs 41
6. Rs 104
7. Food I — 3 units, Food—II 1 unit
   Minimum cost = Rs 2.80
8. Rs 4.6
9. 4 units of A, 4 units of B, Rs 40.00
10. A : 5 days, B : 3 days
11. 30 packages of screw 'A' and 20 packages of screw 'B', Rs 41.00
12. 200 Belts of type A and 600 belts of type B; Max. profit = Rs 1300
13. 20 ordinary models, 10 deluxe models; Max profit = Rs. 400
15. Either 4 machines of type A and 3 machines of type

    or

    6 Machines of type A and no Machine of type B
16. 2 units of A, 3 units of B , profit = Rs 230.
17. $\frac{22}{5}$ chairs and $\frac{9}{5}$ tables ; Max profit = Rs 22.2
18. 24 chairs and 14 tables ; Max profit = Rs 2200
19. 2 kg of A and 4 kg of B ; Min cost = Rs 64
20. 25 units of product A, 100 units of product B; Max profit = Rs 475
21. 30 type A and 60 type B ; Max profit = Rs 2400
22. 2 units of first product, 4 units of second product ; Max profit = 16 monetary units.
23. 360 hard cover edition, 600 paper back edition,
    Max. profit = Rs 2880
24. 2 pills of size A, 8 pills of size B ; Quantity of codeine = 50 grains
25. 19 kg, 13 kg; Rs 254
26. 8 type A, 20 type B, Max profit = Rs 16
27. 48 units of product A, 16 units of product B , Max. profit = Rs 1440
28. 3, Trunks of type A 3, Trunks of Type B Max profit = Rs 165

29. 10500 bottles of A, 34500 bottles of B, max profit = Rs 325500
30. First class tickets = 40, Economy class tickets = 160, Proft = Rs 64000.00
31. Type I Fertilizer 100 kg, Type II Fertilizer = 80 kg, Cost = Rs 92
32. Rs 2000 in Saving Certificates, Rs 10,000 in National Saving Bonds, Income = Rs 1160 per month
33. Type A : 20 trees Type B : 40 trees, Max profit = Rs 2200
34. 4 pedestal lamps 4 wooden shades
35. 3 units of X and 8 units of Y
36. 60 units of type A and 240 units of type B
37. 8 items of type A and 16 items of type B
38. 12 toys of type A and 15 toys of type B
39. 12 units of product A and 6 units of product B.
40. 8 units of A and 16 units of B.
41. Product $A \to 25$ units, Product $B \to 125$ units
    Max. profit = Rs 2375
42. Large Box = 6, Small box = 12
    Maximum profit = Rs 42.
43. Product A — 48 units, Product B — 16 units
    Maximum profit = Rs 1440.
44. At 25 km/hr — 50/3 km, the 40 km/hr — 40/3 km
    Max. Distance = 30 km.
45. 3 trunks of each typements 46. 8 rings and 16 chains 47. 12, 6 48. 20, 10
49. From A : 10 quintals, 50 quintals and 40 quintals to D, E, F respectively.
    From B : 50 quintals, 0 quintal and 0 quintal to D, E, F respectively.
50. From A : 500 litres, 3000 litres, 3500 litres to D, E, F respectively
    From B : 400 litres, 0 litres, 0 litres to D, E, F respectively.
51. From A : 10 packets, 0 packets and 50 packets to P, Q and R respectively
    From B : 30 packets, 40 packets and 0 packets to P, Q and R respectively.
    Minmum cost = Rs 400.

HINTS TO SELECTED PROBLEMS

1. Let 25 x gms of food F_1 and 25 y gms of food F_2 be used to fulfil the minimum requirements of thiamine, phosphorous and iron. Then, the LLP is

$$\begin{aligned} \textit{Minimize} \quad & Z = 20x + 15y \\ \textit{Subject to} \quad & 0.25x + 0.10y \geq 1 \\ & 0.75x + 1.50y \geq 7.50 \\ & 1.60x + 0.80y \geq 10 \\ \textit{and,} \quad & x, y \geq 0 \end{aligned}$$

2. Let x units of food A any y units of food B are used. Then, the LPP is

$$\begin{aligned} \textit{Minimize } Z &= 4x + 3y \\ \textit{Subject to} \quad & 200x + 100y \geq 4000 \\ & x + 2y \geq 50 \\ & 40x + 40y \geq 1400 \\ \textit{and,} \quad & x, y \geq 0 \end{aligned}$$

3. Let x units of food I and y units of food II are used to fulfil minimum daily requirements. Then, the LPP is

Minimize $Z = 0.60x + y$

Subject to $10x + 4y \geq 20$

$5x + 6y \geq 20$

$2x + 6y \geq 12$

$x, y \geq 0$

4. Let x units of food A and y units of food B are combined. The LPP is

Minimize $Z = 0.1x + 0.1y$

Subject to $0.12x + 0.10y \geq 0.5$

$100x + 150y \geq 600$

and, $x, y \geq 0$

5. Let x kg of food X and y kg of food Y are mixed to produce the desired diet. The LPP is

Minimize $Z = 5x + 8y$

Subject to $x + 2y \geq 6$

$x + y \geq 7$

$x + 3y \geq 11$

$2x + y \geq 9$

and, $x, y \geq 0$

8. Let the cereal contain x kg of bran and y kg of rice. Then, the *LPP* is

Minimize $Z = 5x + 4y$

Subject to $x \times \frac{80}{1000} + y \times \frac{100}{1000} \geq \frac{88}{1000}$ or, $20x + 25y \geq 22$

$x \times \frac{40}{1000} + y \times \frac{30}{1000} \geq \frac{36}{1000}$ or, $20x + 15y \geq 18$

$x, y \geq 0$

9. Suppose the manufacturer produces x units of item A and y units of item B. Then, the mathematical form of the given LPP is

Maximize $Z = 6x + 4y$

Subject to $x + 2y \leq 12$

$2x + y \leq 12$

$x + \frac{5}{4}y \geq 5$

and, $x, y \geq 0$

10. We have to minimize the labour cost. This means that the profit is to be maximized. For this, suppose the tailors A and B work for x and y days respectively. Then the LPP is

Maximize $Z = 15x + 20y$

Subject to $6x + 10y \geq 60$

$4x + 4y \geq 32$

and, $x, y \geq 0$

11. Suppose the manufacturer produces x packages of screws A and y packages of screws B in a day. The LPP is

Maximize $Z = 0.7x + y$

Subject to $4x + 6y \leq 240$

$6x + 3y \leq 240$

and, $x, y \geq 0.$

12. Suppose the company produces x belts of type A and y belts of type B. Then, profit $= 2x + 1.5y$

Since the rate of production of belts of type B is 1000 per day. Therefore, time taken to produce y belts of type B is $\frac{y}{1000}$. Also, since each belt of type A requires twice as much time as a belt of type B, the rate of production of belts of type A is 500 per day and consequently total time taken to produce x belts of type A is $\frac{x}{500}$. Thus, we have

$$\frac{x}{500} + \frac{y}{1000} \leq 1 \Rightarrow 2x + y \leq 1000$$

The supply of leather is sufficient only for 800 belts per day.

$\therefore \quad x + y \leq 800$

Since 400 buckles are available for belt A and 700 buckles are available for belt B per day.

$\therefore \quad x \leq 400, y \leq 700$

Thus, the mathematical formulation of the LPP is

Maximize $Z = 2x + 1.5y$

Subject to $2x + y \leq 1000$

$x + y \leq 800$

$x \leq 400$

$y \leq 700$

and, $x, y \geq 0$

13. If the manufacturer makes x_1 deluxe model articles and x_2 ordinary model, then

Maximize $Z = 15x_1 + 10x_2$

Subject to $2x_1 + x_2 \leq 40$

$2x_1 + 3x_2 \leq 80$

and, $x_1, x_2 \geq 0$

14. Let x tea-cups of type A and y tea-cups of type B are manufactured per day. Then, the LPP is

Maximize $Z = 0.75x + 0.50y$

Subject to $12x + 6y \leq 360$

$18x + 0y \leq 360$

$6x + 9y \leq 360$

and, $x, y \geq 0$

15. Let x machines of type A and y machines of type B are bought to maximize the daily output. Then, the LPP is

Maximize $Z = 60x + 40y$

Subject to $1000x + 1200y \leq 7600$

$12x + 8y \leq 72$

and, $x, y \geq 0$

16. Suppose the company produces x goods of type A and y goods of type B. The mathematical form of the LPP is as follows:

Maximize $Z = 40x + 50y$

Subject to $3x + y \le 9$

$x + 2y \le 8$

and, $x, y \ge 0$

20. The LPP is as follows : Max. $Z = 3x_1 + 4x_2$

$4x_1 + 4x_2 \le 500$

$8x_1 + 4x_2 \le 600$

$x_1, x_2 \ge 0$

25. Let x kg of compound A and y kg of compound B are produced. Then, the mathematical formulation of the LPP is as follows :

Minimize $Z = 4x + 6y$

Subject to $x + 2y \ge 80$

$3x + y \ge 75$

$x, y \ge 0$

26. Let x souvenirs of type A and y souvenirs of type B are manufactured. Then, the LPP is as follows :

Maximize $Z = 50x + 60y$

Subject to $5x + 8y \le 200$

$10x + 8y \le 240$

and, $x, y \ge 0$

27. Let x units of product A and y units of product be B produced. Then the mathematical form of the LPP is as follows :

Maximize $Z = 20x + 30y$

Subject to $200x + 300y \ge 10{,}000$

$x \ge 14$

$y \ge 16$

$\frac{x}{2} + y \le 40$

and, $x, y \ge 0$

28. Suppose x trunks of type A and y trunks of type B are manufactured per day. Then, the mathematical form of the LPP is as follows :

Maximize $Z = 30x + 25y$

Subject to $3x + 3y \le 18$

$3x + 2y \le 15$

and, $x, y \ge 0$

29. Let the manufacturer produce x bottles of medicine A and y bottles of medicine B. Then the mathematical form of the LPP is as follows:

Maximize $Z = 8x + 7y$

Subject to $x \le 20000$

$y \le 40000$

$x + y \le 45000$

$\frac{3x}{1000} + \frac{y}{1000} \le 66$

and, $x, y \ge 0$

30. Let x first-class tickets and y economy class tickets are sold. Then the mathematical form of the LPP is

$$\begin{aligned} \textit{Maximize } Z &= 400x + 600y \\ \textit{Subject to } x + y &\le 2000 \\ x &\ge 20 \\ y &\ge 4x \\ \textit{and, } x, y &\ge 0 \end{aligned}$$

31. Let x kg of fertilizer I and y kg of fertilizer II are used. Then the mathematical form of the LPP is as follows :

$$\textit{Minimize } Z = 60x + 40y$$

$$\textit{Subject to } \frac{10x}{100} + \frac{5y}{100} \ge 14$$

$$\frac{6x}{100} + \frac{10x}{100} \ge 14$$

$$\textit{and, } x, y \ge 0$$

32. Suppose Anil invests Rs x in Saving Certicate and Rs y in National Saving Bonds. Then, the mathematical formulation of the LPP is as follows :

$$\textit{Maximize } Z = \frac{8x}{100} + \frac{10y}{100}$$

$$\begin{aligned} \textit{Subject to } x + y &\le 12000 \\ x &\ge 2000 \\ y &\ge 4000 \\ \textit{and, } x, y &\ge 0 \end{aligned}$$

33. Let x trees of type A and y trees of type B are planted. Then, the mathematical formulation of the LPP is as follows :

$$\textit{Maximize } Z = 40x + 60y - (20x + 25y)$$

$$\begin{aligned} \textit{Subject to } 20x + 25y &\le 1400 \\ 10x + 20y &\le 1000 \\ \textit{and, } x, y &\ge 0 \end{aligned}$$

34. Let x lamps and y shades be manufactured by the manufacturer. Then, the mathematical formulation of the LPP is

$$\begin{aligned} \textit{Maximize } Z &= 5x + 3y \\ \textit{Subject to } 2x + y &\le 12 \\ 3x + 2y &\le 20 \\ \textit{and, } x, y &\ge 0 \end{aligned}$$

35. Let x units of X and y units of Y be produced to maximize the revenue. Then, the LPP is

$$\begin{aligned} \textit{Maximize } Z &= 100x + 120y \\ \textit{Subject to } 2x + 3y &\le 30 \\ 3x + y &\le 17 \\ x \ge 0, y &\ge 0 \end{aligned}$$

39. Let x units of A and y units of B be produced per week for maximum profit. Then, the LPP is

Maximize $Z = 6x + 8y$

Subjected to $4x + 2y \le 60$

$2x + 4y \le 48$

$x, y \ge 0.$

49. Suppose godown A supplies x quintals of grain to the ration shop D and y quintals to ration shop E. Then, the mathematical formulation of the LPP is as follows:

Minimize $Z = 6x + 3y + \frac{5}{2}(100 - x - y) + 4(60 - x) + 2(50 - y) + 3(x + y - 60)$

Subject to $x + y \le 100$

$x \le 60$

$y \le 50$

$x + y \ge 60$

and, $x, y \ge 0$

50. Let depot A supply x litres to petrol pump D and y litres to petrol pump E. The mathematical formulation of the LPP is as follows :

Minimize $Z = 7x + 6y + 3(7000 - x - y) + 3(4500 - x) + 4(3000 - y) + 2(x + y - 3000)$

Subject to $x + y \le 7000$

$x \le 4500$

$y \le 300$

$x + y \ge 3000$

and, $x, y \ge 0$

51. Let x packets and y packets be transported from the factory A to the agencies P and Q respectively. Then, the mathematical formulation of the LPP is as follows:

Minimize $Z = 5x + 4y + 3(60 - x - y) + 4(40 - x) + 2(40 - y) + 5(x + y - 30)$

Subject to $x + y \le 60$

$x \le 40$

$y \le 40$

$x + y \ge 30$

and, $x \ge 0, y \ge 0$

MULTIPLE CHOICE QUESTIONS (MCQs)

1. The solution set of the inequation $2x + y > 5$ is
   (a) half plane that contains the origin
   (b) open half plane not containing the origin
   (c) whole xy-plane except the points lying on the line $2x + y = 5$.
   (d) none of these
2. Objective function of a LPP is
   (a) a constraint
   (b) a function to be optimized
   (c) a relation between the variables
   (d) none of these
3. Which of the following sets are convex?
   (a) $\{(x, y) : x^2 + y^2 \ge 1\}$
   (b) $\{(x, y) : y^2 \ge x\}$
   (c) $\{(x, y) : 3x^2 + 4y^2 \ge 5\}$
   (d) $\{(x, y) : y \ge 2, y \le 4\}$

4. Let X_1 and X_2 are optimal solutions of a LPP, then
   (a) $X = \lambda X_1 + (1 - \lambda) X_2, \lambda \in R$ is also an optimal solution
   (b) $X = \lambda X_1 + (1 - \lambda) X_2, 0 \le \lambda \le 1$ gives an optimal solution
   (c) $X = \lambda X_1 + (1 + \lambda) X_2, 0 \le \lambda \le 1$ give an optimal solution
   (d) $X = \lambda X_1 + (1 + \lambda) X_2, \lambda \in R$ gives an optimal solution
5. The maximum value of $Z = 4x + 2y$ subjected to the constraints $2x + 3y \le 18$, $x + y \ge 10; x, y \ge 0$ is
   (a) 36 (b) 40 (c) 20 (d) none of these
6. The optimal value of the objective function is attained at the points
   (a) given by intersection of inequations with the axes only
   (b) given by intersection of inequations with x-axis only
   (c) given by corner points of the feasible region
   (d) none of these
7. The maximum value of $Z = 4x + 3y$ subjected to the constraints $3x + 2y \ge 160$, $5x + 2y \ge 200, x + 2y \ge 80; x, y \ge 0$ is
   (a) 320 (d) 300 (c) 230 (d) none of these
8. Consider a LPP given by
   Min $Z = 6x + 10y$
   Subjected to $x \ge 6;\ y \ge 2;\ 2x + y \ge 10;\ x, y \ge 0$
   Redundant constraints in this LPP are
   (a) $x \ge 0, y \ge 0$ (b) $x \ge 6, 2x + y \ge 10$
   (c) $2x + y \ge 10$ (d) none of these
9. The objective function $Z = 4x + 3y$ can be maximied subjected to the constraints $3x + 4y \le 24, 8x + 6y \le 48, x \le 5, y \le 6; x, y \ge 0$
   (a) at only one point
   (b) at two points only
   (c) at an infinite number of points
   (d) none of these
10. If the constraints in a linear programming problem are changed
    (a) the problem is to be re-evaluated
    (b) solution is not defined
    (c) the objective function has to be modified
    (d) the change in constraints is ignored
11. Which of the following statements is correct?
    (a) Every LPP admits an optimal solution
    (b) A LPP admits unique optimal solution
    (c) If a LPP admits two optimal solutions it has an infinite number of optimal solutions
    (d) The set of all feasible solutions of a LPP is not a converse set
12. Which of the following is not a convex set?
    (a) $\{(x, y) : 2x + 5y < 7\}$ (b) $\{(x, y) : x^2 + y^2 \le 4\}$
    (c) $\{x : |x| = 5\}$ (d) $\{(x, y) : 3x^2 + 2y^2 \le 6\}$
13. By graphical method, the solution of linear programming problem
    Maximize $Z = 3x_1 + 5x_2$
    Subject to $3x_1 + 2x_2 \le 18$
    $x_1 \le 4$
    $x_2 \le 6$
    $x_1 \ge 0, x_2 \ge 0$, is

(a) $x_1 = 2,\ x_2 = 0,\ Z = 6$ (b) $x_1 = 2,\ x_2 = 6,\ Z = 36$

(c) $x_1 = 4,\ x_2 = 3,\ Z = 27$ (d) $x_1 = 4,\ x_2 = 6,\ Z = 42$

14. The region represented by the inequation system $x, y \geq 0, y \leq 6, x + y \leq 3$ is
(a) unbounded in first quadrant
(b) unbounded in first and second quadrants
(c) bounded in first quadrant
(d) none of these

15. The point at which the maximum value of $x + y$, subject to the constraints $x + 2y \leq 70, 2x + y \leq 95, x, y \geq 0$ is obtained, is
(a) (30, 25) (b) (20, 35) (c) (35, 20) (d) (40, 15)

16. The value of objective function is maximum under linear constraints
(a) at the centre of feasible region
(b) at (0, 0)
(c) at any vertex of feasible region
(d) the vertex which is maximum distance from (0, 0)

17. The corner points of the feasible region determined by the following system of linear inequalities:
$2x + y \leq 10, x + 3y \leq 15, x, y \geq 0$ are (0, 0), (5, 0), (3, 4) and (0, 5). Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximukm of Z occurs at both (3, 4) and (0, 5) is
(a) $p = q$ (b) $p = 2q$ (c) $p = 3q$ (d) $q = 3p$

ANSWERS

1. (b) 2. (b) 3. (d) 4. (b) 5. (d) 6. (c) 7. (d) 8. (c)
9. (c) 10. (a) 11. (c) 12. (c) 13. (b) 14. (a) 15. (d) 16. (a)
17. (d)

REVISION EXERCISE

1. Solve the following linear programming problems graphically:

(i) *Maximum* $z = 4x + y$
Subject to $x + y \leq 50$
$3x + y \leq 90$
$x \geq 0, y \geq 0$

(ii) Minimize $Z = 200x + 500y$
Subject to $x + 2y \geq 10$
$3x + 4y \leq 24$
$x \geq 0, y \geq 0$

(iii) Max. and Min. $Z = 3x + 9y$
Subject to $x + 3y \leq 60$
$x + y \geq 10$
$x \leq y$
$x \geq 0, y \geq 0$

(iv) Minimize $Z = 3x + 2y$
Subject to $x + y \geq 8$
$3x + 5y \leq 15$
$x \geq 0, y \geq 0$

(v) Minimize $Z = x + 2y$
Subject to $2x + y \geq 3$
$x + 2y \geq 6$
$x \geq 0, y \geq 0$

(vi) Min.i and Maxi. $Z = 5x + 10y$
Subject to $x + 2y \leq 120$
$x + 2y \geq 60$
$x - 2y \geq 0$
$x \geq 0, y \geq 0$

2. A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs 10,500 and Rs 9,000

respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so at to maximise the total profit of the society?

3. A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs 8000 on each piece of model A and Rs 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realise a maximum profit? What is the maximum profit per week?
4. Reshma wishes to mix two types of food P and Q in such a way that the vitamin cotents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60 kg and Food Q costs Rs 80 kg. Food P contains 3 units/kg of Vitamin A and 5 units/kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.
5. One kind of cake requires 200 g of flour and 25 g of fat, and another kind of cake requires 100 g of flour and 50 g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.
6. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.
   (i) What number of rackets and bats must be made if the factory is to work at full capacity?
   (ii) If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.
7. A merchant plans to sell two types of personal computers–a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectivley. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.
8. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs Rs 4 per unit food and F_2 costs Rs 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minearls. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.
9. There are two types of fertilisers F_1 and F_2. F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs Rs 6 /kg and F_2 costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?
10. A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of

cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

11. A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for atleast 5 hours a day. She produces only two items M and N each requiring the use of all the three machines.
The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Items	*Number of hours required on machines*		
	I	II	III
M	1	2	1
N	2	1	1.25

She makes a profit of Rs 600 and Rs 400 on items M and N respectively. How many of each item should she produce so as to maximise her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

12. There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:

From/To	*Cost (in Rs)*		
	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

13. A farmer mixes two brands P and Q of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing Rs 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?

14. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	*Vitamin A*	*Vitamin B*	*Vitamin C*
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet?

15. A manufacturer makes two types of toys *A* and *B*. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Types of Toys	*Machines*		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum fo 6 hours per day. If the profit on each toy of type *A* is Rs 7.50 and that on each toy of type *B* is Rs 5, show that 15 toys of type *A* and 30 of type *B* should be manufactured in a day to get maximum profit.

16. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

17. A fruit grower can use two types of fertilizer in his garden, brand *P* and *Q*. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine.

	kg per bag	
	Brand P	*Brand Q*
Nitrogen	3	3.5
Phosphoric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

18. A toy company manufactures two types of dolls, *A* and *B*. Market tests and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type *B* is at most half of that for dolls of type *A*. Further, the production level of dolls of type *A* can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs 12 and Rs 16 per doll respectively on dolls *A* and *B*, how many of each should be produced weekly in order to maximise the profit?

ANSWERS

1. (i) Max $Z = 120$ at $x = 30, y = 0$ (ii) Min $Z = 2300$ at $x = 4, y = 3$

(iii) Max $Z = 180$ at $x = 15, y = 15$ and $x = 0, y = 20$

Min $z = 60$ at $x = 5, y = 5$

(iv) No feasible solution

(v) Min $Z = 6$ at all the points on the line segment joining the points $(6, 0)$ and $(0, 3)$

(vi) Min $Z = 300$ at $x = 60, y = 0$

Max $Z = 600$ at all the points on the line segment joining (120, 0) and (60, 30).

2. 30 hectares for crop X, 20 hectares for crop Y, Total profit = Rs 4, 95,000.
3. 12 pieces of Model A, 6 pieces of Model B, Profit = Rs 1,68,000
4. Rs 160 at all points on the like segment joining points $\left(\frac{8}{3}, 0\right)$ and $\left(2, \frac{1}{2}\right)$
5. 30 cakes of one kind and 10 cakes of second kind.
6. (i) Tennis rackets = 4; Cricket bats = 12; Max. Profit = Rs 200
7. 200 units of desktop model and 50 units of portable model.
   Max. Profit = Rs 1150000.
8. Minimize $Z = 4x + 6y$.
   Subject to $3x + 6y \geq 80, 4x + 3y \geq 100, x \geq 0, y \geq 0$.
   Minimum cost = Rs 104.
9. Fertiliser $F_1 = 100$ kg, Fertiliser $F_2 = 80$ kg.
   Minimum cost = Rs 1000.
10. Food P = 15 packets, Food $Q = 20$ packets.
    Minimum amount of vitamin $A = 150$ units.
11. Item $M = 4$, Item $N = 4$, Profit = Rs 4000.
12.

Factory at	*Depot*		
	A	B	C
P	0	5	3
Q	5	0	1

Cost = Rs 1550.

13. Bags of brand $P = 3$, Bags of brand $Q = 15$, Max. amount of vitamin = 285 units.
14. Food $X = 2$ kg, Food $Y = 4$ kg, Least cost = Rs 112.
15. 400 tickets of executive class, 160 tickets of economy class, Profit = Rs 136000.
17. Brand $P = 40$ bags, Brand $Q = 50$ bags, Minimum amount of Nitrogen = 470 kg.
18. 800 dolls of type A, 400 dolls of type B, Max. Profit = Rs 16000.

SUMMARY

1. A general linear programming problem can be stated as follows:
   Given a set of m linear inequalities or equations in n variables, we wish to find non-negative values of these variables which will satisfy these inequalities or equations and maximize or minimize some linear function of the variables.
   The inequalities or equations are called the *constraints* and the function to be maximized or minimized is called the *objective function* which can be of maximization type or minimization type.
   The general form of linear programming problem is maximize (minimize)

$$Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \qquad \text{...(i)}$$

Subjected to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \quad \dots \quad + a_{1n}x_n \quad \{\le, =, \ge\}\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \quad \dots \quad + a_{2n}x_n \quad \{\le, =, \ge\}\ b_2 \\ \dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \quad \dots \quad + a_{mn}x_n \quad \{\le, =, \ge\}\ b_m \end{array}\right\} \qquad \dots(ii)$$

and, $x_1, x_2, x_3, \dots, x_n \ge 0.$ ...(iii)

where,

(i) $x_1, x_2, \dots, x_n$ are the variables whose values we wish to determine and are called the decision variables.

(ii) the linear function Z which is to be maximized or minimized is called the *objective function.*

(iii) the inequalities or equations in (ii) are called the constraints.

(iv) the set of inequalities in (iii) is known as the set of non-negativity restrictions.

(v) b_i $(i = 1, 2, \dots, m)$ represents the requirement or availability of the i^{th} constraint and the column matrix $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is called the requirement vector.

(vi) c_j $(j = 1, 2, \dots, n)$ represents the profit or cost to the objective function of the j^{th} variable x_j and the row matrix $C = [c_1, c_2, \dots, c_n]$ is called the profit (cost) matrix (vector).

(vii) the coefficients a_{ij} $(i = 1, 2, \dots, m; j = 1, 2, \dots, n)$ are known as the technological or substitution coefficients.

(viii) the expression $(\le, =, \ge)$ means that one and only one of the signs $\le, =, \ge$ holds for a particular constraint but the sign may vary from constraint to constraint.

2. A set of values of the decision variables which satisfy the constraints of a linear programming problem (LPP) is called a solution of the LPP.
3. A solution of a LPP which also satisfies the non-negativity restrictions of the problem is called its feasible solution.
   The set of all feasible solutions of a LPP is called the feasible region.
4. A feasible solution which optimizes (maximize or minimize) the objective function of a LPP is called an optimal solution of the LPP.
   A linear programming problem may have many optimal solutions. In fact, if a LPP has two optimal solution, then there are an infinite number of optimal solutions.
5. Let $X_1 = (x_{11}, x_{12})$ and $X_2 = (x_{21}, x_{22})$ be any two points in R^2 (two dimensional plane). Then,
   $X = \lambda X_1 + (1 - \lambda) X_2, \lambda \in R$ is any point on the line joining points X_1 *and* X_2.
   If $0 \le \lambda \le 1$, then X is any point on the line segment joining X_1 and X_2.
   Thus, the set of points given by
   $E = \{X \mid X = \lambda X_1 + (1 - \lambda) X_2, \lambda \in R\}$ is the line joining point X_1 and X_2 and the set $E_1 = \{X \mid X = \lambda X_1 + (1 - \lambda) X_2, 0 \le \lambda \le 1\}$ is the line segment joining X_1 and X_2.
6. A set E in R^2 is said to be convex set if for any two points X_1, X_2 in E, the line segment joining X_1, X_2 is contained in the set i.e. $X = \lambda X_1 + (1 - \lambda) X_2 \in E$ for $0 \le \lambda \le 1$.
   The set of all feasible solutions of a linear programming problem is a convex set.
7. The graphical method for solving linear programming problems is applicable to those problems which involve only two variables. This method is based upon a theorem, called extreme point theorem, which is stated as follows:

If a LPP admits an optimal solution, then at least one of the extreme (or comer) points of the feasible region gives the optimal solution.

8. There are two methods to solved a linear programming problem graphically.
   (i) Corner-point method (ii) Iso-profit or Iso-cost method.
9. To solve a linear programming problem by corner point method, we follow the following steps:
   (i) Formulate the given LPP in mathematical form if it is not given in mathematical form.
   (ii) Convert all inequations into equations and draw their graphs. To draw the graph of a linear equation, put $y = 0$ in it and obtain a point on X-axis. Similarly, by putting $x = 0$ obtain a point on y-axis. Join these two points to obtain the graph of the equation.
   (iii) Determine the region represented by each inequation. To determiine the region represented by an inequation replace x and y both by zero, if the inequation reduces to a valid statement, then the region containing the origin is the region represented by the given inequation. Otherwise, the region not containing the origin is the region represented by the given inequation.
   (iv) Obtain the region in xy-plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The polygonal region so obtained is the feasible region and is known as the convex polygon of the set of all feasible solutions of the LPP.
   (v) Determine the coordinates of the vertices (corner points) of the convex polygon obtained in step (ii). These vertices are known as the extreme points of the set of all feasible solutions of the LPP.
   (vi) Obtain the values of the objective function at each of the vertices of the convex polygon. The point where the objective function attains its optimum (maximum or minimum) value is the optimal solution of the given LPP.
10. To solve a linear programming problem by Iso-profit or Iso-cost method, we follow the following steps:
   (i) Formulate the given LPP in mathematical form, if it is not given so.
   (ii) Obtain the region in xy-plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The polygonal region so obtained is the convex set of all feasible solutions of the given LPP and it is also known as the feasible region.
   (iii) Determine the coordinates of the vertices (Corner points) of the feasible region obtained in step (ii).
   (iv) Give some convenient value to Z and draw the line so obtained in xy-plane.
   (v) If the objective function is of maximization type, then draw lines parallel to the line in step (iv) and obtain a line which is farthest from the origin and has at least one point common to the feasible region.
   If the objective function is of minimization type, then draw lines parallel to the line in step (iv) and obtain a line which is nearest to the origin and has at least one point common to the feasible region.
   (vi) Find the coordinates of the common point (s) obtained in step (v). The point (s) so obtained determine the optimal solution (s) and the value (s) of the objective function at these point (s) give the optimal solution.

30

PROBABILITY

30.1 INTRODUCTION

There are three aproaches to theory of probability, namely Experimental or Empirical approach, Classical approach and Axiomatic approach. In class IX, we have learnt about experimental approach. The classical approach has been discussed in class X. The axiomatic approach, formulated by Russian Mathematician A.N. Kolmogorov (1903-1987), has been discussed in class XI. We have also established the equivalence between the axiomatic theory of probability and the classical theory of probability in case of equally likely outcomes. On the basis of this relationship we obtained probabilities of events associated with discrete sample spaces. We have also studied addition theorem of probability. In continuation of these, we will introduce the concept of conditional probability which will be useful in obtaining multiplication rule of probability. The same will be used to derive a formula for the conditional probability. All these results will be helpful in understanding total probability theorem and Baye's theorem which will be introduced in the end of the chapter.

30.2 RECAPITULATION

Let us recall important terms and concepts which we have studied in earlier classes.

RANDOM EXPERIMENT If an experiment, when repeated under identical conditions, do not produce the same outcome everytime but the outcome in a trial is one of the several possible outcomes, then such an experiment is called a random experiment or a probabilistic experiment.

ELEMENTARY EVENT *If a random experiment is performed, then each of its outcomes is known as an elementary event.*

SAMPLE SPACE *The set of all possible outcomes of a random experiment called the sample space associated with it.*

EVENT *A subset of the sample space associated with a random experiment is called an event.*

OCCURRENCE OF AN EVENT *An event associated to a random experiment is said to occur if any one of the elementary events belonging to it is an outcome.*

Corresponding to every event A, associated to a random experiment, we define an event "not A denoted by $\overline{A}$" which is said to occur when and only when A does not occur.

CERTAIN (OR SURE) EVENT *An event associated with a random experiment is called a certain event if it always occurs whenever the experiment is performed.*

IMPOSSIBLE EVENT *An event associated with a random experiment is called an impossible event if it never occurs whenever the experiment is performed.*

COMPOUND EVENT *An event associated with a random experiment is a compound event, if it is the disjoint union of two or more elementary events.*

MUTUALLY EXCLUSIVE EVENTS *Two or more events associated with a random experiment are said to be mutually exclusive or impossible events if the occurrence of any one of them prevents the occurrence of all others, i.e. if no two or more of them can occur simultaneously in the same trial.*

EXHAUSTIVE EVENTS *Two or more events associated with a random experiment are exhaustive if their union is the sample space.*

FAVOURABLE ELEMENTARY EVENTS *Let S be the sample space associated with a random experiment and A be an event associated with the experiment. Then, elementary events belonging to A are knwon as favourble elementary events to the event A.*

Thus, an elementary event E is favourable to an event A, if the occurrence of E ensures the happening or occurrence of event A.

Events associated to a random experiment are generally described verbally and it is very important to have the ability of conversion of verbal description to equivalent set theoretic notations. Following table provides verbal description of some events and their equivalent set theoretic notations for ready reference.

Verbal description of the event	*Equivalent set theoretic notation*
Not A	$\overline{A}$
A or B (at least one of A or B)	$A \cup B$
A and B	$A \cap B$
A but not B	$A \cap \overline{B}$
Neither A nor B	$\overline{A} \cap \overline{B}$
At least one of A, B or C	$A \cup B \cup C$
Exactly one of A and B	$(A \cap \overline{B}) \cup (\overline{A} \cap B)$
All three of A, B and C	$A \cap B \cap C$
Exactly two of A, B and C	$(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$

PROBABILITY OF AN EVENT *If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happening or occurrence of A is denoted by P (A) and is defined as the ratio* $\frac{m}{n}$.

Thus, $\quad P(A) = \frac{m}{n}$

If $P(A) = 1$, then A is called the certain event and A is called an impossible event if $P(A) = 0$.

Also, $\quad P(A) + P(\overline{A}) = 1$

The odds in favour of occurrence of the event A are defined by $m : (n - m)$ i.e., $P(A) : P(\overline{A})$ and the odds against the occurrence of A are defined by $(n - m) : m$ i.e., $P(\overline{A}) : P(A)$.

ILLUSTRATION 1 *Twelve balls are distributed among three boxes, find the probability that the first box will contain three balls.*

SOLUTION Since each ball can be put into any one of the three boxes. So, the total number of ways in which 12 balls can be put into three boxes it 3^{12}

Out of 12 balls, 3 balls can be chosen in ${}^{12}C_3$ ways. Put these three balls in the first box. Now, remaining 9 balls are to be put in the remaining two boxes. This can be done in 2^9 ways.

So, the total number of ways in 3 balls can be put in the first box and the remaining 9 in other two boxes is ${}^{12}C_3 \times 2^9$.

Hence, required probability $= \dfrac{{}^{12}C_3 \times 2^9}{3^{12}}$

ILLUSTRATION 2 *Find the probability that the birth days of six different persons will fall in exactly two calender months.*

SOLUTION Since each person can have his birth day in any one of the 12 calender months. So, there are 12 options for each person.

$\therefore$ Total number of ways in which 6 persons can have their birth days

$= 12 \times 12 \times 12 \times 12 \times 12 \times 12 = 12^6.$

Out of 12 months, 2 months can be chosen in ${}^{12}C_2$ ways.

Now, birth days of six persons can fall in these two months in 2^6 ways. Out of these 2^6 ways, there are two ways when all six birth days fall in one month. So, there are $(2^6 - 2)$ ways in which six birth days fall in the chosen 2 months.

$\therefore$ Number of ways in which six birth days fall in exactly two calender months

$= {}^{12}C_2 \times (2^6 - 2)$

Hence, required probability $= \dfrac{{}^{12}C_2 \times (2^6 - 2)}{12^6} = \dfrac{341}{12^5}$

ILLUSTRATION 3 *If each element of a second order determinant is either zero or one, what is the probability that the value the determinant is non-negative?*

SOLUTION In a 2×2 determinant there are 4 elements and each element can take 2 values. So, total number of 2×2 determinants with elements 0 and 1 is $2^4 = 16$. Out of these determinants the values of the following determinants are negative:

$$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

Hence, required probability $= \dfrac{13}{16}$

ILLUSTRATION 4 *Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary six faced die. Find the probability that the equation will have real roots.*

SOLUTION Since each of the coefficients a, b, and c can take the values from 1 to 6. Therefore,

Total number of equations $= 6 \times 6 \times 6 = 216.$

The roots of the equation $ax^2 + bx + c = 0$ will be real if $b^2 - 4ac \geq 0 \Rightarrow b^2 \geq 4ac.$

The favourable number of elementary events can be enumerated as follows:

ac	a	c	$4ac$	b (so that $b^2 \geq 4\,ac$)	No. of ways
1	1	1	4	2, 3, 4, 5, 6	$1 \times 5 = 5$
2	1 2	2 1	8	3, 4, 5, 6	$2 \times 4 = 8$
3	1 3	3 1	12	4, 5, 6	$2 \times 3 = 6$
4	1 4 2	4 1 2	16	4, 5, 6	$3 \times 3 = 9$
5	1 5	5 1	20	5, 6	$2 \times 2 = 4$
6	1 6 2 3	6 1 3 2	24	5, 6	$4 \times 2 = 8$
7	ac is not possible				0
8	2 4	4 2	32	6	$2 \times 1 = 2$
9	3	3	36	6	1
					Total $= 43.$

Since $b^2 \geq 4\,ac$ and since the maximum value of b^2 is 36, therefore ac = 10, 11, 12 ... etc. is not possible.

$\therefore$ Total number of favourable elementary events = 43.

Hence, required probability $= \frac{43}{216}$.

ILLUSTRATION 5 *Two numbers b and c are chosen at random with replacement from the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9. Find the probability that $x^2 + bx + c > 0$ for all $x \in R$.*

SOLUTION Since b and c both can assume values from 1 to 9. So, total number of ways of choosing b and c is $9 \times 9 = 81$.

Now, $x^2 + bx + c > 0$ for all $x \in R$

$\Rightarrow$ Disc < 0 i.e. $b^2 - 4c < 0$

The following table shows the possible values of b and c for which $b^2 - 4c < 0$

c	b	Total
1	1	1
2	1, 2	2
3	1, 2, 3	3
4	1, 2, 3	3
5	1, 2, 3, 4	4
6	1, 2, 3, 4	4
7	1, 2, 3, 4, 5	5
8	1, 2, 3, 4, 5	5
9	1, 2, 3, 4, 5	5
		32

So, favourable number of favourable elementary events $= 32$.

Hence, required probability $= 32/81$.

ILLUSTRATION 6 *A die is rolled thrice, find the probability of getting a larger number each time than the previous number.*

SOLUTION We have,

Total number of elementary events $= 6 \times 6 \times 6 = 216$.

Clearly, the second number has to be greater than unity. If the second number is $i\,(i > 1)$, then the first can be chosen in $(i-1)$ ways and the third $(6-i)$ ways. So, three numbers can be chosen in $(i-1) \times 1\,(6-i)$ ways. But, the second number can vary from 2 to 5. Therefore,

Favourable number of elementary events

$$= \sum_{i=2}^{5} (i-1)(6-i) = 1 \times 4 + 2 \times 3 + 3 \times 2 + 4 \times 1 = 20.$$

Hence, required probability $= \dfrac{20}{216} = \dfrac{5}{54}$.

ILLUSTRATION 7 *In how many ways, can three girls and nine boys be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats? Now, if all the seating arrangements are equally likely, what is the probability of 3 girls sitting together in a back row on adjacent seats?*

SOLUTION Each van has 7 seats. So, there are 14 numbered seats in two vans.

The total number of ways in which 3 girls and 9 boys can sit on these seats is ${}^{14}C_{12} \times 12!$

So, total number of seating arrangements $= {}^{14}C_{12} \times 12!$

In a van 3 girls can choose adjacent seats in the back row in two ways (1, 2, 3, or 2, 3, 4). So, the number of ways in which 3 girls can sit in the back row on adjacent seats is 2 (3!) ways. The number of ways in which 9 boys can sit on the remaining 11 seats is ${}^{11}C_9 \times 9!$ ways.

So, the number of ways in which 3 girls and 9 boys can sit in two vans

$$= 2(3!) \times {}^{11}C_9 \times 9! + 2\,(3!) \times {}^{11}C_9 \times 9!$$

Hence, required probability $= \dfrac{4 \times 3! \times {}^{11}C_9 \times 9!}{{}^{14}C_{12} \times 12!} = \dfrac{1}{91}$

ILLUSTRATION 8 *If the letters of the word 'ATTRACTION' are written down at random, find the probability that*

(i) *all the T's occur together,* (ii) *no two T's occur together.*

SOLUTION The total number of arrangements of the letters of the word 'ATTRACTION' is $\dfrac{10!}{3!\,2!}$.

(i) Considering three T's as one letter there are 8 letters consisting of two identical A's. These 8 letters can be arranged in $\dfrac{8!}{2!}$ ways.

Hence, required probability $= \dfrac{\dfrac{8!}{2!}}{\dfrac{10!}{3!\,2!}} = \dfrac{3!\,8!}{10!} = \dfrac{1}{15}$

Hence, required probability $= \dfrac{\frac{8!}{2!}}{\frac{10!}{3!\,2!}} = \dfrac{3!\,8!}{10!} = \dfrac{1}{15}$

(ii) Other than 3 T's there are 7 letters which can be arranged in $\frac{7!}{2!}$ ways. There are 8 places, 6 between the 7 letters and one on extreme left and the other on extreme right. To separate three T's, we arrange them in these 8 places. This can be done in $^8C_3$ ways. Therefore,

Number of ways in which no two T's are together $= \frac{7!}{2!} \times {^8C_3}$

Hence, required probability $= \dfrac{\frac{7!}{2!} \times {^8C_3}}{\frac{10!}{3!\,2!}} = \dfrac{7}{15}$

ILLUSTRATION 9 *What is the probability that for S's come consecutively in the word 'MISSISSIPPI'?*

SOLUTION The total number of arrangements of the letters of the word 'MISSISSIPPI' is

$$\frac{11!}{4!\,4!\,2!}$$

Considering 4 S's as one letter, there are 8 letters which can be arranged in arrow in $\frac{8!}{4!\,2!}$ ways.

So, the number of ways in which 4 S's come together $= \dfrac{8!}{4!\,2!}$

Hence, required probability $= \dfrac{8!/4!\,2!}{11!/4!\,4!\,2!} = \dfrac{8! \times 4!}{11!} = \dfrac{4}{165}$

ADDITION THEOREM *It A and B are two events associated with a random experiment, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

If A, B, C are three events associated with a random experiment, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

If A, B, C are mutually exclusive events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

For any two events A and B

(i) Probability of occurrence of A only $= P(A \cap \overline{B}) = P(A) - P(A \cap B)$

(ii) Probability of occurrence of B only $= P(\overline{A} \cap B) = P(B) - P(A \cap B)$

(iii) Probability of occurrence of exactly one of A and B

$$= P(A \cap \overline{B}) + P(\overline{A} \cap B) = P(A) + P(B) - 2P(A \cap B) = P(A \cup B) - P(A \cap B).$$

ILLUSTRATION 10 *A basket contains 20 apples and 10 oranges out of which 5 apples and 3 oranges are defective. If a person takes out 2 at random what is the probability that either both are apples or both are good?*

SOLUTION Out of 30 items, two can be selected in $30C_2$ ways.

So, total number of elementary events $= 30C_2$.

Consider the following events:

A = Getting two apples ; B = Getting two good items

Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ...(i)

There are 20 apples, out of which 2 can be drawn in $20C_2$ ways.

$$\therefore \quad P(A) = \frac{{}^{20}C_2}{{}^{30}C_2}$$

There are 8 defective pieces and the remaining 22 are good. Out of 22 good pieces, two can be selected in 22C2 ways.

$$\therefore \quad P(B) = \frac{{}^{22}C_2}{{}^{30}C_2}$$

Since there are 15 pieces which are good apples out of which 2 can be selected in $15C_2$ ways. Therefore,

$$P(A \cap B) = \text{Probability of getting 2 pieces which are good apples} = \frac{{}^{15}C_2}{{}^{30}C_2}$$

Now,

Required probability $= P(A) + P(B) - P(A \cap B)$ [From (i)]

$$\Rightarrow \quad \text{Required probability} = \frac{{}^{20}C_2}{{}^{30}C_2} + \frac{{}^{22}C_2}{{}^{30}C_2} - \frac{{}^{15}C_2}{{}^{30}C_2} = \frac{316}{435}$$

ILLUSTRATION 11 *The probability that a person will get an electric contract is $\frac{2}{5}$ and the probability that he will not get plumbing contract is $\frac{4}{7}$. If the probability of getting at least one contract is $\frac{2}{3}$, what is the probability that he will get both?*

SOLUTION Consider the following events :

A = Person gets an electric contract, B = Person gets plumbing contract

We have,

$$P(A) = \frac{2}{5}, \; P(\overline{B}) = \frac{4}{7} \text{ and } P(A \cup B) = \frac{2}{3}$$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow \quad \frac{2}{3} = \frac{2}{5} + \left(1 - \frac{4}{7}\right) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) = \frac{2}{5} + \frac{3}{7} - \frac{2}{3} = \frac{17}{105}$$

ILLUSTRATION 12 *Let A, B, C be three events. If the probability of occuring exactly one event out of A and B is $1 - x$, out of B and C is $1 - 2x$, out of C and A is $1 - x$, and that of occuring three events simultaneously is x^2, then prove that the probability that atleast one out of A, B, C will occur is greater than 1/2.*

SOLUTION We have,

$P(A) + P(B) - 2P(A \cap B) = 1 - x$, ...(i)

$$P(B) + P(C) - 2P(B \cap C) = 1 - 2x, \quad ...(ii)$$

$$P(C) + P(A) - 2P(C \cap A) = 1 - x \quad ...(iii)$$

and, $$P(A \cap B \cap C) = x^2 \quad ...(iv)$$

Adding (i), (ii) and (iii), we get

$$P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = \frac{3-4x}{2} \quad ...(v)$$

Now,

Probability that atleast one out of A, B, C will occur

$= P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

$= \dfrac{3-4x}{2} + x^2$ [Using (iv) and (v)]

$= x^2 - 2x + \dfrac{3}{2} = (x-1)^2 + \dfrac{1}{2} > \dfrac{1}{2}$

ILLUSTRATION 13 *For the three events A, B and C, P (exactly one of the events A or B occurs) = P (exactly one of the events B or C occurs) = P (exactly one of the events C and A occurs) = p and P (all the three events occur simultaneously) = p^2, where $0 < p < 1/2$. Then, find the probability of occurrence of at least one of the three events A, B, and C.*

SOLUTION We have,

P (exactly one of the events A or B occurs) = p,

P (exactly one of the events B or C occurs) = p,

P (exactly one of the events C or A occurs) = p,

and, P (all the three events occur simultaneously) = p^2

i.e., $$P(A) + P(B) - 2P(A \cap B) = p, \quad ...(i)$$

$$P(B) + P(C) - 2P(B \cap C) = p, \quad ...(ii)$$

$$P(C) + P(A) - 2P(A \cap B) = p \quad ...(iii)$$

and, $$P(A \cap B \cap C) = p^2 \quad ...(iv)$$

Adding (i), (ii) and (iii), we get

$$P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) = \frac{3p}{2} \quad ...(v)$$

Now,

Required probability

$= P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

$= \dfrac{3p}{2} + p^2 = \dfrac{3p + 2p^2}{2}$

ILLUSTRATION 14 *The probabilities that a student pass in Mathematics, Physics and Chemistry are m, p and c respectively. Of these subjects, the student has a 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Find the value of $p + m + c$.*

SOLUTION Let A, B and C be three events given by

A = The student passes in Mathematics,

B = The student passes in Physics,

C = The student passes in Chemistry,

It is given that

$$P(A) = p, P(B) = m \text{ and } P(C) = c$$

$$P(A \cup B \cup C) = \frac{75}{100} \qquad ...(i)$$

$$P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C) = \frac{50}{100} \qquad ...(ii)$$

$$P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C) = \frac{40}{100} \qquad ...(iii)$$

From (ii) and (iii), we get

$$P(A \cap B \cap C) = \frac{1}{10} \qquad ...(iv)$$

Adding (ii) and (iii), we get

$$2\{P(A \cap B) + P(B \cap C) + P(C \cap A)\} - 5P(A \cap B \cap C) = \frac{9}{10}$$

$$\Rightarrow \quad 2\{P(A \cap B) + P(B \cap C) + P(C \cap A)\} = \frac{9}{10} + \frac{5}{10}$$

$$\Rightarrow \quad P(A \cap B) + P(B \cap C) + P(C \cap A) = \frac{7}{10} \qquad ...(v)$$

From (i), we have

$$P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) = \frac{3}{4}$$

$$\Rightarrow \quad p + m + c - \frac{7}{10} + \frac{1}{10} = \frac{3}{4} \qquad \text{[Using (iv) and (v)]}$$

$$\Rightarrow \quad p + m + c = \frac{27}{20}$$

30.3 CONDITIONAL PROBABILITY

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of event A under the condition that B has already occurred and $P(B) \neq 0$, is called the conditional probability and it is denoted by $P(A/B)$. Thus, we have

$P(A/B)$ = Probability of occurrence of A given that B has already occurred.

Similarly, $P(B/A)$ when $P(A) \neq 0$ is defined as the probability of occurrence of event B when A has already occurred.

In fact, the meanings of symbols $P(A/B)$ and $P(B/A)$ depend on the nature of the events A and B and also on the nature of the random experiment. These two symbols have the following meaning also.

$P(A/B)$ = Probability of occurrence of A when B occurs

OR

Probability of occurrence of A when B is taken as the sample space

OR

Probability of occurrence of A with respect to B.

and, $P(B/A)$ = Probability of occurrence of B when A occurs

OR

Probability of occurrence of B when A is taken as the sample space.

OR

Probability of occurrence of B with respect to A.

In order to understand the meaning of conditional probability let us consider the following illustrations.

ILLUSTRATION 1 *Let there be a bag containing 5 white and 4 red balls. Two balls are drawn from the bag one after the other without replacement. Consider the following events:*

A = Drawing a white ball in the first draw, B = Drawing a red ball in the second draw.

Now,

$P(B/A)$ = Probability of drawing a red ball in second draw given that a white ball has already been drawn in the first draw

$\Rightarrow$ $P(B/A)$ = Probability of drawing a red ball from a bag containing 4 white and 4 red balls

$\Rightarrow$ $P(B/A) = \frac{4}{8} = \frac{1}{2}$

For this random experiment $P(A/B)$ is not meaningful because A cannot occur after the occurrence of event B.

In the above illustration events A and B were subsets of two different sample spaces as they are outcomes of two different trials which are performed one after the other.

ILLUSTRATION 2 *Consider the random experiment of throwing a pair of dice and two events associated with it given by*

A = The sum of the numbers on two dice is 8 = $\{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\}$

B = There is an even number on first die.

$= \{(2, 1), ..., (2, 6), (4, 1), ..., (4, 6), (6, 1), ..., (6, 6)\}$

In this case, events A and B are the subsets of the same sample space. So, we have the following meanings for $P(A/B)$.

We have,

$P(A/B)$ = Probability of occurrence of A when B occurs

or

Probability of occurrence A when B is taken as the sample space.

$$\Rightarrow \quad P(A/B) = \frac{\text{Number of elementary events in } B \text{ which are favourable to } A}{\text{Number of elementary events in } B}$$

$$\Rightarrow \quad P(A/B) = \frac{\text{Number of elementary events favourable to } A \cap B}{\text{Number of elementary events favourable to } B}$$

$$\Rightarrow \quad P(A/B) = \frac{3}{18}$$

$P(B/A)$ = Probability of occurrence of B when A occurs.

or

Probability of occurrence of B when A is taken as the sample space

$$\Rightarrow \quad P(B/A) = \frac{\text{Number of elementary events in } A \text{ which are favourable to } B}{\text{Number of elementary events in } A}$$

$$\Rightarrow \quad P(B/A) = \frac{\text{Number of elementary events favourable to } A \cap B}{\text{Number of elementary events favourable to } A}$$

$$\Rightarrow \quad P(B/A) = \frac{3}{5}$$

ILLUSTRATION 3 *A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?*

SOLUTION Consider the following events :

A = Number 4 appears at least once, B = The sum of the numbers appearing is 6.

$\therefore$ Required probability

$= P(A/B)$

= Probability of occurrence of A when B is taken as the sample space

$$= \frac{\text{Number of elementary events favourable to } A \text{ which are favourable to } B}{\text{Number of elementary events favourable to } B}$$

$$= \frac{\text{Number of elementary events favourable to } (A \cap B)}{\text{Number of elementary events favourable to } B}$$

$$= \frac{2}{5}$$

It follows from illustrations 2 and 3 that if A and B are two events associated with the same sample space 5 of a random experiment, then

$$P(A/B) = \frac{\text{Number of elementary events favourable } A \cap B}{\text{Number of elementary events favourable to } B}$$

$$\Rightarrow \quad P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} \qquad \left[\begin{array}{r}\text{Dividing numerator and} \\ \text{denominator by } n(S)\end{array}\right]$$

$$\Rightarrow \quad P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Similarly, we have

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

These results are derived in the next section by using multiplication theorem on probability.

ILLUSTRATIVE EXAMPLES

Type I **EXAMPLES BASED ON** $P(A/B) = \frac{n(A \cap B}{n(B)}, P(B/A) = \frac{n(A \cap B)}{n(A)}$

EXAMPLE 1 *A fair dice is rolled. Consider the following events:*

$A = \{1, 3, 5\}, B = \{2, 3\}$, *and* $C = \{2, 3, 4, 5\}$. *Find* [NCERT]

(i) $P(A/B)$ *and* $P(B/A)$ (ii) $P(A/C)$ *and* $P(C/A)$

(iii) $P(A \cup B/C)$ *and* $P(A \cap B/C)$

SOLUTION We have, $n(A) = 3, n(B) = 2, n(c) = 4$

Clearly,

$$A \cap B = \{3\}, A \cap C = \{3, 5\}, A \cup B \cap C = \{2, 3, 5\} \text{ and } A \cap B \cap C = \{3\}$$

$$\Rightarrow \quad n(A \cap B) = 1, n(A \cap C) = 2, n(A \cup B \cap C) = 3 \text{ and } n(A \cap B \cap C) = 1$$

Now,

(i) $P(A/B) = \dfrac{n(A \cap B)}{n(B)} \Rightarrow P(A/B) = \dfrac{1}{2}$

$P(B/A) = \dfrac{n(A \cap B)}{n(A)} \Rightarrow P(B/A) = \dfrac{1}{3}$

(ii) $P(A/C) = \dfrac{n(A \cap C)}{n(C)} \Rightarrow P(A/C) = \dfrac{2}{4} = \dfrac{1}{2}$

$P(C/A) = \dfrac{n(A \cap C)}{n(A)} \Rightarrow P(C/A) = \dfrac{2}{3}$

(iii) $P(A \cup B/C) = \dfrac{n(A \cup B \cap C)}{n(C)} \Rightarrow P(A \cup B/C) = \dfrac{3}{4}$

(iv) $P(A \cap B/C) = \dfrac{n(A \cap B \cap C)}{n(C)} \Rightarrow P(A \cap B/C) = \dfrac{1}{4}$

EXAMPLE 2 *A coin is tossed three times. Find P (E/F) in each of the following:*

(i) E = *Head on the third toss*, F = *Heads on first two tosses*

(ii) E = *At least two heads*, F = *At most two heads*

(iii) E = *At most two tails*, F = *At least one tail* **[NCERT]**

SOLUTION The sample space associated to the given random experiment is given by

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

(i) We have,

$$E = \{HHH, HTH, THH, TTH\}; \; F = \{HHH, HHT\}$$

$$\therefore \quad E \cap F = \{HHH\}$$

Clearly, $n(E \cap F) = 1$ and $n(F) = 2$

$$\therefore \quad P(E/F) = \frac{n(E \cap F)}{n(F)} = \frac{1}{2}$$

(ii) We have,

$$E = \{HHH, HHT, HTH, THH\}, \; F = \{TTT, THT, TTH, HTT, THH, HTH, HHT\}$$

$$\therefore \quad E \cap F = \{HHT, HTH, THH\}$$

Clearly, $n(E \cap F) = 3$ and $n(F) = 7$

$$\therefore \quad P(E/F) = \frac{n(E \cap F)}{n(F)} = \frac{3}{7}$$

(iii) We have,

$$E = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$$

$$F = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

$$\therefore \quad E \cap F = \{HHT, HTH, THH, HTT, THT, TTH\}$$

Clearly, $n(E \cap F) = 6$ and $n(F) = 7$

$$\therefore \quad P(E/F) = \frac{n(E \cap F)}{n(F)} = \frac{6}{7}$$

EXAMPLE 3 *Two coins are tossed once. Find P (E/F) in each of the following:*

(i) *E = Tail appears on one coin, F = One coin shows head*

(ii) *E = No tail appears, F = No head appears.* **[NCERT]**

SOLUTION The sample space associated to the random experiment of tossing two coins is given by

$$S = \{HH, HT, TH, TT\}$$

(i) We have,

$$E = \{HT, TH, TT\}, F = \{HT, TH\}$$

$$\therefore \quad E \cap F = \{HT, TH\}$$

Clearly, $n(E \cap F) = 2$ and $n(F) = 2$

$$\therefore \quad P(E/F) = \frac{n(E \cap F)}{n(F)} = \frac{2}{2} = 1$$

(ii) We have,

$$E = \{HH\}, F = \{TT\}$$

$$\therefore \quad E \cap F = \{\ \} = \phi$$

Clearly, $n(E \cap F) = 0$ and $n(F) = 1$

$$\therefore \quad \text{Required probability} = \frac{n(E \cap F)}{n(F)} = \frac{0}{1} = 0$$

EXAMPLE 4 *Mother, father and son line up at random for a family picture. Find P (A/B), if A and B are defined as follows:*

A = Son on one end, B = Father in the middle **[NCERT]**

SOLUTION Total number of ways in which Mother (M), Father (F) and Son (S) can be lined up at random in one of the following ways:

$$MFS, MSF, FMS, FSM, SFM, SMF$$

We have,

$$A = \{SMF, SFM, MFS, FMS\} \text{ and } B = \{MFS, SFM\}$$

$$\therefore \quad A \cap B = \{MFS, SFM\}$$

Clearly, $n(A \cap B) = 2$ and $n(B) = 2$

$$\therefore \quad \text{Required probability} = P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{2} = 1$$

EXAMPLE 5 *A and B are two events such that P (A) ≠ 0. Find P (B/A), if*

(i) *A is a subset of B* (ii) $A \cap B = \phi$

SOLUTION (i) If A is a subset of B, then

$$A \cap B = A \Rightarrow n(A \cap B) = n(A)$$

$$\therefore \quad P(B/A) = \frac{n(A \cap B)}{n(A)} = 1$$

(ii) If $A \cap B = \phi$, then $n(A \cap B) = 0$

$$\therefore \quad P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{0}{n(A)} = 0$$

EXAMPLE 6 *A couple has two children. Find the probability that*

(i) ***both the children are boys, if it is known that the older child is a boys.*** **[NCERT]**

(ii) ***both the children are girls, if it is known that the older child is a girl.***

(iii) ***both the children are boys, if it is known that at least one of the children is a boy.***

SOLUTION Let B_i and G_i $(i = 1, 2)$ stand for the event that i^{th} child be a boy and a girl respectively. Then, the sample space associated to the random experiment is

$$A = \{B_1 B_2, B_1 G_2, G_1, B_2, G_1 G_2\}$$

(i) Consider the following events:

$$A = \text{Both the children are boys} = \{B_1 B_2\}$$

$$B = \text{The older child is a boy} = \{B_1 B_2, B_1 G_2\}$$

$\therefore$ $A \cap B = \{B_1 B_2\}$. Clearly, $n(A \cap B) = 1$ and $n(B) = 2$

$\therefore$ $$\text{Required probability} = P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{2}$$

(ii) Consider the following events:

$$A = \text{Both the children are girls} = \{G_1 G_2\}$$

$$B = \text{The older child is a girl} = \{G_1, G_2, G_1 B_2\}$$

$\therefore$ $A \cap B = \{G_1 G_2\}$

Clearly, $n(A \cap B) = 1$ and $n(B) = 2$

$\therefore$ $$\text{Required probability} = \frac{n(A \cap B)}{n(B)} = \frac{1}{2}$$

(iii) Consider the following events:

$$A = \text{Both the children are boys} = \{B_1 B_2\}$$

$$B = \text{At least one of the children is a boy} = \{B_1 B_2, B_1 G_2, G_1 B_2\}$$

$\therefore$ $A \cap B = \{B_1 B_2\}$

Clearly, $n(A \cap B) = 1$ and $n(B) = 3$

$\therefore$ $$\text{Required probability} = \frac{n(A \cap B)}{n(B)} = \frac{1}{3}$$

EXAMPLE 7 *A pair of dice is thrown. If the two numbers appearing on them are different, find the probability (i) the sum of the numbers is 6 (ii) the sum of the numbers is at least 4 (iii) the sum of the numbers is 4 or less.*

SOLUTION Consider the following events:

A = Numbers appearing on two dice are different

B = The sum of the numbers on two dice is 6

C = The sum of the numbers on two dice is 4 or less

D = The sum of the numbers on two dice is 4.

Clearly,

$$A = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,4), (3,5), (3,6), (4,1), (4,2)(4,3), (4,5), (4,6), (5,1), (5,2)\ (5,3), (5,4), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$$

$$B = \{(1,5), (5,1), (2,4)\ (4,2), (3,3)\}$$

$$C = \{(1,1), (1,2), (2,1), (1,3), (3,1), (2,2)\}$$

$$D = \{(1,3), (3,1), (2,2)\}$$

Clearly, $A \cap B = \{(1,5), (5,1), (2,4), (4,2)\}$

$$A \cap C = \{(1,2), (2,1), (1,3), (3,1)\}$$

$$A \cap D = \{(1,3), (3,1)\}$$

$\therefore$ $n(A \cap B) = 4, n(A \cap C) = 4, n(A \cap D) = 2$ and $n(A) = 30$

(i) Required probability $= P(B/A) = \dfrac{n(A \cap B)}{n(A)} = \dfrac{4}{30} = \dfrac{2}{15}$

(ii) Required probability $= P(C/A) = \dfrac{n(A \cap C)}{n(A)} = \dfrac{4}{30} = \dfrac{2}{15}$

(iii) Required probability $= P(D/A) = \dfrac{n(A \cap D)}{n(A)} = \dfrac{2}{30} = \dfrac{1}{15}$

EXAMPLE 8 *A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the probability that the number 4 has appeared at least once?* **[NCERT]**

SOLUTION Consider the following events:

A = Sum of the numbers appearing on two dice is 6

$= \{(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)\}$

B = Number 4 has appeared at least once

$= \{(1, 4), (4, 1), (2, 4), (4, 2), (3, 4), (4, 3), (4, 4), (4, 5), (5, 4), (4, 6), (6, 4)\}$

Required probability $= P(B/A) = \dfrac{n(A \cap B)}{n(A)} = \dfrac{2}{5}$

EXAMPLE 9 *A die is thrown three times. Events A and B are defined as below:*

A = *4 on third die*

B = 6 *on the first* and 5 *on the second throw*

Find the probability of A given that B has already occurred. **[NCERT]**

SOLUTION The sample space S associated to the random experiment of throwing three dice has $6 \times 6 \times 6 = 216$ elements.

We have,

$$\begin{aligned} A = \{&(1, 1, 4), (1, 2, 4), (1, 3, 4), (1, 4, 4), (1, 5, 4), (1, 6, 4), (2, 1, 4),\\ &(2, 1, 4), (2, 2, 4), (2, 3, 4), (2, 4, 4), (2, 5, 4), (2, 6, 4)\\ &\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\ &(6, 1, 4), (6, 2, 4), (6, 3, 4), (6, 4, 4), (6, 5, 4), (6, 6, 4)\end{aligned}$$

$B = \{(6, 5, 1), (6, 5, 2), (6, 5, 3), (6, 5, 4), (6, 5, 5), (6, 5, 6)\}$

$A \cap B = \{(6, 5, 4)\}$

$\therefore \quad P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{1}{6}$

EXAMPLE 10 *A black and a red dice are rolled in order. Find the conditional probability of obtaining*

(i) a sum grater than 9, given that the black die resulted in a 5.

(ii) *a sum 8, given that the red die resulted in a number less than 4.* **[NCERT]**

SOLUTION (i) Consider the following events:

A = Getting a sum greater than 9, B = Black die resulted in a 5

Clearly, $A = \{(5, 5), (6, 4), (4, 6), (6, 5), (5, 6), (6, 6)\}$

$B = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

$\therefore \quad A \cap B = \{(5, 5), (5, 6)\}$

We have, $n(A \cap B) = 2$, $n(A) = 6$ and $n(B) = 6$

Required probability $= P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{2}{6} = \dfrac{1}{3}$

(ii) Consider the following events:

A = Getting 8 as the sum

B = Red die resulted in a number less than 4

Clearly, $A = \{(2, 6), (6, 2)(3, 5), (5, 3), (4, 4)\}$

$B = \{(6, 1), (6, 2), (6, 3), (5, 1), (5, 2), (5, 3), (4, 1), (4, 2), (4, 3), (3, 1), (3, 2), (3, 3), (2, 1), (2, 2), (2, 3), (1, 1), (1, 2), (1, 3)\}$

$\therefore \quad A \cap B = \{(6, 2), (5, 3)\}$

We have, $n(A \cap B) = 2, n(B) = 18$

Required probability $= P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{2}{18} = \dfrac{1}{9}$

EXAMPLE 11 *Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail' given that 'at least one die shows a three'.* **[NCERT]**

SOLUTION The sample space S associated to the given random experiment is given by

$S = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6), (1, H), (1, T), (2, H), (2, T), (4, H), (4, T), (5, H), (5, T)\}$

Consider the following events:

A = The coin shows a tail, B = At least one die shows a three

Clearly, $A = \{(1, T), (2, T), (4, T), (5, T)\}$

$B = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (6, 3)\}$

$\therefore \quad A \cap B = \phi$

We have, $n(A \cap B) = 0, n(A) = 4$ and $n(B) = 7$

Required probability $= P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{0}{7} = 0$

EXAMPLE 12 *In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in class XII given that the chosen student is a girl?* **[NCERT]**

SOLUTION Let A be the event that a student chosen randomly studies in class XII and B be the event that the randomly chosen student is a girl.

We have to find $P(A/B)$.

Clearly, $n(A \cap B) = 10\%$ of $430 = 430 \times \dfrac{10}{100} = 43$ and, $n(B) = 430$

$\therefore$ Required probability $= P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{43}{430} = \dfrac{1}{10}$

EXAMPLE 13 *An instructor has a question bank consisting of 300 easy True/false questions, 200 difficult True/False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be any easy question given that it is a multiple choice question?* **[NCERT]**

SOLUTION Let A be the event that selected question is an easy question and B be the event that the question selected is a multiple choice question.

We have,

$n(A) = 300 + 500 = 800, n(B) = 500 + 400 = 900$

$A \cap B$ = Selected question is an easy multiple choice question

$\therefore \quad n(A \cap B) = 500$

Required probability $= P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{500}{900} = \dfrac{5}{9}$

Type II EXAMPLES BASED ON $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ AND $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

EXAMPLE 14 *Given that A and B are two events such that $P(A) = 0.6$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$, find $P(A/B)$ and $P(B/A)$.* [NCERT]

SOLUTION We have,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \text{ and } P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A/B) = \frac{0.2}{0.3} = \frac{2}{3} \text{ and } P(B/A) = \frac{0.2}{0.6} = \frac{1}{3}$$

EXAMPLE 15 *If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find*

(i) $P(A \cap B)$ (ii) $P(A/B)$ (iii) $P(B/A)$ [NCERT]

SOLUTION (i) We have,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11}$$

(ii) We have,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A/B) = \frac{\frac{4}{11}}{\frac{5}{11}} = \frac{4}{5}$$

(iii) We have,

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(B/A) = \frac{\frac{4}{11}}{\frac{6}{11}} = \frac{2}{3}$$

EXAMPLE 16 *Evaluate $P(A \cup B)$, if $2P(A) = P(B) = \frac{5}{13}$ and $P(A/B) = \frac{2}{5}$* [NCERT]

SOLUTION We have,

$$2P(A) = P(B) = \frac{5}{13} \Rightarrow P(A) = \frac{5}{26} \text{ and } P(B) = \frac{5}{13}$$

Now, $\quad P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$\Rightarrow \quad \frac{2}{5} = \frac{P(A \cap B)}{\frac{5}{13}} \Rightarrow P(A \cap B) = \frac{2}{13}$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{11}{26}$$

Type III PROBLEMS BASED UPON THE MEANING OF CONDITIONAL PROBABILITY

EXAMPLE 17 *Two integers are selected at random from integers 1 to 11. If the sum is even, find the probability that both the numbers are odd.*

SOLUTION Out of integers from 1 to 11, there are 5 even integers and 6 odd integers. Consider the following events:

A = Both the numbers chosen are odd,

B = The sum of the numbers chosen is even,

$\therefore$ Required probability

$= P(A/B)$

= Probability that the two numbers chosen are odd if it is given that the sum of the numbers chosen is even.

$$= \frac{{}^6C_2}{{}^5C_2 + {}^6C_2} \qquad \left[\begin{array}{l} \because \text{ The number of ways of getting the sum} \\ \text{as an even number } = {}^5C_2 + {}^6C_2. \\ \text{The number of ways of selecting} \\ \text{two odd numbers } = {}^6C_2 \end{array}\right]$$

$$= \frac{3}{5}$$

EXAMPLE 18 *A die is thrown three times, if the first throw is a four, find the chance of getting 15 as the sum.*

SOLUTION Consider the folliwing events :

A = Getting 15 as the sum in a throw of three dice, B = Getting 4 on the first die

$\therefore$ Required probability

$= P(A/B)$

= Probability of getting 15 as the sum of the numbers if there is 4 on the first die.

$$= \frac{2}{36} \qquad \left[\begin{array}{l} \because \text{ Total number of ways} = 1 \times 6 \times 6 = 36 \\ \text{There are two favourable elementary} \\ \text{events viz. } (4,6,5), (4,5,6). \end{array}\right]$$

$$= \frac{1}{18}$$

EXERCISE 30.1

1. Ten cards numbered 1 through 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?
2. Assume that each child born is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl? [NCERT]
3. Given that the two numbers appearing on throwing two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.
4. A coin is tossed three times, if head occurs on first two tosses, find the probability of getting head on third toss.

5. A die is thrown three times, find the probability that 4 appears on the third toss if it is given that 6 and 5 appear respectively on first two tosses.
6. Compute $P(A/B)$, if $P(B) = 0.5$ and $P(A \cap B) = 0.32$
7. If $P(A) = 0.4$, $P(B) = 0.3$ and $P(B/A) = 0.5$, find $P(A \cap B)$ and $P(A/B)$.
8. If A and B are two events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{5}$ and $P(A \cup B) = \frac{11}{30}$, find $P(A/B)$ and $P(B/A)$.
9. A couple has two children. Find the probability that both the children are (i) males, if it is known that at least one of the children is male. (ii) females, if it is known that the elder child is a female. **[CBSE 2010]**

ANSWERS

1. $\frac{4}{7}$ 2. (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ 3. $\frac{1}{15}$ 4. $\frac{1}{2}$ 5. $\frac{1}{6}$ 6. $\frac{16}{25}$

7. $0.2, \frac{2}{3}$ 8. $\frac{5}{6}, \frac{1}{2}$ 9. (i) $\frac{1}{3}$ (ii) $\frac{1}{2}$

HINTS TO SELECTED PROBLEMS

1. The sample space associated to the given random experiment is given by

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Consider the following events:

A = Number on the card drawn is even $= \{2, 4, 6, 8, 10\}$

B = Number on the card drawn is greater than $3 = \{4, 5, 6, 7, 8, 9, 10\}$

$\therefore$ Required probability $= P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{4}{7}$

30.4 MULTIPLICATION THEOREMS ON PROBABILITY

In this section, we shall discuss some theorems which are helpful in computing the probabilities of simultaneous occurrences of two or more events associated with a random experiment.

THEOREM 1 *If A and B are two events associated with a random experiment, then*

$$P(A \cap B) = P(A)\, P(B/A), \text{ if } P(A) \neq 0$$

or,

$$P(A \cap B) = P(B)\, P(A/B), \text{ if } P(B) \neq 0.$$

PROOF Let S be the sample space associated with the given random experiment. Suppose S contains n elementary events. Let m_1, m_2 and m be the number of elementary events favourable to A, B and $A \cap B$ respectively. Then,

$$P(A) = \frac{m_1}{n},\ P(B) = \frac{m_2}{n} \text{ and } P(A \cap B) = \frac{m}{n}.$$

Since m_1 elementary events are favourable to A out of which m are favourable to B.

Therefore, $P(B/A) = \dfrac{m}{m_1}$.

Similarly, we have

$$P(A/B) = \frac{m}{m_2}$$

Now, $$P(A \cap B) = \frac{m}{n}$$

$$\Rightarrow \quad P(A \cap B) = \frac{m}{m_1} \cdot \frac{m_1}{n} = P(B/A) \cdot P(A) \quad \text{...(i)}$$

and, $$P(A \cap B) = \frac{m}{n}$$

$$\Rightarrow \quad P(A \cap B) = \frac{m}{m_2} \cdot \frac{m_2}{n} = P(A/B)\, P(B) \quad \text{...(ii)}$$

Q.E.D.

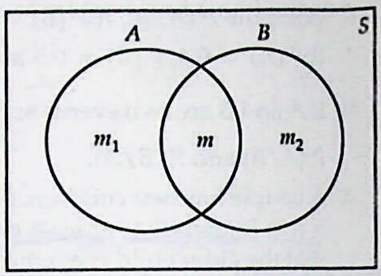

Fig. 30.1

NOTE 1 *From (i) and (ii) in the above theorem, we find that*

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \text{ and } P(A/B) = \frac{P(A \cap B)}{P(B)}$$

REMARK *If A and B are independent events, then $P(A/B) = P(A)$ and $P(B/A) = P(B)$. Therefore,*

$$P(A \cap B) = P(A)\, P(B)$$

Also, $$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A)\, P(B) \\ &= 1 - [1 - P(A) - P(B) + P(A)\, P(B)] \\ &= 1 - [(1 - P(\overline{A}))\,(1 - P(\overline{B}))] \\ &= 1 - P(\overline{A})\, P(\overline{B}) \end{aligned}$$

THEOREM 2 *(Extension of multiplication theorem) if $A_1, A_2, ..., A_n$ are n events associated with a random experiment, then*

$$P(A_1 \cap A_2 \cap A_3 .... \cap A_n)$$

$$= P(A_1)\, P(A_2/A_1)\; P(A_3/A_1 \cap A_2) \,.......\, P(A_n/A_1 \cap A_2 \cap .... \cap A_{n-1}),$$

where $P(A_i/A_1 \cap A_2 ... \cap A_{i-1})$ represents the conditional probability of the occurrence of event A_i, given that the events $A_1, A_2, ..., A_{i-1}$ have already occurred.

PROOF This theorem can be proved by using the principle of mathematical induction.

PARTICULAR CASE If A, B, C are three events associated with a random experiment, then

$$P(A \cap B \cap C) = P(A)\, P(B/A)\, P(C/A \cap B).$$

ILLUSTRATIVE EXAMPLES

Type I **FIND THE PROBABILITY OF SIMULTANEOUS OCCURRENCE OF TWO OR MORE EVENTS BY USING MULTIPLICATION THEOREM**

EXAMPLE 1 *A bag contains 10 white and 15 black balls. Two balls are drawn in succession without replacement. What is the probability that first is white and second is black?*

SOLUTION Consider the following events :

A = getting a white ball in first draw, B = getting a black ball in second draw.

Required probability

$= $ Probability of getting a white ball in first draw and a black ball in second draw

$= P(A \text{ and } B)$

$= P(A \cap B)$

$= P(A)\, P(B/A)$ [By Multiplication Theorem] ...(i)

Now, $P(A) = \dfrac{{}^{10}C_1}{{}^{25}C_1} = \dfrac{10}{25} = \dfrac{2}{5}$

and, $P(B/A) = $ Probability of getting a black ball in second draw when a white ball has already been in first draw

$\Rightarrow \quad P(B/A) = \dfrac{{}^{15}C_1}{{}^{24}C_1} = \dfrac{15}{24} = \dfrac{5}{8}$ $\left[\because \text{24 balls are left after drawing a white ball in first–draw out of which 15 are black}\right]$

Substituting these values in (i), we have

Required probability $= P(A \cap B) = P(A)\, P(B/A) = \dfrac{2}{5} \times \dfrac{5}{8} = \dfrac{1}{4}$

EXAMPLE 2 *Find the probability of drawing a diamond card in each of the two consecutive draws from a well shuffled pack of cards, if the card drawn is not replaced after the first draw.*

[CBSE 2002C]

SOLUTION Let A be the event of drawing a diamond card in the first draw and B be the even of drawing a diamond card in the second draw. Then,

$$P(A) = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{13}{52} = \frac{1}{4}.$$

After drawing a diamond card in first draw 51 cards are left out of which 12 cards are diamond cards.

$\therefore \quad P(B/A) = $ Probability of drawing a diamod card in second draw when a diamond card has already been drawn in first draw

$\Rightarrow \quad P(B/A) = \dfrac{{}^{12}C_1}{{}^{51}C_1} = \dfrac{12}{51} = \dfrac{4}{17}.$

Now, Required probability $= P(A \cap B) = P(A)\, P(B/A) = \dfrac{1}{4} \times \dfrac{4}{17} = \dfrac{1}{17}$

EXAMPLE 3 *A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one without replacement, find the probability of getting all white balls.*

SOLUTION Let A, B, C, D denote events of getting a white ball in first, second, third and fourth draw respectively. Then,

Required probability

$= P(A \cap B \cap C \cap D)$

$= P(A)\, P(B/A)\, P(C/A \cap B)\, P(D/A \cap B \cap C)$ ...(i)

Now, $P(A) = $ Probability of drawing a white ball in first draw $= \dfrac{5}{20} = \dfrac{1}{4}$

When a white ball is drawn in the first draw there are 19 balls left in the bag, out of which 4 are white.

$\therefore \quad P(B/A) = \dfrac{4}{19}$

Since the ball drawn is not replaced, therefore after drawing a white ball in second draw there are 18 balls left in the bag, out of which 3 are white.

$$\therefore \qquad P(C/A\cap B) = \frac{3}{18} = \frac{1}{6}$$

After drawing a white ball in third draw there are 17 balls left in the bag, out of which 2 are white.

$$\therefore \qquad P(D/A\cap B\cap C) = \frac{2}{17}$$

Hence, Required probability

$$= P(A\cap B\cap C\cap D)$$

$$= P(A)\,P(B/A)\,P(C/A\cap B)\,P(D/A\cap B\cap C) = \frac{1}{4}\times\frac{4}{19}\times\frac{1}{6}\times\frac{2}{17} = \frac{1}{969}$$

EXAMPLE 4 *A bag contains 19 tickets, numbered from 1 to 19. A ticket is drawn and then another ticket is drawn without replacement. Find the probability that both tickets will show even numbers.*

SOLUTION Let A be the event of drawing an even numbered ticket in first draw and B be the event of drawing an even numbered ticket in the second draw. Then,

$$\text{Required probability} = P(A\cap B) = P(A)\,P(B/A) \qquad \text{...(i)}$$

Since there are 19 tickets, numbered 1 to 19, in the bag out of which 9 are even numbered viz. 2, 4, 6, 8, 10, 12, 14, 16, 18. Therefore,

$$P(A) = \frac{9}{19}$$

Since the ticket drawn in the first draw is not replaced, therefore second ticket drawn is from the remaining 18 tickets, out of which 8 are even numbered.

$$\therefore \qquad P(B/A) = \frac{8}{18} = \frac{4}{9}.$$

Hence, Required probability $= P(A\cap B)$

$$= P(A)\,P(B/A) \qquad \text{[From (i)]}$$

$$= \frac{9}{19}\times\frac{4}{9} = \frac{4}{19}$$

EXAMPLE 5 *An urn contains 5 white and 8 black balls. Two successive drawings of three balls at a time are made such that the balls are not replaced before the second draw. Find the probability that the first draw gives 3 white balls and second draw gives 3 black balls.*

SOLUTION Consider the following events :

A = drawing 3 white balls in first draw,

B = drawing 3 black balls in the second draw.

$$\text{Required probabilty} = P(A\cap B) = P(A)\,P(B/A) \qquad \text{...(i)}$$

Now, $$P(A) = \frac{^5C_3}{^{13}C_3} = \frac{10}{286} = \frac{5}{143}$$

After drawing 3 white balls in first draw 10 balls are left in the bag, out of which 8 are black balls.

$$\therefore \qquad P(B/A) = \frac{^8C_3}{^{10}C_3} = \frac{56}{120} = \frac{7}{15}$$

Hence, Required probability $= P(A\cap B)$

$\Rightarrow$ Required probability $= P(A)\,P(B/A)$ [From (i)]

$\Rightarrow$ Required probability $= \dfrac{5}{143} \times \dfrac{7}{15} = \dfrac{7}{429}$

EXAMPLE 6 *Two balls are drawn from an urn containing 2 white, 3 red and 4 black balls one by one without replacement. What is the probability that at least one ball is red?*

SOLUTION Let A be the event of not getting a red ball in first draw and B be the event of not getting a red ball in second draw. Then,

Required probability

$= $ Probability that at least one ball is red

$= 1 -$ Probability that none is red

$= 1 - P(A \text{ and } B) = 1 - P(A \cap B)$

$= 1 - P(A)\,P(B/A)$ ...(i)

Now,

$P(A) =$ Probability of not getting a red ball in first draw

$\Rightarrow$ $P(A) =$ Probability of getting an other colour (white or black) ball in first draw

$\Rightarrow$ $P(A) = \dfrac{6}{9} = \dfrac{2}{3}$

When another colour ball is drawn in first draw there are 5 other colour (white or black) balls and 3 red balls, out of which one other colour ball can be drawn in ${}^5C_1$ ways.

$\therefore$ $P(B/A) = \dfrac{5}{8}$

Substituting these values in (i), we have

Required probability $= 1 - P(A)\,P(B/A) = 1 - \dfrac{2}{3} \times \dfrac{5}{8} = \dfrac{7}{12}$

EXAMPLE 7 *A bag contains n white and n red balls. Pairs of balls are drawn without replacement until the bag is empty. Show that the probability that each pair consists of one white and one red ball is $\dfrac{2^n}{{}^{2n}C_n}$.*

SOLUTION Let $A_i\ (i = 1, 2, \ldots, n)$ be the event of getting one white and one red ball in ith draw. Then,

Required probability

$= P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)$

$= P(A_1)\,P(A_2/A_1)\,P(A_3/A_1 \cap A_2) \ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1})$ ...(i)

Now, $P(A_1) = \dfrac{{}^nC_1 \times {}^nC_1}{{}^{2n}C_2} = \dfrac{n^2}{{}^{2n}C_2}$

$P(A_2/A_1) = \dfrac{{}^{n-1}C_1 \times {}^{n-1}C_1}{{}^{2n-2}C_2} = \dfrac{(n-1)^2}{{}^{2n-2}C_2}$

$P(A_3/A_1 \cap A_2) = \dfrac{{}^{n-2}C_1 \times {}^{n-2}C_1}{{}^{2n-4}C_2} = \dfrac{(n-2)^2}{{}^{2n-4}C_2}$ and so on.

Finally, $P(A_{n-1}/A_1 \cap A_2 \cap \ldots \cap A_{n-2}) = \dfrac{{}^2C_1 \times {}^2C_1}{{}^4C_2} = \dfrac{2^2}{{}^4C_2}$

and, $P(A_n/A_1 \cap A_2 \ldots \cap A_{n-1}) = \dfrac{1}{{}^2C_2}$

Substituting these values in (i), we have

Required probability

$$= \frac{n^2}{{}^{2n}C_2} \times \frac{(n-1)^2}{{}^{2n-2}C_2} \times \frac{(n-2)^2}{{}^{2n-4}C_2} \times \ldots \times \frac{2^2}{{}^4C_2} \times \frac{1}{{}^2C_2}$$

$$= \frac{(n!)^2}{\frac{(2n)(2n-1)}{2} \times \frac{(2n-2)(2n-3)}{2} \times \ldots \times \frac{4 \times 3}{2} \times \frac{2 \times 1}{2}} = \frac{(n!)^2\, 2^n}{(2n)!} = \frac{2^n}{{}^{2n}C_n}$$

EXAMPLE 8 *To test the quality of electric bulbs produced in a factory, two bulbs are randomly selected from a large sample without replacement. If either bulb is defective, the entire lot is rejected. Suppose a sample of 200 bulbs contains 5 defective bulbs. Find the probability that the sample will be rejected.*

SOLUTION Clearly, the sample will be rejected if at least one of the two bulbs is defective. Consider the following events:

A = First bulb is defective, B = Second bulb is defective.

$\therefore$ Required probability

$= P(A \cup B)$

$= 1 - P(\overline{A \cup B})$

$= 1 - P(\overline{A} \cap \overline{B})$

$= 1 - P(\overline{A})\, P(\overline{B}/\overline{A}) = 1 - \dfrac{195}{200} \times \dfrac{194}{199} = 1 - \dfrac{3783}{3980} = \dfrac{197}{3980}$.

EXERCISE 30.2

1. From a pack of 52 cards, two are drawn one by one without replacement. Find the probability that both of them are kings.
2. From a pack of 52 cards, 4 are drawn one by one without replacement. Find the probability that all are aces.
3. Find the chance of drawing 2 white balls in succession from a bag containing 5 red and 7 white balls, the ball first drawn not being replaced.
4. A bag contains 25 tickets, numbered from 1 to 25. A ticket is drawn and then another ticket is drawn without replacement. Find the probability that both tickets will show even numbers.
5. From a deck of cards, three cards are drawn on by one without replacement. Find the probability that each time it is a card of spade.
6. Two cards are drawn without replacement from a pack of 52 cards. Find the probability that
   (i) both are kings (ii) the first is a king and the second is an ace
   (iii) the first is a heart and second is red.

7. A bag contains 20 tickets, numbered from 1 to 20. Two tickets are drawn without replacement. What is the probability that the first ticket has an even number and the second an odd number.

8. (i) An urn contains 3 white, 4 red and 5 black balls. Two balls are drawn one by one without replacement. What is the probability that at least one ball is black?

(ii) A bag contains 4 white, 7 black and 5 red balls. Three balls are drawn one after the other without replacement. Find the probability that the balls drawn are white, black and red respectively.

9. A bag contains 5 white, 7 red and 3 black balls. If three balls are drawn one by one without replacement, find the probability that none is red. **[CBSE 2002C]**

10. A card is drawn from a well-shuffled deck of 52 cards and then a second card is drawn. Find the probability that the first card is a heart and the second card is a diamond if the first card is not replaced.

11. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black? **[NCERT]**

12. Three cards are drawn successively, without replacement from a pack of 52 well skuffled cards. What is the probability that first two cards are kings and third cards drawn is an ace? **[NCERT]**

13. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale otherwise it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

ANSWERS

1. $\frac{1}{221}$ 2. $\frac{1}{270725}$ 3. $\frac{7}{22}$ 4. $\frac{11}{50}$ 5. $\frac{11}{850}$

6. (i) $\frac{1}{221}$ (ii) $\frac{4}{663}$ (iii) $\frac{25}{204}$ 7. $\frac{5}{19}$ 8. (i) $\frac{15}{22}$ (ii) $\frac{1}{24}$

9. $\frac{8}{65}$ 10. $\frac{13}{204}$ 11. $\frac{3}{7}$ 12. $\frac{2}{5525}$ 13. $\frac{44}{91}$

HINTS TO SELECTED PROBLEMS

10. Consider the following events:

A = Getting heart card in first draw

B = Getting a diamond card in second draw

Required probability $= P(A \cap B) = P(A)\, P(B/A) = \frac{13}{52} \times \frac{13}{51} = \frac{13}{204}$

11. Let B_i $(i = 1, 2)$ denote the event of getting a black ball in the ith draw.

Required probability $= P(B_1 \cap B_2) = P(B_1)\, P(B_2/B_1) = \frac{10}{15} \times \frac{9}{14} = \frac{3}{7}$

12. Consider the following events:

K_1 = Getting a king in first draw

K_2 = Getting a king in second draw

A_3 = Getting an ace in third draw.

Since the ball drawn is not replaced, therefore after drawing a white ball in second draw there are 18 balls left in the bag, out of which 3 are white.

$$\therefore \quad P(C/A \cap B) = \frac{3}{18} = \frac{1}{6}$$

After drawing a white ball in third draw there are 17 balls left in the bag, out of which 2 are white.

$$\therefore \quad P(D/A \cap B \cap C) = \frac{2}{17}$$

Hence, Required probability

$$= P(A \cap B \cap C \cap D)$$

$$= P(A)\,P(B/A)\,P(C/A \cap B)\,P(D/A \cap B \cap C) = \frac{1}{4} \times \frac{4}{19} \times \frac{1}{6} \times \frac{2}{17} = \frac{1}{969}$$

EXAMPLE 4 *A bag contains 19 tickets, numbered from 1 to 19. A ticket is drawn and then another ticket is drawn without replacement. Find the probability that both tickets will show even numbers.*

SOLUTION Let A be the event of drawing an even numbered ticket in first draw and B be the event of drawing an even numbered ticket in the second draw. Then,

$$\text{Required probability} = P(A \cap B) = P(A)\,P(B/A) \quad \text{...(i)}$$

Since there are 19 tickets, numbered 1 to 19, in the bag out of which 9 are even numbered viz. 2, 4, 6, 8, 10, 12, 14, 16, 18. Therefore,

$$P(A) = \frac{9}{19}$$

Since the ticket drawn in the first draw is not replaced, therefore second ticket drawn is from the remaining 18 tickets, out of which 8 are even numbered.

$$\therefore \quad P(B/A) = \frac{8}{18} = \frac{4}{9}.$$

Hence, Required probability $= P(A \cap B)$

$$= P(A)\,P(B/A) \quad \text{[From (i)]}$$

$$= \frac{9}{19} \times \frac{4}{9} = \frac{4}{19}$$

EXAMPLE 5 *An urn contains 5 white and 8 black balls. Two successive drawings of three balls at a time are made such that the balls are not replaced before the second draw. Find the probability that the first draw gives 3 white balls and second draw gives 3 black balls.*

SOLUTION Consider the following events :

A = drawing 3 white balls in first draw,

B = drawing 3 black balls in the second draw.

$$\text{Required probabilty} = P(A \cap B) = P(A)\,P(B/A) \quad \text{...(i)}$$

Now, $$P(A) = \frac{{}^5C_3}{{}^{13}C_3} = \frac{10}{286} = \frac{5}{143}$$

After drawing 3 white balls in first draw 10 balls are left in the bag, out of which 8 are black balls.

$$\therefore \quad P(B/A) = \frac{{}^8C_3}{{}^{10}C_3} = \frac{56}{120} = \frac{7}{15}$$

Hence, Required probability $= P(A \cap B)$

$\Rightarrow$ Required probability $= P(A)\,P(B/A)$ [From (i)]

$\Rightarrow$ Required probability $= \frac{5}{143} \times \frac{7}{15} = \frac{7}{429}$

EXAMPLE 6 *Two balls are drawn from an urn containing 2 white, 3 red and 4 black balls one by one without replacement. What is the probability that at least one ball is red?*

SOLUTION Let A be the event of not getting a red ball in first draw and B be the event of not getting a red ball in second draw. Then,

Required probability
= Probability that at least one ball is red
= 1 – Probability that none is red
$= 1 - P(A \text{ and } B) = 1 - P(A \cap B)$
$= 1 - P(A)\,P(B/A)$ ...(i)

Now,

$P(A)$ = Probability of not getting a red ball in first draw

$\Rightarrow$ $P(A)$ = Probability of getting an other colour (white or black) ball in first draw

$\Rightarrow$ $P(A) = \frac{6}{9} = \frac{2}{3}$

When another colour ball is drawn in first draw there are 5 other colour (white or black) balls and 3 red balls, out of which one other colour ball can be drawn in ${}^5C_1$ ways.

$\therefore$ $P(B/A) = \frac{5}{8}$

Substituting these values in (i), we have

Required probability $= 1 - P(A)\,P(B/A) = 1 - \frac{2}{3} \times \frac{5}{8} = \frac{7}{12}$

EXAMPLE 7 *A bag contains n white and n red balls. Pairs of balls are drawn without replacement until the bag is empty. Show that the probability that each pair consists of one white and one red ball is* $\frac{2^n}{{}^{2n}C_n}$.

SOLUTION Let A_i $(i = 1, 2, \ldots, n)$ be the event of getting one white and one red ball in ith draw. Then,

Required probability

$= P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)$

$= P(A_1)\,P(A_2/A_1)\,P(A_3/A_1 \cap A_2) \ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1})$ ...(i)

Now, $P(A_1) = \frac{{}^nC_1 \times {}^nC_1}{{}^{2n}C_2} = \frac{n^2}{{}^{2n}C_2}$

$P(A_2/A_1) = \frac{{}^{n-1}C_1 \times {}^{n-1}C_1}{{}^{2n-2}C_2} = \frac{(n-1)^2}{{}^{2n-2}C_2}$

$P(A_3/A_1 \cap A_2) = \frac{{}^{n-2}C_1 \times {}^{n-2}C_1}{{}^{2n-4}C_2} = \frac{(n-2)^2}{{}^{2n-4}C_2}$ and so on.

Finally, $P(A_{n-1}/A_1 \cap A_2 \cap \ldots \cap A_{n-2}) = \dfrac{{}^2C_1 \times {}^2C_1}{{}^4C_2} = \dfrac{2^2}{{}^4C_2}$

and, $P(A_n/A_1 \cap A_2 \ldots \cap A_{n-1}) = \dfrac{1}{{}^2C_2}$

Substituting these values in (i), we have

Required probability

$$= \frac{n^2}{{}^{2n}C_2} \times \frac{(n-1)^2}{{}^{2n-2}C_2} \times \frac{(n-2)^2}{{}^{2n-4}C_2} \times \ldots \times \frac{2^2}{{}^4C_2} \times \frac{1}{{}^2C_2}$$

$$= \frac{(n!)^2}{\frac{(2n)(2n-1)}{2} \times \frac{(2n-2)(2n-3)}{2} \times \ldots \times \frac{4 \times 3}{2} \times \frac{2 \times 1}{2}} = \frac{(n!)^2 \, 2^n}{(2n)!} = \frac{2^n}{{}^{2n}C_n}$$

EXAMPLE 8 *To test the quality of electric bulbs produced in a factory, two bulbs are randomly selected from a large sample without replacement. If either bulb is defective, the entire lot is rejected. Suppose a sample of 200 bulbs contains 5 defective bulbs. Find the probability that the sample will be rejected.*

SOLUTION Clearly, the sample will be rejected if at least one of the two bulbs is defective. Consider the following events:

A = First bulb is defective, B = Second bulb is defective.

$\therefore$ Required probability

$= P(A \cup B)$

$= 1 - P(\overline{A \cup B})$

$= 1 - P(\overline{A} \cap \overline{B})$

$= 1 - P(\overline{A})\, P(\overline{B}/\overline{A}) = 1 - \dfrac{195}{200} \times \dfrac{194}{199} = 1 - \dfrac{3783}{3980} = \dfrac{197}{3980}$.

EXERCISE 30.2

1. From a pack of 52 cards, two are drawn one by one without replacement. Find the probability that both of them are kings.
2. From a pack of 52 cards, 4 are drawn one by one without replacement. Find the probability that all are aces.
3. Find the chance of drawing 2 white balls in succession from a bag containing 5 red and 7 white balls, the ball first drawn not being replaced.
4. A bag contains 25 tickets, numbered from 1 to 25. A ticket is drawn and then another ticket is drawn without replacement. Find the probability that both tickets will show even numbers.
5. From a deck of cards, three cards are drawn on by one without replacement. Find the probability that each time it is a card of spade.
6. Two cards are drawn without replacement from a pack of 52 cards. Find the probability that
   (i) both are kings (ii) the first is a king and the second is an ace
   (iii) the first is a heart and second is red.

7. A bag contains 20 tickets, numbered from 1 to 20. Two tickets are drawn without replacement. What is the probability that the first ticket has an even number and the second an odd number.

8. (i) An urn contains 3 white, 4 red and 5 black balls. Two balls are drawn one by one without replacement. What is the probability that at least one ball is black?

(ii) A bag contains 4 white, 7 black and 5 red balls. Three balls are drawn one after the other without replacement. Find the probability that the balls drawn are white, black and red respectively.

9. A bag contains 5 white, 7 red and 3 black balls. If three balls are drawn one by one without replacement, find the probability that none is red. **[CBSE 2002C]**

10. A card is drawn from a well-shuffled deck of 52 cards and then a second card is drawn. Find the probability that the first card is a heart and the second card is a diamond if the first card is not replaced.

11. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black? **[NCERT]**

12. Three cards are drawn successively, without replacement from a pack of 52 well skuffled cards. What is the probability that first two cards are kings and third cards drawn is an ace? **[NCERT]**

13. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale otherwise it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

ANSWERS

1. $\frac{1}{221}$ 2. $\frac{1}{270725}$ 3. $\frac{7}{22}$ 4. $\frac{11}{50}$ 5. $\frac{11}{850}$

6. (i) $\frac{1}{221}$ (ii) $\frac{4}{663}$ (iii) $\frac{25}{204}$ 7. $\frac{5}{19}$ 8. (i) $\frac{15}{22}$ (ii) $\frac{1}{24}$

9. $\frac{8}{65}$ 10. $\frac{13}{204}$ 11. $\frac{3}{7}$ 12. $\frac{2}{5525}$ 13. $\frac{44}{91}$

HINTS TO SELECTED PROBLEMS

10. Consider the following events:

A = Getting heart card in first draw

B = Getting a diamond card in second draw

Required probability $= P(A \cap B) = P(A)\,P(B/A) = \frac{13}{52} \times \frac{13}{51} = \frac{13}{204}$

11. Let B_i $(i = 1, 2)$ denote the event of getting a black ball in the ith draw.

Required probability $= P(B_1 \cap B_2) = P(B_1)\,P(B_2/B_1) = \frac{10}{15} \times \frac{9}{14} = \frac{3}{7}$

12. Consider the following events:

K_1 = Getting a king in first draw

K_2 = Getting a king in second draw

A_3 = Getting an ace in third draw.

Required probability $= P(K_1 \cap K_2 \cap A_3)$

$= P(K_1)\, P(K_2/K_1)\, P(A_3/K_1 \cap K_2) = \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}$

13. Let G_i $(i = 1, 2, 3)$ denote the event of getting a good orange in ith draw.

Required probability $= P(G_1 \cap G_2 \cap G_3)$

$= P(G_1)\, P(G_2/G_1)\, P(G_3/G_1 \cap G_2) = \frac{12}{15} \times \frac{11}{14} \times \frac{10}{13} = \frac{44}{91}$

30.5 MORE ON CONDITIONAL PROBABILITY

In section 30.3, we have introduced the concept of conditional probability which has been used in multiplication theorem of probability.

In this section, we will obtain a formula for finding the conditional probability.

If A and B are two events associated with a random experiment, then

$$P(A \cap B) = P(A)\, P(B/A), \text{ if } P(A) \neq 0 \qquad \text{[Multiplication theorem]}$$

or, $$P(A \cap B) = P(B)\, P(A/B), \text{ if } P(B) \neq 0$$

$\therefore$ $$P(A/B) = \frac{P(A \cap B)}{P(B)} \text{ and } P(B/A) = \frac{P(A \cap B)}{P(A)}$$

30.5.1 PROPERTIES OF CONDITIONAL PROBABILITY

Following are some properties of conditional probability which are stated and proved as theorems.

THEOREM 1 *Let A and B be two events associated with sample space S, then*

$$0 \leq P(A/B) \leq 1$$

SOLUTION We know that

$$A \cap B \subset B$$

$\Rightarrow$ $$P(A \cap B) \leq P(B)$$

$\Rightarrow$ $$\frac{P(A \cap B)}{P(B)} \leq 1, \text{ if } P(B) \neq 0 \qquad [\because P(B) > 0]$$

Also, $$P(A \cap B) \geq 0 \text{ and } P(B) > 0$$

$\therefore$ $$\frac{P(A \cap B)}{P(B)} \geq 0$$

Thus, we have

$$0 \leq \frac{P(A \cap B)}{P(B)} \leq 1$$

Hence, $$0 \leq P(A/B) \leq 1$$

THEOREM 2 *If A is an event associated with the sample space S of a random experiment, then*

$$P(S/A) = P(A/A) = 1$$

PROOF We have,

$$P(S/A) = \frac{P(S \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

Also, $$P(A/A) = \frac{P(A \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

Hence, $$P(S/A) = P(A/A) = 1$$

THEOREM 3 *Let A and B be two events associated with a random experiment and S be the sample space. If C is an event such that P (C) ≠ 0, then*

$$P\left((A \cup B)/C\right) = P(A/C) + P(B/C) - P\left((A \cap B)/C\right)$$

In particular, if A and B are mutually exclusive events, then

$$P\left((A \cap B)/C\right) = P(A/C) + P(B/C)$$

PROOF We have,

$$P\left((A \cup B)/C\right) = \frac{P\{(A \cup B) \cap C\}}{P(C)}$$

$$\Rightarrow \quad P\left((A \cup B)/C\right) = \frac{P\{(A \cap C) \cup (B \cap C)\}}{P(C)}$$

$$\Rightarrow \quad P\left((A \cup B)/C\right) = \frac{P(A \cap C) + (B \cap C) - P\{A \cap C) \cap (B \cap C)\}}{P(C)}$$

$$\Rightarrow \quad P\left((A \cup B)/C\right) = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}$$

$$\Rightarrow \quad P\left((A \cup B)/C\right) = \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

$$\Rightarrow \quad P\left((A \cup B)/C\right) = P(A/C) + P(B/C) - P\left((A \cap B)/C\right)$$

If A and B are mutually exclusive events, then

$$P\left((A \cap B)/C\right) = 0$$

$$\therefore \quad P(A \cap B)/C) = P(A/C) + P(B/C)$$

THEOREM 4 *If A and B are two events associated with a random experiment, then*

$$P(\overline{A}/B) = 1 - P(A/B)$$

PROOF We know that

$$P(S/B) = 1 \qquad \text{[See Theorem 2]}$$

$$\Rightarrow \quad P\left((A \cup \overline{A})/B\right) = 1$$

$$\Rightarrow \quad P(A/B) + P(\overline{A}/B) = 1 \qquad [\because A \text{ and } \overline{A} \text{ are mutually exclusive}]$$

$$\Rightarrow \quad P(\overline{A}/B) = 1 - P(A/B)$$

Following examples will illustrate the applications of the formulas and properties for conditional probability.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If A and B are two events such that P (A) = 0.5, P (B) = 0.6 and P (A ∪ B) = 0.8, find P (A/B) and P (B/A).*

SOLUTION We have,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore \quad P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.5 + 0.6 - 0.8 = 0.3$$

Now, $P(A/B) = \dfrac{P(A \cap B)}{P(B)} \Rightarrow P(A/B) = \dfrac{0.3}{0.6} = \dfrac{1}{2}$

and, $\quad P(B/A) = \dfrac{P(A \cap B)}{P(A)} \Rightarrow P(B/A) = \dfrac{0.3}{0.5} = \dfrac{3}{5}$

EXAMPLE 2 *If A and B are two events such that $P(A) = 0.3$, $P(B) = 0.6$ and $P(B/A) = 0.5$, find $P(A/B)$ and $P(A \cup B)$.*

SOLUTION We have,

$$P(A \cap B) = P(A)\, P(B/A) = 0.3 \times 0.5 = 0.15$$

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A/B) = \frac{0.15}{0.6} = \frac{1}{4}$$

Now, $\quad P(A \cup B) = P(A$

$\Rightarrow \quad P(A \cup B) = 0.3$

EXAMPLE 3 *I[illegible] $= 0.5$, then find $P(A/B)$ and $P(A \cup B)$.*

We have,

$\therefore$ [illegible] 0.3 $\quad [\because P(A) + P(\overline{A}) = 1]$

Now,

$\Rightarrow \quad 0.5 =$ [illegible] $P(A \cap B) = 0.15$

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.7} = \frac{3}{14}$$

and, $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.7 - 0.15 = 0.85$

EXAMPLE 4 *If A and B are two events associated with a random experiment such that $P(A) = 0.8$, $P(B) = 0.5$, $P(B/A) = 0.4$, find*
(i) $P(A \cap B)$ (ii) $P(A/B)$ (iii) $P(A \cup B)$

SOLUTION (i) We have,

$$P(B/A) = 0.4$$

$$\Rightarrow \quad \frac{P(B \cap A)}{P(A)} = 0.4$$

$$\Rightarrow \quad \frac{P(A \cap B)}{0.8} = 0.4 \qquad [\because P(A) = 0.8 \text{ (given)}]$$

$$\Rightarrow \quad P(A \cap B) = 0.32$$

(ii) We have,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A/B) = \frac{0.32}{0.5} = 0.64$$

(iii) We have,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cup B) = 0.8 + 0.5 - 0.32 = 0.98$$

EXAMPLE 5 *A fair die is rolled. Consider the events $A = \{1, 3, 5\}$, $B = \{2, 3\}$ and $C = \{2, 3, 4, 5\}$. Find*
(i) $P(A/B)$ and $P(B/A)$ (ii) $P(A/C)$ and $P(C/A)$
(iii) $P(A \cup B/C)$ and $P(A \cap B/C)$

SOLUTION We have,

$$S = \{1, 2, 3, 4, 5, 6\},\ A = \{1, 3, 5\}, B = \{2, 3\} \text{ and } C = \{2, 3, 4, 5\}$$

$$\Rightarrow \quad n(S) = 6, n(A) = 3, n(B) = 2 \text{ and } n(C) = 4$$

$$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{2}{6} = \frac{1}{3}, P(C) = \frac{4}{6} = \frac{2}{3}, P(A \cap B) = \frac{1}{6},$$

$$P(A \cap C) = \frac{2}{6} = \frac{1}{3}, P(B \cap C) = \frac{2}{6} = \frac{1}{3}, P(A \cap B \cap C) = \frac{1}{6} \text{ and } P(A \cup B) = \frac{4}{6} = \frac{2}{3}$$

(i) $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2} \text{ and, } P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

(ii) $$P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \text{ and, } P(C/A) = \frac{P(A \cap C)}{P(A)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

(iii) We have,

$$P(A \cup B/C) = \frac{P((A \cup B) \cap C)}{P(C)}$$

$$\Rightarrow \quad P(A \cup B/C) = \frac{P((A \cap C) \cup (B \cap C))}{P(C)}$$

$$\Rightarrow \quad P(A \cup B/C) = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)} = \frac{\frac{1}{3} + \frac{1}{3} - \frac{1}{6}}{\frac{2}{3}} = \frac{3}{4}$$

and, $$P(A \cap B/C) = \frac{P\left((A \cap B) \cap C\right)}{P(C)} = \frac{P(A \cap B \cap C)}{P(C)} = \frac{\frac{1}{6}}{\frac{2}{3}} = \frac{1}{4}$$

EXAMPLE 6 *If $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$, find $P(\overline{A}/\overline{B})$ and $P(\overline{B}/\overline{A})$.*

SOLUTION We have,

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$$

$$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$$

$$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = 1 - \{P(A) + P(B) - P(A \cap B)\}$$

$$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = 1 - \left\{\frac{3}{8} + \frac{1}{2} - \frac{1}{4}\right\} = \frac{3}{8}$$

$$P(\overline{A}) = 1 - P(A) = \frac{5}{8} \text{ and } P(\overline{B}) = 1 - P(B) = \frac{1}{2}$$

$$\therefore \quad P(\overline{A}/\overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{3/8}{1/2} = \frac{3}{4} \text{ and } P(\overline{B}/\overline{A}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{A})} = \frac{3/8}{5/8} = \frac{3}{5}$$

EXAMPLE 7 *A die is rolled twice and the sum of the numbers appearing on them is observed to be 7. What is the conditional probability that the number 2 has appeared at least once?*

SOLUTION Consider the following events :

A = Getting number 2 at least once;

B = Getting 7 as the sum of the numbers on two dice.

We have,

$$A = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (1, 2), (3, 2), (4, 2), (5, 2) (6, 2)\}$$

and,

$$B = \{(2, 5), (5, 2), (6, 1), (1, 6), (3, 4), (4\ 3)\}$$

$$\therefore \quad P(A) = \frac{11}{36}, \ P(B) = \frac{6}{36} \text{ and } P(A \cap B) = \frac{2}{36}$$

So, $\quad$ Required probability $= P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{2/36}{6/36} = \dfrac{1}{3}$

EXAMPLE 8 *A black and a red die are rolled.*

(i) *Find the conditional probablity of obtaining a sum greater than 9, given that the black die resulted in a 5.*

(ii) *Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.*

SOLUTION Consider the following events:

A = Getting a sum greater than 9

B = Getting 5 on black die

C = Getting 8 as the sum

D = Getting a number less than 4 on red die.

We have,

$$A = \{(4, 6), (6, 4), (5, 5), (6, 5), (5, 6), (6, 6)\}$$

$$B = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$$

$$C = \{(2, 6), (6, 2), (4, 4), (3, 5), (5, 3)\}$$

and, $\quad D = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (1, 2), (2, 2), (3, 2)$
$\quad (4, 2), (5, 2), (6, 2), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3)\}$

$$\therefore \quad P(A) = \frac{6}{36} = \frac{1}{6}, P(B) = \frac{6}{36} = \frac{1}{6}, P(C) = \frac{5}{36}, P(D) = \frac{18}{36} = \frac{1}{2}$$

$$P(A \cap B) = \frac{2}{36} = \frac{1}{18}, P(C \cap D) = \frac{2}{36} = \frac{1}{18}$$

(i) $\quad$ Required probability $= P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{18}}{\frac{1}{6}} = \dfrac{1}{3}$

(ii) $\quad$ Required probability $= P(C/D) = \dfrac{P(C \cap D)}{P(D)} = \dfrac{\frac{1}{8}}{\frac{1}{2}} = \dfrac{1}{9}$

EXAMPLE 9 *Two integers are selected at random from integers 1 through 11. If the sum is even, find the probability that both the numbers are odd.*

SOLUTION Out of integers from 1 to 11, there are 5 even integers and 6 odd integers. Consider the following events :

A = Both the numbers chosen are odd

B = The sum of the numbers chosen is even.

Since the sum of two integers is even if either both are even or both are odd.

$$\therefore \quad P(A) = \frac{^6C_2}{^{11}C_2},\ P(B) = \frac{^6C_2 + {^5C_2}}{^{11}C_2} \text{ and } P(A \cap B) = \frac{^6C_2}{^{11}C_2}$$

Now,

Required probability $= P(A/B)$

$$\Rightarrow \quad \text{Required probabilit} = \frac{P(A \cap B)}{P(B)} = \frac{^6C_2/{^{11}C_2}}{\frac{^6C_2 + {^5C_2}}{^{11}C_2}} = \frac{^6C_2}{^6C_2 + {^5C_2}} = \frac{15}{15+10} = \frac{3}{5}$$

EXAMPLE 10 *A couple has 2 children. Find the probability that both are boys, if it is known that (i) one of the children is a boy; (ii) the older child is a boy.* [CBSE 2010]

SOLUTION Let B_i and G_i stand for ith child be a boy and girl respectively. Then the sample space can be expressed as

$S = \{B_1 B_2, B_1 G_2, G_1 B_2, G_1 G_2\}$

Consider the following events :

A = Both the children are boys; B = One of the children is a boy;

C = The older child is a boy.

Then, $A = B_1 B_2$, $B = B_1 G_2, B_1 B_2, G_1 B_2$ and $C = B_1 B_2, B_1 G_2$

So, $A \cap B = \{B_1 B_2\}$ and $A \cap C = \{B_1 B_2\}$

(i) Required probability $= P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/4}{3/4} = \dfrac{1}{3}$

(ii) Required probability $= P(A/C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{1/4}{2/4} = \dfrac{1}{2}$

EXAMPLE 11 *Consider a random experiment in which a coin is tossed and if the coin shows head it is tossed again but if it shows a tail then a die is tossed. If 8 possible outcomes are equally likely, find the probability that the die shows a number greater than 4 if it is known that the first throw of the coin results in a tail.*

SOLUTION The sample space S associated with the given random experiment is

$S = \{(H, H), (H, T), (T, 1), (T, 2), (T, 3) (T, 4), (T, 5), (T, 6)\}$.

Let A be the event that the die shows a number greater than 4 and B be the event that the first throw of the coin results in a tail. Then,

$A = \{(T, 5) (T, 6)\}$ and $B = \{\{(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

$\therefore$ Required probability $= P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{n(A \cap B)}{n(B)} = \dfrac{2}{6} = \dfrac{1}{3}$

EXAMPLE 12 *A coin is tossed twice and the four possible outcomes are assumed to be equally likely. If A is the event, 'both head and tail have appeared', and B be the event, 'at most one tail is observed', find P (A), P (B), P (A/B) and P (B/A).*

SOLUTION Here, $S = \{HH, HT, TH, TT\}$, $A = \{HT, TH\}$ and $B = \{HH, HT, TH\}$.

$\therefore \quad A \cap B = \{HT, TH\}$.

Now, $\quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{2}{4} = \dfrac{1}{2}, P(B) = \dfrac{n(B)}{n(S)} = \dfrac{3}{4}$

and, $\quad P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{2}{4} = \dfrac{1}{2}$

$\therefore \quad P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/2}{3/4} = \dfrac{2}{3}$ and $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/2}{1/2} = 1.$

EXAMPLE 13 *A bag contains 3 red and 4 black balls and another bag has 4 red and 2 black balls. One bag is selected at random and from the selected bag a ball is drawn. Let A be the event that the first bag is selected, B be the event that the second bag is selected and C be the event that the ball drawn is red. Find P(A), P(B), P (C/A) and P (C/B).*

SOLUTION There are two bags. Therefore,

$$P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{2}$$

Now, $\quad P(C/A) =$ Probability of drawing a red ball when first is selected

$\Rightarrow \quad P(C/A) =$ Probability of drawing a red ball from first bag $= \dfrac{3}{7}$

and, $\quad P(C/B) =$ Probability of drawing a red ball from second bag $= \dfrac{4}{6} = \dfrac{2}{3}$.

EXAMPLE 14 *A coin is tossed, then a die is thrown. Find the probability of obtaining a '6' given that head came up.*

SOLUTION The sample space S associated to the given random experiment is given by

$$S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$$

Consider the following events :

A = Getting head on the coin, $\quad B$ = Getting 6 on the dice.

We have,

$$A = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)\} \text{ and } B = \{(H,6), (T,6)\}$$

$\therefore \quad$ Required probability $= P(B/A)$

$\Rightarrow \quad$ Required probability $= \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/12}{6/12} = \dfrac{1}{6}$

EXAMPLE 15 *Two coins are tossed. What is the probability of coming up two heads if it is known that at least one head comes up.*

SOLUTION Consider the following events :

A = Getting at least one head, B = Getting two heads.

We have,

$$A = \{HT, TH, HH\} \text{ and } B = \{HH\}$$

$\therefore \quad$ Required probability $= P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/4}{3/4} = \dfrac{1}{3}$

EXAMPLE 16 *An instructor has a test bank consisting of 300 easy True/False questions, 200 difficult True/False questions, 500 easy multiple choice questions (MCQ) and 400 difficult multiple choice questions. If a questions is selected at random from the test bank, what is the probability that it will be an easy question given that it is a multiple choice question.*

SOLUTION Consider the following events:

$E =$ The question selected is an easy question

$D =$ The question selected is a difficult question

$T =$ The question selected is a True/False question

$M =$ The question selected is a multiple choice question.

We have,

Total number of questions $= 300 + 200 + 500 + 400 = 1400$

$\therefore \quad P(E) = \frac{800}{1400} = \frac{4}{7}, P(D) = \frac{600}{1400} = \frac{3}{7}, P(T) = \frac{500}{1400} = \frac{5}{14},$

$P(M) = \frac{900}{1400} = \frac{9}{14}$ and $P(E \cap M) = \frac{500}{1400} = \frac{5}{14}$

$\therefore \quad$ Required probability $= P(E/M) = \frac{P(E \cap M)}{P(M)} = \frac{\frac{5}{14}}{\frac{9}{14}} = \frac{5}{9}$

EXAMPLE 17 *Consider the experiment of tossing a coin. If the coin shows head toss it again but if it shows tail then throw a die. Find the conditional probability of the event 'the die shows a number greater than 4, given that 'there is at least one tail'.* **[NCERT]**

SOLUTION The outcomes of the experiment can be represented in the following tree diagram.

The sample space S of the experiment is given as

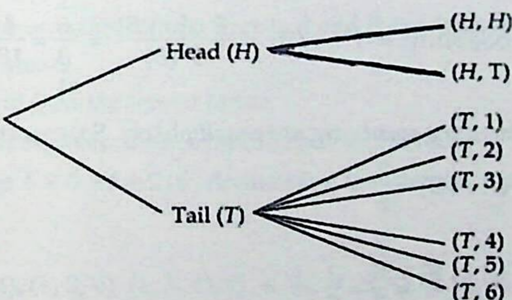

Fig. 30.2

$S = \{(H, H), (H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

The probabilities of these elementary events are

$P\{(H, H)\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, P\{(H, T)\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}, P\{(T, 1)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12},$

$P\{(T, 2\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}, P\{(T, 3)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}, P\{(T, 4\} = \frac{1}{2} \times \frac{1}{6} = 12,$

$P\{(T, 5)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$ and, $P\{(T, 6)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}.$

Consider the following events:

$A =$ The die shows a number greater than 4

$B =$ There is at least one tail.

We have,

$$A = \{(T, 5), (T, 6)\}, B = \{(H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$$

and, $A \cap B = \{(T, 5)\ (T, 6)\}$

$$\therefore \quad P(B) = P\{(H, T)\} + P\{T, 1)\} + P\{(T, 2)\} + P\{(T, 3)\} + P\{(T, 4)\} + P\{(T, 5)\} + P\{(T, 6)\}$$

$$\Rightarrow \quad P(B) = \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{4}$$ [See Fig. 30.3]

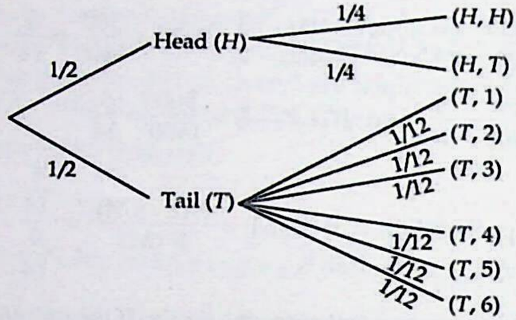

Fig. 30.3

and, $$P(A \cap B) = P\{(T, 5)\} + P\{(T, 6)\} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

$\therefore$ Required probability $= P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{6}}{\frac{3}{4}} = \dfrac{4}{18} = \dfrac{2}{9}$

REMARK *Here, the elementary events are not equally likely. So, we cannot say that*

$$P(B) = \frac{7}{8}, P(A \cap B) = \frac{2}{8}$$

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{8}}{\frac{7}{8}} = \frac{2}{7}$$

EXAMPLE 18 *Consider the experiment of throwing a die, if a multiple of 3 comes up throw the die again and if any other number comes toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 2'.*

SOLUTION The sample space of the experiment is given by

$$S = \{(3, 1), (3, 2)\ (3, 3), (3, 4), (3, 5), (3, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)$$
$$(1, H), (1, T), (2, H), (2, T), (4, H), (4, T), (5, H), (5, T)\}$$

The probabilities of the elementary events are:

$$P\{(3, 1)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}, P\{(3, 2)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}, P\{(3, 3)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},$$

$$P\{(3, 4)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}, P\{(3, 5)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}, P\{(3, 6)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},$$

$$P\{(6,1)\}=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36},\ P\{(6,2)\}=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36},\ P\{(6,3)\}=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36},$$

$$P\{(6,4)\}=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36},\ P\{(6,5)\}=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36},\ P\{(6,6)\}=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36},$$

$$P\{(1,H)\}=\frac{1}{6}\times\frac{1}{2}=\frac{1}{12},\ P\{(1,T)\}=\frac{1}{6}\times\frac{1}{2}=\frac{1}{12},\ P\{(2,H)\}=\frac{1}{6}\times\frac{1}{2}=\frac{1}{12},$$

$$P\{(2,T)\}=\frac{1}{6}\times\frac{1}{2}=\frac{1}{12},\ P\{(4,H)\}=\frac{1}{6}\times\frac{1}{2}=\frac{1}{12},\ P\{(4,T)\}=\frac{1}{6}\times\frac{1}{2}=\frac{1}{12},$$

$$P\{(5,H)\}=\frac{1}{6}\times\frac{1}{2}=\frac{1}{12},\ P\{(5,T)\}=\frac{1}{6}\times\frac{1}{2}=\frac{1}{12}$$

Clearly, the elementary events are not equally likely.

Consider the following events:

A = The coin shows a tail

B = At least one die shows a 2.

We have,

$$A=\{(1,T),(2,T),(4,T),(5,T)\},\ B=\{(3,2),(6,2),(2,H),(2,T)\}$$

and, $A\cap B=\{(2,T)\}$

$\therefore$ $P(B)=P\{(3,2)\}+P\{(6,2)\}+P\{(2,H)\}+P\{(2,T)\}=\frac{1}{36}+\frac{1}{36}+\frac{1}{12}+\frac{1}{12}=\frac{2}{9}$

and, $P(A\cap B)=P\{(2,T)\}=\frac{1}{12}=\frac{1}{12}$

$\therefore$ Required probability $=P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{12}}{\frac{2}{9}}=\frac{9}{24}=\frac{3}{8}$

EXAMPLE 19 *A die is thrown three times. Events A and B are defined as follows:*

A : *4 on the third throw*

B : *6 on the first* and *5 on the second throw.*

Find the probability of A given that B has already occurred.

SOLUTION There are $6\times6\times6=216$ elementary events associated with the random experiment.

We have,

$A=\{(1,1,4),(1,2,4),(1,3,4),(1,4,4),(1,5,4),(1,6,4),(2,1,4),(2,2,4),(2,3,4),$
$(2,4,4),(2,5,4),(2,6,4),(3,1,4),(3,2,4),(3,3,4),(3,4,4),(3,5,4),(3,6,4)$
$(4,1,4),(4,2,4),(4,3,4),(4,4,4),(4,5,4),(4,6,4),(5,1,4),(5,2,4),(5,3,4),$
$(5,4,4),(5,5,4),(5,6,4),(6,1,4),(6,2,4),(6,3,4),(6,4,4),(6,5,4),(6,6,4)\}$

$B=\{(6,5,1),(6,5,2),(6,5,3),(6,5,4),(6,5,5),(6,5,6)\}$

and, $A\cap B=\{(6,5,4)\}$

$\therefore$ $P(A\cap B)=\frac{1}{216}$ and $P(B)=\frac{6}{216}$

Now, $P(A/B)=\frac{P(A\cap B)}{P(B)}$

$\Rightarrow$ $P(A/B)=\frac{\frac{1}{216}}{\frac{6}{216}}=\frac{1}{6}$

EXAMPLE 20 *In a hostel 60% of the students read Hindi newspaper, 40% read English newspaper and 20 % read both Hindi and English newspapers. A student is selected at random.*

(i) *Find the probability that she reads neither Hindi nor English news papers.*

(ii) *If she reads Hindi newspaper, find the probability that she reads English newspaper.*

(iii) *If she reads English newspaper, find the probability that she reads Hindi newspaper.*

SOLUTION Consider the following events :

H = Student reads Hindi newspaper.

E = Student reads English newspaper.

We have,

$$P(H) = \frac{60}{100} = \frac{3}{5}, P(E) = \frac{40}{100} = \frac{2}{5} \text{ and } P(H \cap E) = \frac{20}{100} = \frac{1}{5}$$

(i) Required probability $= P(\overline{H} \cap \overline{E})$

$$= P(\overline{H \cup E})$$

$$= 1 - P(H \cup E)$$

$$= 1 - \{P(H) + P(E) - P(H \cap E)\}$$

$$= 1 - \left\{\frac{3}{5} + \frac{2}{5} - \frac{1}{5}\right\} = 1 - \frac{4}{5} = \frac{1}{5}$$

(ii) Required probability $= P(E/H) = \dfrac{P(H \cap E)}{P(H)} = \dfrac{\frac{1}{5}}{\frac{3}{5}} = \dfrac{1}{3}$

(iii) Required probability $= P(H/E) = \dfrac{P(H \cap E)}{P(E)} = \dfrac{\frac{1}{5}}{\frac{2}{5}} = \dfrac{1}{2}$

EXAMPLE 21 *An electronic assembly consists of two sub-systems say A and B. From previous testing procedures, the following probabilities are assumed to be known.*

$$P(A \text{ fails}) = 0.2, P(B \text{ fails alone}) = 0.15, P(A \text{ and } B \text{ fail}) = 0.15.$$

Evaluate the following probabilities:

(i) $P(A \text{ fails}/B \text{ has failed})$ (ii) $P(A \text{ fails alone})$

SOLUTION Consider the following events:

$E = A$ fails, $F = B$ fails.

We have,

$P(A \text{ fails}) = 0.2 \Rightarrow P(E) = 0.2$

$P(A \text{ and } B \text{ fails}) = 0.15 \Rightarrow P(E \cap F) = 0.15$

$P(B \text{ fails alone}) = 0.15$

$\Rightarrow$ $P(\overline{E} \cap F) = 0.15$

$\Rightarrow$ $P(F) - P(E \cap F) = 0.15$

$\Rightarrow$ $P(F) = P(E \cap F) + 0.15$

$\Rightarrow$ $P(F) = 0.15 + 0.15 = 0.30$

(i) $P(A \text{ fails}/B \text{ has failed}) = P(E/F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{0.15}{0.30} = \dfrac{1}{2}$

(ii) $P(A \text{ fails alone}) = P(E \cap \overline{F}) = P(E) - P(E \cap F) = 0.2 - 0.15 = 0.05$

EXAMPLE 22 *Three distinguishable balls are distributed in three cells. Find the conditional probability that all the three occupy the same cell, given that at least two of them are in the same cell.*

SOLUTION Since each ball can be place in a cell in three ways. Therefore, three distinct balls can be placed in three cells in $3 \times 3 \times 3 = 27$ ways.

Consider the following events:

$E =$ All balls are in the same cell.

$F =$ At least two balls are in the same cell.

All balls can be placed in the same cell in three ways.

$\therefore \quad P(E) = \frac{3}{27}$

$P(F) = 1 - P$ (Balls are placed in distinct cells)

$\Rightarrow \quad P(F) = 1 - \frac{3!}{27} = 1 - \frac{6}{27} = \frac{21}{27}$

Clearly, $E \subset F$

$\therefore \quad E \cap F = E$

$\Rightarrow \quad P(E \cap F) = P(E) = \frac{3}{27}$

$$\text{Required probability } = P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{3}{27}}{\frac{21}{27}} = \frac{1}{7}$$

EXERCISE 30.3

1. If $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, find $P(A/B)$.
2. If A and B are events such that $P(A) = 0.6$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$, find $P(A/B)$ and $P(B/A)$.
3. If A and B are two events such that $P(A \cap B) = 0.32$ and $P(B) = 0.5$, find $P(A/B)$.
4. If $P(A) = 0.4$, $P(B) = 0.8$, $P(B/A) = 0.6$. Find $P(A/B)$ and $P(A \cup B)$.
5. If A and B are two events such that
   (i) $P(A) = 1/3$, $P(B) = 1/4$ and $P(A \cup B) = 5/12$, find $P(A/B)$ and $P(B/A)$.
   (ii) $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find $P(A \cap B)$, $P(A/B)$, $P(B/A)$
   (iii) $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, find $P(A'/B)$.
6. If A and B are two events such that $2P(A) = P(B) = \frac{5}{13}$ and $P(A/B) = \frac{2}{5}$, find $P(A \cup B)$.
7. If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find
   (i) $P(A \cap B)$ (ii) $P(A/B)$ (iii) $P(B/A)$
8. A coin is tossed three times. Find $P(A/B)$ in each of the following:
   (i) $A =$ Heads on third toss, $B =$ Heads on first two tosses
   (ii) $A =$ At least two heads, $B =$ At most two heads
   (iii) $A =$ At most two tails, $B =$ At least one tail.

9. Two coins are tossed once. Find $P(A/B)$ in each of the following:
   (i) A = Tail appears on one coin, B = One coin shows head.
   (ii) A = No tail appears, B = No head appears.

10. A die is thrown three times. Find $P(A/B)$ and $P(B/A)$, if

    A = 4 appears on the third toss

    B = 6 and 5 appear respectively on first two tosses.

11. Mother, father and son line up at random for a family picture. If A and B are two events given by

    A = Son on one end, B = Father in the middle, find $P(A/B)$ and $P(B/A)$.

12. A dice is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?
13. Two dice are thrown. Find the probability that the numbers appeared has the sum 8, if it is known that the second die always exhibits 4.
14. A pair of dice is thrown. Find the probability of getting 7 as the sum, if it is known that the second die always exhibits an odd number.
15. A pair of dice is thrown. Find the probability of getting 7 as the sum if it is known that the second die always exhibits a prime number.
16. A die is rolled. If the outcome is an odd number, what is the probability that it is prime?
17. A pair of dice is thrown. Find the probability of getting the sum 8 or more, if 4 appears on the first die.
18. Find the probability that the sum of the numbers showing on two dice is 8, given that at least one die does not show five.
19. Two numbers are selected at random from integers 1 through 9. If the sum is even, find the probability that both the numbers are odd.
20. A die is thrown twice and the sum of the numbers appearing is observed to be 8. What is the conditional probability that the number 5 has appeared at least once? **[CBSE 2003]**
21. Two dice are thrown and it is known that the first die shows a 6. Find the probability that the sum of the numbers showing on the dice is 7.
22. A pair of dice is thrown. Let E be the event that the sum is greater than or equal to 10 and F be the event "5 appears on the first-die". Find $P(E/F)$. If F is the event "5 appears on at least one die", find $P(E/F)$.
23. The probability that a student selected at random from a class will pass in Mathematics is $\frac{4}{5}$, and the probability that he/she passes in Mathematics and Computer Science is $\frac{1}{2}$. What is the probability that he/she will pass in Computer Science if it is known that he/she has passed in Mathematics?
24. The probability that a certain person will buy a shirt is 0.2, the probability that he wili buy a trouser is 0.3, and the probability that he will buy a shirt given that he buys a trouser is 0.4. Find the probability that he will buy both a shirt and a trouser. Find also the probability that he will buy a trouser given that he buys a shirt.

25. In a school there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in class XII given that the chosen student is a girl?

26. Ten cards numbered 1 through 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?

ANSWERS

1. $\frac{4}{9}$ 2. $\frac{2}{3}, \frac{1}{3}$ 3. 0.64 4. 0.3, 0.96

5. (i) $\frac{2}{3}, \frac{1}{2}$ (ii) $\frac{4}{11}, \frac{4}{5}, \frac{2}{3}$ (iii) $\frac{5}{9}$ 6. $\frac{11}{26}$

7. (i) $\frac{4}{11}$ (ii) $\frac{4}{5}$ (iii) $\frac{2}{3}$

8. (i) $\frac{1}{2}$ (ii) $\frac{3}{7}$ (iii) $\frac{6}{7}$ 9. (i) 1 (ii) 0

10. $\frac{1}{6}, \frac{1}{36}$ 11. $1, \frac{1}{2}$ 12. $\frac{2}{5}$ 13. $\frac{1}{6}$ 14. $\frac{1}{6}$ 15. $\frac{1}{6}$ 16. $\frac{2}{3}$ 17. $\frac{1}{2}$

18. $\frac{3}{25}$ 19. $\frac{5}{8}$ 20. $\frac{2}{5}$ 21. $\frac{1}{6}$ 22. $\frac{1}{3}, \frac{3}{11}$ 23. $\frac{5}{8}$ 24. 0.12, 0.6

26. $\frac{4}{7}$

HINTS TO SELECTED PROBLEMS

6. We have, $P(A \cap B) = P(B)\, P(A/B) = \frac{5}{13} \times \frac{2}{5} = \frac{2}{13}$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{11}{26}$$

7. (i) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11}$

(ii) $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{4}{11}}{\frac{5}{11}} = \frac{4}{5}$$

(iii) $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{4}{11}}{\frac{6}{11}} = \frac{2}{3}$$

8. (i) We have,

$A = \{HHH, HTH, THH, TTH\}$; $B = \{HHH, HHT\}$

$\therefore \quad P(A/B) = \frac{1}{2}$

(ii) We have,

$A = \{HHH, HTH, THH, HHT\}$,

$B = \{TTT, TTH, HTT, THT, HHT, THH, HTH\}$

$\therefore \quad P(A/B) = \frac{3}{7}$

(iii) $A = \{HHH, HTH, THH, HHT, THT, HTT, TTH\}$

$B = \{THH, HTH, HHT, TTH, THT, HTT, TTT\}$

$\therefore \quad P(A/B) = \frac{6}{7}$

9. (i) We have,

$A = \{TH, HT\}, B = \{HT, TH\} \quad \therefore \quad P(A/B) = 1$

(ii) We have,

$A = \{HH\}, B = \{TT\} \quad \therefore \quad P(A/B) = 0$

10. We have,

$A = \{(1,1,4), (1,2,4), (1,3,4), (1,4,4), (1,5,4), (1,6,4), (2,1,4), (2,2,4), (2,3,4),$
$(2,4,4), (2,5,4), (2,6,4), (3,1,4), (3,2,4), (3,3,4), (3,4,4), (3,5,4), (3,6,4),$
$(4,1,4), (4,2,4), (4,3,4), (4,4,4), (4,5,4), (4,6,4), (5,1,4), (5,2,4), (5,3,4),$
$(5,4,4), (5,5,4), (5,6,4), (6,1,4), (6,2,4), (6,3,4), (6,4,4), (6,5,4), (6,6,4)\}$

$B = \{(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5), (6,5,6)\}$

$P(A/B) = \frac{1}{6}, P(B/A) = \frac{1}{36}.$

11. The sample space S is given by

$S = \{MFS, MSF, FSM, FMS, SMF, SFM\}$

We have,

$A = \{MFS, FMS\ SMF, SFM\}, B = \{MFS, SFM\}$

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{6}}{\frac{2}{6}} = 1 \quad P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{6}}{\frac{4}{6}} = \frac{1}{2}$$

19. Let A = Getting two odd numbers, B = Getting the sum as an even number.

Required probability

$$= P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{{}^5C_2/{}^9C_2}{({}^4C_2 + {}^5C_2)/{}^9C_2} = \frac{{}^5C_2}{{}^4C_2 + {}^5C_2} = \frac{10}{16}$$

25. A = Student chosen randomly studies in class XII

B = Randomly chosen student is a girl.

$P(B) = \frac{430}{1000}$ and $P(A \cap B) = \frac{43}{1000}$

$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = 0.1$

30.6 INDEPENDENT EVENTS

DEFINITION *Events are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence or non-occurrence of the other.*

Suppose a bag contains 6 white and 3 red balls. Two balls are drawn from the bag one after the other. Consider the events

A = drawing a white ball in first draw;

B = drawing a red ball in second draw.

If the ball drawn in the first draw is not replaced back in the bag, then events A and B are dependent events because $P(B)$ is increased or decreased according as the first draw results as a white or a red ball. If the ball drawn in first draw is replaced back in the bag, then A and B are independent events because $P(B)$ remains same whether we get a white ball or a red ball in first draw i.e. $P(B) = P(B/A)$ and $P(B) = P(B/\overline{A})$.

It is evident from the above discussion that if A and B are two independent events associated with a random experiment, then

$$P(A/B) = P(A) \text{ and } P(B/A) = P(B)$$

and vice-versa.

THEOREM 1 *If A and B are independent events associated with a random experiment, then*

$$P(A \cap B) = P(A)\, P(B)$$

i.e., the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities.

PROOF By multiplication theorem, we have

$$P(A \cap B) = P(A)\, P(B/A).$$

Since A and B are independent events, therefore $P(B/A) = P(B)$.

Hence, $P(A \cap B) = P(A)\, P(B)$.

Q.E.D.

THEOREM 2 *If $A_1, A_2, \ldots, A_n$ are independent events associated with a random experiment, then*

$$P(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) = P(A_1)\, P(A_2) \ldots P(A_n)$$

PROOF By multiplication theorem, we have

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

Since $A_1, A_2, \ldots, A_{n-1}, A_n$ are independent events. Therefore,

$P(A_2/A_1) = P(A_2)$, $P(A_3/A_1 \cap A_2) = P(A_3), \ldots, P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1}) = P(A_n)$

Hence, $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2) \ldots P(A_n)$.

Q.E.D.

PAIRWISE INDEPENDENT EVENTS *Let $A_1, A_2, \ldots, A_n$ be n events associated to a random experiment. These events are said to be pairwise independent if*

$$P(A_i \cap A_j) = P(A_i)\, P(A_j) \text{ for } i \neq j;\ i, j = 1, 2, \ldots, n$$

MUTUALLY INDEPENDENT EVENTS *Let $A_1, A_2, \ldots, A_n$ be n events associated to a random experiment. These events are said to be mutually independent if the probability of the simultaneous occurrence of any finite number of them is equal to the product of their separate probabilities i.e.*

$$P(A_i \cap A_j) = P(A_i)\, P(A_j), \text{ for } i \neq j;\ i, j = 1, 2, \ldots, n$$

$$P(A_i \cap A_j \cap A_k) = P(A_i)\, P(A_j)\, P(A_k), \quad \text{for } i \neq j \neq k;\ i, j, k = 1, 2, \ldots, n$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$P(A_1 \cap A_2 \ldots \cap A_n) = P(A_1)\, P(A_2) \ldots P(A_n)$$

REMARK 1 *If $A_1, A_2 \dots A_n$ are pairwise independent events, then the total number of conditions for their pairwise independence is ${}^nC_2$ whereas for their mutual independences there must be ${}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^n - n - 1$ condition.*

REMARK 2 *It follows from the above definitions that mutually independent events are always pairwise independent but the converse need not be true as illustrated below:*

ILLUSTRATION 1 *A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as*

A : "the first bulb is defective",

B : "the second bulb is non-defective",

C : "the two bulbs are both defective or both non-defective."

Determine whether (i) A, B, C are pairwise independent, (ii) A, B, C are mutually independent.

SOLUTION We have,

$$P(A) = \frac{1}{2}, P(B) = \frac{1}{2}, P(C) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

$P(A \cap B) =$ Probability that the first is defective and the second is non-defective

$$\Rightarrow \quad P(A \cap B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A) \cdot P(B)$$

$P(B \cap C) =$ Probability that both the bulbs are non-defective

$$\Rightarrow \quad P(B \cap C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(B)\,P(C)$$

and, $P(A \cap C) =$ Probability that both the bulbs are defective

$$\Rightarrow \quad P(A \cap C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A)\,P(C)$$

Hence, A, B, C are pairwise independent.

Now, $P(A \cap B \cap C) =$ Probability that the first bulb is defective and the second is non-defective and the first and second are both defective or both non-defective

$$\Rightarrow \quad P(A \cap B \cap C) = 0$$

and, $P(A) \cdot P(B) \cdot P(C) = \dfrac{1}{8}$

Clearly, $P(A \cap B \cap C) \neq P(A)\,P(B)\,P(C)$

Thus, A, B, C are not mutually independent.

REMARK 3 *In case of two events only associated to a random experiment, there is no distinction between their mutual independence and pairwise independence.*

THEOREM 3 *If A and B are independent events associated with a random experiment, then prove that*

(i) $\overline{A}$ and B are independent events

(ii) A and $\overline{B}$ are independent events

(iii) $\overline{A}$ and $\overline{B}$ are also independent events. **[NCERT]**

SOLUTION Since A and B are independent events. Therefore,

$$P(A \cap B) = P(A)\,P(B) \quad \text{...(i)}$$

(i) It is evident from the Venn-diagram that $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive events such that

$$(A \cap B) \cup (\overline{A} \cap B) = B$$

Therefore, by addition theorem on probability, we have

$$P(A \cap B) + P(\overline{A} \cap B) = P(B)$$

$$\Rightarrow \quad P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(\overline{A} \cap B) = P(B) - P(A)\,P(B) \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad P(\overline{A} \cap B) = P(B)\,[1 - P(A)]$$

$$\Rightarrow \quad P(\overline{A} \cap B) = P(B)\,P(\overline{A})$$

$$\Rightarrow \quad P(\overline{A} \cap B) = P(\overline{A})\,P(B)$$

Thus, $\quad P(\overline{A} \cap B) = P(\overline{A})\,P(B).$

Hence, $\overline{A}$ and B are independent events.

(ii) It is clear from the Venn-diagram that $A \cap \overline{B}$ and $A \cap B$ are mutually exclusive events such that

$$(A \cap \overline{B}) \cup (A \cap B) = A$$

So, by addition theorem on probability, we have

$$P(A \cap \overline{B}) + P(A \cap B) = P(A)$$

$$\Rightarrow \quad P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap \overline{B}) = P(A) - P(A)\,P(B) \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad P(A \cap \overline{B}) = P(A)\,[1 - P(B)] = P(A)\,P(\overline{B})$$

Thus, $\quad P(A \cap \overline{B}) = P(A)\,P(\overline{B}).$

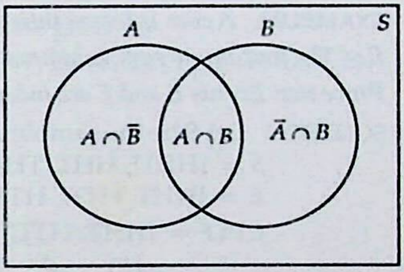

Fig. 30.4

Hence, A and $\overline{B}$ are independent events.

(iii) We have to show that $\overline{A}$ and $\overline{B}$ are independent events if A and B are independent events. For this it is sufficient to show that

$$P(\overline{A} \cap \overline{B}) = P(\overline{A})\,P(\overline{B})$$

We have,

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$$

$$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$$

$$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = 1 - [P(A) + P(B) - P(A \cap B)] \qquad \text{[By Add. Theorem]}$$

$$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = 1 - [P(A) + P(B) - P(A)\,P(B)] \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = (1 - P(A)) - P(B)\,(1 - P(A))$$

$$\Rightarrow \quad P(\overline{A} \cap \overline{B}) = (1 - P(A))\,(1 - P(B)) = P(\overline{A})\,P(\overline{B})$$

Hence, $\overline{A}$ and $\overline{B}$ are independent events.

<u>REMARK 4</u> *In what follows the term independent events will mean mutually independent events.*

<u>REMARK 5</u> *If A and B are independent events associated to a random experiment, then*

Probability of occurrence of at least one $= P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$
$$= P(A) + P(B) - P(A)\,P(B)$$
$$= 1 - [1 - P(A) - P(B) + P(A) + P(B)]$$
$$= 1 - (1 - P(A))(1 - P(B)) = 1 - P(\overline{A})\,P(\overline{B})$$

REMARK 6 *If $A_1, A_2, \ldots A_n$ are independent events associated with a random experiment, then Probability of occurrence of at least one*

$$= P(A_1 \cup A_2 \cup \ldots \cup A_n)$$
$$= 1 - P(\overline{A_1 \cup A_2 \cup \ldots \cup A_n})$$
$$= 1 - P(\overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n) = 1 - P(\overline{A_1})\,P(\overline{A_2}) \ldots P(\overline{A_n})$$

INDEPENDENT EXPERIMENTS *Two random experiments are independent if for every pair of events A and B where A is associated with the first experiment and B with the second experiment, we have*

$$P(A \cap B) = P(A)\,P(B)$$

ILLUSTRATIVE EXAMPLES

Type I PROBLEMS ON PROVING THE INDEPENDENCE OR DEPENDENCE OF EVENTS

EXAMPLE 1 *A coin is tossed thrice and all eight outcomes are equally likely.*
E : "The first throw results in head" F : "The last throw results in tail"
Prove that Events E and F are independent.

SOLUTION Let S be the sample space associated with the given experiment. Then,

$$S = \{HHH, HHT, THH, HTH, TTH, HTT, THT, TTT\}$$
$$E = \{HHT, HTH, HTT, HHH\},\ F = \{HHT, HTT, THT, TTT\}$$
$$E \cap F = \{HHT, HTT\}$$

$\therefore$ $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{4}{8} = \dfrac{1}{2}$, $P(F) = \dfrac{4}{8} = \dfrac{1}{2}$ and $P(E \cap F) = \dfrac{2}{8} = \dfrac{1}{4}$.

Clearly, $P(E \cap F) = \dfrac{1}{4} = P(E)\,P(F)$.

Hence, E and F are independent events.

EXAMPLE 2 *An unbiased die is thrown twice. Let the event A be 'odd number on the first throw' and B the event 'odd number on the second throw'. Check the independence of events A and B.* **[NCERT]**

SOLUTION We have,

Total number of elementary events = 36

An odd number on the first throw means an odd number on first throw and any number on second throw. Therefore, favourable number of elementary events to event A is $3 \times 6 = 18$.

$\therefore$ $P(A) = \dfrac{18}{36} = \dfrac{1}{2}$

Similarly, $P(B) = \dfrac{18}{36} = \dfrac{1}{2}$

Now,

$$P(A \cap B) = P(\text{Getting an odd number on both throws})$$

$\Rightarrow$ $P(A \cap B) = \dfrac{9}{36} = \dfrac{1}{4} = \dfrac{1}{2} \times \dfrac{1}{2} = P(A)\,P(B)$

Hence, A and B are independent events.

EXAMPLE 3 *Three coins are tossed. Consider the event E 'three heads or three tails', F 'at least two heads' and G 'at most two heads'. Of the pairs (E, F), (E, G) and (F, G) which are independent? Which are dependent?* **[NCERT]**

SOLUTION The sample space of the experiment is given by

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

Clearly, $E = \{HHH, TTT\}$, $F = \{HHH, HHT, HTH, THH\}$

and, $G = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Also, $E \cap F = \{HHH\}$, $E \cap G = \{TTT\}$, $F \cap G = \{HHT, HTH, THH\}$

$\therefore \quad P(E) = \frac{2}{8} = \frac{1}{4}, P(F) = \frac{4}{8} = \frac{1}{2}, P(G) = \frac{7}{8}$

$P(E \cap F) = \frac{1}{8}, P(E \cap G) = \frac{1}{8}$ and $P(F \cap G) = \frac{3}{8}$

Clearly, $P(E \cap F) = P(E)\,P(F)$, but $P(E \cap G) \neq P(E)\,P(G)$ and $P(F \cap G) \neq P(F)\,P(G)$

So, E and F are independents, E and G are dependent events and F and G are also dependent events.

EXAMPLE 4 *A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die.'. Check whether A and B are independent event or not.* **[NCERT]**

SOLUTION The sample space related to the experiment is given by

$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5)\ (T, 6)\}$

We have,

$A = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6)\}$

$B = \{(H, 3), (T, 3)\}$

$A \cap B) = \{(H, 3)\}$

$\therefore \quad P(A) = \frac{6}{12} = \frac{1}{2}, P(B) = \frac{2}{12} = \frac{1}{6}$ and $P(A \cap B) = \frac{1}{12}$

Clearly, $P(A \cap B) = P(A)\,P(B)$.

Hence, A and B are independent events.

EXAMPLE 5 *A die is marked 1, 2, 3, in red and 4, 5, 6 in green is toosed. Let A be the event 'number is even' and B be the event 'number is red'. Are A and B independent?* **[NCERT]**

SOLUTION We have,

$A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$ and $A \cap B = \{2\}$

$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{3}{6} = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{6}$

Clearly, $P(A \cap B) \neq P(A)\,P(B)$.

So, A and B are not independent events.

EXAMPLE 6 *Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\text{not } A \text{ or not } B) = \frac{1}{4}$. State whether A and B are independent?* **[NCERT]**

SOLUTION We have,

$P(\text{not } A \text{ or not } B) = \frac{1}{4}$

$\Rightarrow \quad P(\overline{A} \cup \overline{B}) = \frac{1}{4} \Rightarrow P(\overline{A \cap B}) = \frac{1}{4} \Rightarrow 1 - P(A \cap B) = \frac{1}{4} \Rightarrow P(A \cap B) = \frac{3}{4}$

Clearly, $P(A \cap B) \neq P(A)\,P(B)$.

So, A and B are not independent events.

EXAMPLE 7 *An urn contains four tickets with numbers 112, 121, 211, 222 and one ticket is drawn. Let A_i $(i = 1, 2, 3)$ be the event that the i^{th} digit of the number on ticket drawn is 1. Discuss the independence of the events A_1, A_2, A_3.*

SOLUTION We have,

$P(A_1)$ = Probability that the first digit of the number on the drawn ticket is 1.

$\Rightarrow \quad P(A_1) = \frac{2}{4} = \frac{1}{2}$

$P(A_2)$ = Probability that the second digit of the number on the drawn ticket is

$\Rightarrow \quad P(A_2) = \frac{2}{4} = \frac{1}{2}$

$P(A_3)$ = Probability that the third digit of the number on the drawn ticket is 1.

$\Rightarrow \quad P(A_3) = \frac{2}{4} = \frac{1}{2}$

$P(A_1 \cap A_2)$ = Probability that first and second digits of the number on the drawn ticket are each equal to 1.

$\Rightarrow \quad P(A_1 \cap A_2) = \frac{1}{4}$

$P(A_2 \cap A_3) = \frac{1}{4}$

$P(A_1 \cap A_3) = \frac{1}{4}$

and, $P(A_1 \cap A_2 \cap A_3)$ = Probability that all the digits of the number on the drawn ticket are unity

$\Rightarrow \quad P(A_1 \cap A_2 \cap A_3) = 0$

We observe that

$$P(A_1 \cap A_2) = P(A_1)\,P(A_2),$$

$$P(A_2 \cap A_3) = P(A_2)\,P(A_3),\ P(A_3 \cap A_1) = P(A_3)\,P(A_1)$$

But, $P(A_1 \cap A_2 \cap A_3) \neq P(A_1)\,P(A_2)\,P(A_3)$.

Hence, A_1, A_2 and A_3 are pairwise independent but not mutually independent.

EXAMPLE 8 *A die is thrown once. If A is the event "the number appearing is a multiple of 3" and B is the event "the number appearing is even". Are the events A and B independent?* **[NCERT]**

SOLUTION We have,

$$P(A) = \frac{2}{6} = \frac{1}{3},\ P(B) = \frac{3}{6} = \frac{1}{2}$$

and,

$P(A \cap B) = P$ (Number appearing is even and a multiple of 3)

$\Rightarrow \quad P(A \cap B) = P$ (Number appearing is 6) $= \frac{1}{6}$

Clearly, $P(A \cap B) = P(A) \times P(B)$

Hence, A and B are independent events.

EXAMPLE 9 *In the two dice experiment, if A is the event of getting the sum of the numbers on dice as 11 and B is the event of getting a number other than 5 on the first die, find P (A and B). Are A and B independent events?*

SOLUTION We have,
Total number of elementary events = 36
Number of elementary events favourable to $A = 2$
Number of elementary events favourable to $B = 30$

$$\therefore \quad P(A) = \frac{2}{36} = \frac{1}{18}, P(B) = \frac{30}{36} = \frac{5}{6}$$

Now,

$P(A \cap B) = P$ (Getting the sum of the numbers on dice as 11 when 5 does not occur on first die)

$$= \frac{1}{36}$$

Clearly, $P(A \cap B) = \frac{1}{36} \neq \frac{1}{18} \times \frac{5}{6} = P(A)\, P(B)$

So, A and B are not independent events.

EXAMPLE 10 *Given that the events A and B are such that* $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ *and* $P(B) = p$. *Find p, if they are (i) mutually exclusive, (ii) independent.* [NCERT]

SOLUTION (i) If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

$$\Rightarrow \quad \frac{3}{5} = \frac{1}{2} + p$$

$$\Rightarrow \quad p = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}$$

(ii) If A, B are independent events, then

$$P(A \cap B) = P(A)\, P(B) = \frac{1}{2} p$$

$$\therefore \quad P(A \cup B) = \frac{3}{5}$$

$$\Rightarrow \quad P(A) + P(B) - P(A \cap B) = \frac{3}{5}$$

$$\Rightarrow \quad \frac{1}{2} + p - \frac{p}{2} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{p}{2} = \frac{3}{5} - \frac{1}{2}$$

$$\Rightarrow \quad p = \frac{1}{5}$$

EXAMPLE 11 *If A and B are two events such that* $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ *and* $P(A \cap B) = \frac{1}{8}$, *find P (not A and not B).* [NCERT]

SOLUTION We have,

$$P(A \cap B) = \frac{1}{8} = \frac{1}{4} \times \frac{1}{2} = P(A)\, P(B)$$

So, A and B are independent events.
Now,

$$P(\textit{not } A \text{ and } \textit{not } B) = P(\overline{A} \cap \overline{B})$$

$$\Rightarrow \quad P(\textit{not } A \text{ and } \textit{not } B) = P(\overline{A})\; P(\overline{B}) \quad [\because \overline{A} \text{ and } \overline{B} \text{ are indepenent events}]$$

$$\Rightarrow \quad P(\textit{not } A \text{ and } \textit{not } B) = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{2}\right) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$$

Type II **BASED UPON THE FORMULA $P(A \cap B) = P(A)\,P(B)$ FOR INDEPENDENT EVENTS**

EXAMPLE 12 *Events E and F are independent. Find P (F), if P (E) = 0.35 and P (E ∪ F) = 0.6*

SOLUTION We have,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$\Rightarrow \quad P(E \cup F) = P(E) + P(F) - P(E)\,P(F) \qquad \left[\begin{array}{l} \because E \text{ and } F \text{ are independent} \\ \therefore P(E \cap F) = P(E)\,P(F) \end{array}\right]$$

$$\Rightarrow \quad P(E \cup F) = P(E) + P(F)\,(1 - P(E))$$

$$\Rightarrow \quad 0.6 = 0.35 + P(F)\,(1 - 0.35) \qquad \text{[Substituting the values]}$$

$$\Rightarrow \quad 0.25 = (0.65)\,P(F)$$

$$\Rightarrow \quad P(F) = \frac{0.25}{0.65} = \frac{5}{13}$$

EXAMPLE 13 *If P (A) = 0.4, P (B) = p, P (A ∪ B) = 0.6 and A and B are given to be independent events, find the value of p.*

SOLUTION Since A and B are independent events. Therefore,

$$P(A \cap B) = P(A)\,P(B).$$

Now, $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A)\,P(B)$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B)\,(1 - P(A))$$

$$\Rightarrow \quad 0.6 = 0.4 + p\,(1 - 0.4)$$

$$\Rightarrow \quad 0.2 = 0.6\,p \Rightarrow p = 1/3.$$

EXAMPLE 14 *Let A and B be two independent events. The probability of their simultaneous occurrence is 1/8 and the probability that neither occurs is 3/8. Find P (A) and P (B).*

SOLUTION Let $P(A) = x$ and $P(B) = y$.

We have,

$$P(A \cap B) = 1/8 \text{ and } P(\overline{A} \cap \overline{B}) = 3/8.$$

Now, $$P(A \cap B) = 1/8 \Rightarrow P(A)\,P(B) = 1/8 \Rightarrow xy = 1/8 \qquad \text{...(i)}$$

Since A and B are independent events. Therefore, so are $\overline{A}$ and $\overline{B}$.

Thus, $$P(\overline{A} \cap \overline{B}) = \frac{3}{8}$$

$$\Rightarrow \quad P(\overline{A})\,P(\overline{B}) = \frac{3}{8}$$

$$\Rightarrow \quad (1 - x)(1 - y) = \frac{3}{8}$$

$$\Rightarrow \quad 1 - x - y + xy = \frac{3}{8}$$

$$\Rightarrow \quad x + y - xy = \frac{5}{8}$$

$$\Rightarrow \quad x + y - \frac{1}{8} = \frac{5}{8} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad x + y = \frac{3}{4} \qquad \text{...(ii)}$$

Now, $$(x - y)^2 = (x + y)^2 - 4xy$$

$$\Rightarrow \quad (x - y)^2 = \frac{9}{16} - 4 \times \frac{1}{8} = \frac{1}{16} \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad x - y = \pm\frac{1}{4}$$

CASE I *When* $x - y = \frac{1}{4}$:

In this case, we have

$$x - y = \frac{1}{4} \text{ and } x + y = \frac{3}{4} \Rightarrow x = \frac{1}{2} \text{ and } y = \frac{1}{4} \Rightarrow P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{4}.$$

CASE II *When* $x - y = -\frac{1}{4}$:

In this case, we have

$$x - y = -\frac{1}{4} \text{ and } x + y = \frac{3}{4} \Rightarrow x = \frac{1}{4} \text{ and } y = \frac{1}{2} \Rightarrow P(A) = \frac{1}{4} \text{ and } P(B) = \frac{1}{2}$$

Hence, $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{4}$ or, $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{2}$.

Type III FINDING THE PROBABILITY OF SIMULTANEOUS OCCURRENCE FOR INDEPENDENT EVENTS

EXAMPLE 15 *A bag contains 5 white, 7 red and 4 black balls. If four balls are drawn one by one with replacement, what is the probability that none is white?*

SOLUTION Let A, B, C and D denote the events of not getting a white ball in first, second, third and fourth draw respectively. Since the balls are drawn with replacement. Therefore, A, B, C and D are independent events such that

$$P(A) = P(B) = P(C) = P(D)$$

There are 16 balls out of which 11 are not white. Therefore, $P(A) = 11/16$.

Now, Required probability

$$= P(A \cap B \cap C \cap D)$$

$$= P(A)\,P(B)\,P(C)\,P(D) = \left(\frac{11}{16}\right)^4$$

EXAMPLE 16 *A class consists of 80 students; 25 of them are girls and 55 boys; 10 of them are rich and the remaining poor; 20 of them are fair complexioned. What is the probability of selecting a fair complexioned rich girl?*

SOLUTION Consider the following events :

A = Selecting a fair complexioned student;
B = Selecting a rich student;
C = Selecting a girl.

We have, $P(A) = \frac{20}{80} = \frac{1}{4}$, $P(B) = \frac{10}{80} = \frac{1}{8}$ and $P(C) = \frac{25}{80} = \frac{5}{16}$

Since A, B, C are independent events. Therefore.,

$$\text{Required probability} = P(A \cap B \cap C)$$

$$\Rightarrow \quad \text{Required probability} = P(A)\,P(B)\,P(C)$$

$$\Rightarrow \quad \text{Required probability} = \frac{1}{4} \times \frac{1}{8} \times \frac{5}{16} = \frac{5}{512}$$

EXAMPLE 17 *A bag contains 3 red and 5 black balls and a second bag contains 6 red and 4 black balls. A ball is draw from each bag. find the probability that both are (i) red (ii) black.*

Type II **BASED UPON THE FORMULA P (A ∩ B) = P (A) P (B) FOR INDEPENDENT EVENTS**

EXAMPLE 12 *Events E and F are independent. Find P (F), if P (E) = 0.35 and P (E ∪ F) = 0.6*

SOLUTION We have,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$\Rightarrow \quad P(E \cup F) = P(E) + P(F) - P(E)\,P(F) \qquad \left[\begin{array}{l}\because E \text{ and } F \text{ are independent} \\ \therefore P(E \cap F) = P(E)\,P(F)\end{array}\right]$$

$$\Rightarrow \quad P(E \cup F) = P(E) + P(F)\,(1 - P(E))$$

$$\Rightarrow \quad 0.6 = 0.35 + P(F)\,(1 - 0.35) \qquad \text{[Substituting the values]}$$

$$\Rightarrow \quad 0.25 = (0.65)\,P(F)$$

$$\Rightarrow \quad P(F) = \frac{0.25}{0.65} = \frac{5}{13}$$

EXAMPLE 13 *If P (A) = 0.4, P (B) = p, P (A ∪ B) = 0.6 and A and B are given to be independent events, find the value of p.*

SOLUTION Since A and B are independent events. Therefore,

$$P(A \cap B) = P(A)\,P(B).$$

Now, $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A)\,P(B)$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B)\,(1 - P(A))$$

$$\Rightarrow \quad 0.6 = 0.4 + p\,(1 - 0.4)$$

$$\Rightarrow \quad 0.2 = 0.6\,p \Rightarrow p = 1/3.$$

EXAMPLE 14 *Let A and B be two independent events. The probability of their simultaneous occurrence is 1/8 and the probability that neither occurs is 3/8. Find P (A) and P (B).*

SOLUTION Let $P(A) = x$ and $P(B) = y$.

We have,

$$P(A \cap B) = 1/8 \text{ and } P(\overline{A} \cap \overline{B}) = 3/8.$$

Now, $$P(A \cap B) = 1/8 \Rightarrow P(A)\,P(B) = 1/8 \Rightarrow xy = 1/8 \qquad \text{...(i)}$$

Since A and B are independent events. Therefore, so are $\overline{A}$ and $\overline{B}$.

Thus, $$P(\overline{A} \cap \overline{B}) = \frac{3}{8}$$

$$\Rightarrow \quad P(\overline{A})\,P(\overline{B}) = \frac{3}{8}$$

$$\Rightarrow \quad (1 - x)(1 - y) = \frac{3}{8}$$

$$\Rightarrow \quad 1 - x - y + xy = \frac{3}{8}$$

$$\Rightarrow \quad x + y - xy = \frac{5}{8}$$

$$\Rightarrow \quad x + y - \frac{1}{8} = \frac{5}{8} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad x + y = \frac{3}{4} \qquad \text{...(ii)}$$

Now, $$(x - y)^2 = (x + y)^2 - 4xy$$

$$\Rightarrow \quad (x - y)^2 = \frac{9}{16} - 4 \times \frac{1}{8} = \frac{1}{16} \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad x - y = \pm\frac{1}{4}$$

CASE I *When* $x - y = \frac{1}{4}$:

In this case, we have

$$x - y = \frac{1}{4} \text{ and } x + y = \frac{3}{4} \Rightarrow x = \frac{1}{2} \text{ and } y = \frac{1}{4} \Rightarrow P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{4}.$$

CASE II *When* $x - y = -\frac{1}{4}$:

In this case, we have

$$x - y = -\frac{1}{4} \text{ and } x + y = \frac{3}{4} \Rightarrow x = \frac{1}{4} \text{ and } y = \frac{1}{2} \Rightarrow P(A) = \frac{1}{4} \text{ and } P(B) = \frac{1}{2}$$

Hence, $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{4}$ or, $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{2}$.

Type III FINDING THE PROBABILITY OF SIMULTANEOUS OCCURRENCE FOR INDEPENDENT EVENTS

EXAMPLE 15 *A bag contains 5 white, 7 red and 4 black balls. If four balls are drawn one by one with replacement, what is the probability that none is white?*

SOLUTION Let A, B, C and D denote the events of not getting a white ball in first, second, third and fourth draw respectively. Since the balls are drawn with replacement. Therefore, A, B, C and D are independent events such that

$$P(A) = P(B) = P(C) = P(D)$$

There are 16 balls out of which 11 are not white. Therefore, $P(A) = 11/16$.

Now, Required probability

$$= P(A \cap B \cap C \cap D)$$

$$= P(A)\,P(B)\,P(C)\,P(D) = \left(\frac{11}{16}\right)^4$$

EXAMPLE 16 *A class consists of 80 students; 25 of them are girls and 55 boys; 10 of them are rich and the remaining poor; 20 of them are fair complexioned. What is the probability of selecting a fair complexioned rich girl?*

SOLUTION Consider the following events :

A = Selecting a fair complexioned student;

B = Selecting a rich student;

C = Selecting a girl.

We have, $P(A) = \frac{20}{80} = \frac{1}{4}$, $P(B) = \frac{10}{80} = \frac{1}{8}$ and $P(C) = \frac{25}{80} = \frac{5}{16}$

Since A, B, C are independent events. Therefore.,

Required probability $= P(A \cap B \cap C)$

$\Rightarrow$ Required probability $= P(A)\,P(B)\,P(C)$

$\Rightarrow$ Required probability $= \frac{1}{4} \times \frac{1}{8} \times \frac{5}{16} = \frac{5}{512}$

EXAMPLE 17 *A bag contains 3 red and 5 black balls and a second bag contains 6 red and 4 black balls. A ball is draw from each bag. find the probability that both are (i) red (ii) black.*

SOLUTION (i) Let A be the event that a red ball is drawn from first bag and B be the event that a red ball is drawn from the second bag. Then, A and B are independent events such that

$$P(A)=\frac{3}{8} \text{ and } P(B)=\frac{6}{10}.$$

$\therefore$ Required probability $= P(A \cap B) = P(A)\,P(B) = \frac{3}{8} \times \frac{6}{10} = \frac{9}{40}.$

(ii) Let A and B be the events of drawing a black ball from first and second bag respectively. Then, A and B are independent events such that $P(A)=5/8$ and $P(B)=4/10$.

$\therefore$ Required probability $= P(A \cap B) = P(A)\,P(B) = \frac{5}{8} \times \frac{4}{10} = \frac{1}{4}.$

EXAMPLE 18 *A police-man fires four bullets on a dacoit. The probability that the dacoit will be killed by one bullet is 0.6. What is the probability that the dacoit is still alive?*

SOLUTION Let A_i; $i = 1, 2, 3, 4$ be the event that the dacoit is not killed by the i^{th} bullet. Then, $P(A_i) = 1 - 0.6 = 0.4$.

Now, Probability that the dacoit is still alive

$= P(A_1 \cap A_2 \cap A_3 \cap A_4)$

$= P(A_1)\,P(A_2)\,P(A_3)\,P(A_4)$ [Since all 4 shots are independent]

$= (0.4)^4 = 0.0256.$

EXAMPLE 19 *Two dice are thrown. Find the probability of getting an odd number on the first die and a multiple of 3 on the other.*

SOLUTION Consider the following events :

A = getting an odd number on first die,

B = getting a multiple of 3 on the second die.

We have, $A = \{1, 3, 5\}$ and $B = \{3, 6\}$

$\therefore$ $P(A) = \frac{3}{6} = \frac{1}{2}$ and $P(B) = \frac{2}{6} = \frac{1}{3}$

Now, Required probability $= P(A \cap B)$

$= P(A)\,P(B)$

$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ [A and B are independent events]

Type IV **FINDING THE PROBABILITY OF OCCURRENCE OF AT LEAST ONE EVENT FOR INDEPENDENT EVENTS**

EXAMPLE 20 *A bag contains 5 white, 7 red and 8 black balls. Four balls are drawn one by one with replacement, what is the probability that at least one is white?*

SOLUTION Let A_i be the event that ball drawn in ith draw is white $1 \le i \le 4$.

Since the balls are drawn with replacement. Therefore, A_1, A_2, A_3, A_4 are independent events such that

$$P(A_i) = \frac{5}{20} = \frac{1}{4},\ i = 1, 2, 3, 4.$$

Now, Required probability

$= P(A_1 \cup A_2 \cup A_3 \cup A_4)$

$$= 1 - P(\overline{A}_1)\, P(\overline{A}_2)\, P(\overline{A}_3)\, P(\overline{A}_4) \qquad [\because A_1, A_2, A_3, A_4 \text{ are independent}]$$

$$= 1 - \left(\frac{3}{4}\right)^4$$

EXAMPLE 21 *A problem in mathematics is given to 3 students whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. What is the probability that the problem is solved?*

SOLUTION Let A, B, C be the respective events of solving the problem. Then, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(C) = \frac{1}{4}$.

Clearly A, B, C are independent event and the problem is solved if at least one student solves it.

$\therefore$ Required probability

$$= P(A \cup B \cup C)$$

$$= 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C})$$

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

EXAMPLE 22 *A can solve 90% of the problems given in a book and B can solve 70%. What is the probability that at least one of them will solve the problem, selected at random from the book?*

SOLUTION Let E and F be the events defined as follows :

$$E = A \text{ solves the problem}, F = B \text{ solves the problem.}$$

Clearly, E and F are independent events such that

$$P(E) = \frac{90}{100} = \frac{9}{10} \text{ and } P(F) = \frac{70}{100} = \frac{7}{10}$$

Now,

Required probability $= P(E \cup F)$

$\Rightarrow$ Required probability $= 1 - P(\overline{E})\, P(\overline{F})$ $\qquad [\because E$ and F independent events$]$

$\Rightarrow$ Required probability $= 1 - \left(1 - \frac{9}{10}\right)\left(1 - \frac{7}{10}\right) = 1 - \frac{1}{10} \times \frac{3}{10} = 0.97$

EXAMPLE 23 *The odds against A solving a certain problem are 4 to 3 and the odds in favour of B solving the same problem are 7 to 5. Find the probability that the problem will be solved.*

SOLUTION We are given that

$$P(A) = \frac{3}{4+3} = \frac{3}{7} \qquad [\because \text{odds against are 4 to 3}]$$

and, $$P(B) = \frac{7}{7+5} = \frac{7}{12} \qquad [\because \text{odds in favour are 7 to 5}]$$

The problem will be solved if at least one of them solves the problem, that is, we are interested to find $P(A \cup B)$. Since A and B are independent events.

$$\therefore \quad P(A \cup B) = 1 - P(\overline{A})\, P(\overline{B}) = 1 - \left(1 - \frac{3}{7}\right)\left(1 - \frac{7}{12}\right) = \frac{16}{21}$$

EXAMPLE 24 *The probability that a teacher will give an unannounced test during any class meeting is 1/5. If a student is absent twice, what is the probability that he will miss at least one test?*

SOLUTION Let E_i be the event that the student misses i^{th} test $(i = 1, 2)$. Then E_1 and E_2 are independent events such that

$$P(E_1) = \frac{1}{5} = P(E_2).$$

$\therefore$ Required probability $= P(E_1 \cup E_2)$

$\Rightarrow$ Required probability $= 1 - P(\overline{E}_1)\, P(\overline{E}_2)$ $\quad [\because E_1, E_2$ are independent$]$

$\Rightarrow$ Required probability $= 1 - \left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{5}\right) = \frac{9}{25}.$

EXAMPLE 25 *A machine operates if all of its three components function. The probability that the first component fails during the year is 0.14, the second component fails is 0.10 and the third component fails is 0.05. What is the probability that the machine will fail during the year?*

SOLUTION Consider the following events:

A = First component of the machine fails during the year

B = Second component of the machine fails during the year

C = Third component of the machine fails during the year

We have,

$$P(A) = 0.14,\ P(B) = 0.10 \text{ and } P(C) = 0.05$$

Clearly, the machine will fail if at least one of its three components fails during the year.

$\therefore$ Required probability $= P(A \cup B \cup C)$

$\Rightarrow$ Required probability $= 1 - P\,\overline{(A \cup B \cup C)}$

$\Rightarrow$ Required probability $= 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$

$\Rightarrow$ Required probability $= 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C})$ $\quad \left[\because A, B, C \text{ are independent events}\right]$

$\Rightarrow$ Required probability $= 1 - (1 - 0.14)(1 - 0.10)(1 - 0.05)$

$\Rightarrow$ Required probability $= 1 - (0.86)(0.90)(0.95) = 0.2647$

EXAMPLE 26 *If two switches S_1 and S_2 have respectively 90% and 80% chances of working. Find the probabilities that each of the following circuits will work.*

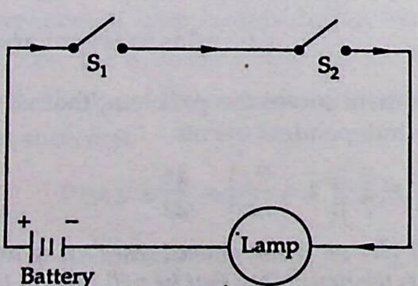

Fig. 30.5

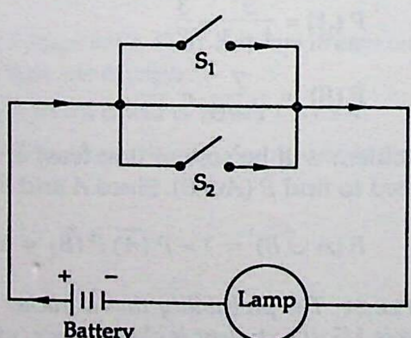

Fig. 30.6

SOLUTION Consider the following events :

A = Switch S_1 works, B = Switch S_2 works.

We have,

$$P(A) = \frac{90}{100} = \frac{9}{10} \text{ and } P(B) = \frac{80}{100} = \frac{8}{10}$$

(i) The circuit will work if the current flows in the circuit. This is possible only when both the switches work together. Therefore,

Required probability $= P(A \cap B)$

$\Rightarrow$ Required probability $= P(A)\,P(B)$ [$\because$ A and B are independent events]

$\Rightarrow$ Required probability $= \dfrac{9}{10} \times \dfrac{8}{10}$

$\Rightarrow$ Required probability $= \dfrac{72}{100} = \dfrac{18}{25}$

(ii) The circuit will work if the current flows in the circuit. This is possible only when at least one of the two switches S_1, S_2 works. Therefore,

Required Probability $= P(A \cup B)$

$\Rightarrow$ Required Probability $= 1 - P(\overline{A})\,P(\overline{B})$ [$\because$ A, B are independent events]

$\Rightarrow$ Required Probability $= 1 - \left(1 - \dfrac{9}{10}\right)\left(1 - \dfrac{8}{10}\right)$

$\Rightarrow$ Required Probability $= 1 - \dfrac{1}{10} \times \dfrac{2}{10} = \dfrac{49}{50}$

EXAMPLE 27 *What is the probability that series circuit in Fig. 30.7 with three switches S_1, S_2 and S_3 with probabilities $\frac{1}{3}$, $\frac{1}{2}$ and $\frac{3}{4}$ respectively, of functioning will work?*

SOLUTION Consider the following events :

A = Switch S_1 works, B = Switch S_2 works and C = Switch S_3 works

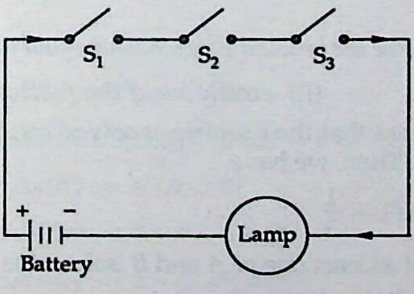

Fig. 30.7

We have,

$$P(A) = \frac{1}{3},\ P(B) = \frac{1}{2} \text{ and } P(C) = \frac{3}{4}$$

Clearly, current flows through the circuit if switches S_1, S_2 and S_3 work together.

$\therefore$ Required probability $= P(A \cap B \cap C)$

$\Rightarrow$ Required probability $= P(A)\,P(B)\,P(C)$ [$\because$ A, B, C are independent events]

$\Rightarrow$ Required probability $= \frac{1}{3} \times \frac{1}{2} \times \frac{3}{4} = \frac{1}{8}$

EXAMPLE 28 *A coin is tossed and a die is thrown. Find the probability that the outcome will be a head or a number greater than 4, or both.*

SOLUTION Let A be the event of getting head in a single toss of a coin and B be the event of getting a number greater than 4 in a throw of a die. Then,

$$P(A) = \frac{1}{2} \text{ and } P(B) = \frac{2}{6} = \frac{1}{3}$$

$\therefore$ Required probability $= P(A \cup B)$

$\Rightarrow$ Required probability $= 1 - P(\overline{A})\,P(\overline{B})$ [$\because$ A and B are independent events]

$\Rightarrow$ Required probability $= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)$

$\Rightarrow$ Required probability $= \frac{2}{3}$

EXAMPLE 29 *In two successive throws of a pair of dice, determine the probability of getting a total of 8 each time.*

SOLUTION Let A denote the event of getting a total of 8 in first throw and B be the event of getting a total of 8 in second throw. Then,

$$P(A) = \frac{5}{36} \text{ and } P(B) = \frac{5}{36}$$

$\therefore$ Required probability $= P(A \cap B)$

$\Rightarrow$ Required probability $= P(A)\,P(B)$ [$\because$ A, B are independent events]

$\Rightarrow$ Required probability $= \frac{5}{36} \times \frac{5}{36} = \frac{25}{1296}$

EXAMPLE 30 *Probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that*

(i) *the problem is solved* (ii) *exactly one of them solves the problem.* [NCERT]

SOLUTION Let E be the event that the problem is solved by A and F be the event that the problem is solved by B. Then, we have

$$P(E) = \frac{1}{2} \text{ and } P(F) = \frac{1}{3}$$

(i) The problem is solved if at least one of A and B solves the problem. Therefore,

Required probability $= P(E \cup F)$

$\Rightarrow$ Required probability $= 1 - P(\overline{F})\,P(\overline{F})$ [$\because$ A, B are independent events]

$\Rightarrow$ Required probability $= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = \frac{2}{3}$

(ii) Required probability $= P(E) + P(F) - 2\,P(E \cap F)$

$\Rightarrow$ Required probability $= P(E) + P(F) - 2\,P(E)\,P(F)$

$\Rightarrow$ Required probability $= \frac{1}{2}+\frac{1}{3}-2\times\frac{1}{2}\times\frac{1}{3}$

$\Rightarrow$ Required probability $= \frac{5}{6}-\frac{1}{3} = \frac{1}{2}$

EXAMPLE 31 *A scientist has to make a decision on each of the two independent events I and II. Suppose the probability of error in making decision on event I is 0.02 and that on event II is 0.05. Find the probability that the scientist will make the correct decision on*

(i) *both the events* (ii) *only one event*

SOLUTION Consider the following events :

A = Scientist will make the correct decision on event I

B = Scientist will make the correct decision on event II

We have,

$P(A) = 1-0.02 = 0.98$ and $P(B) = 1-0.05 = 0.95$

(i) Required probability $= P(A\cap B)$

$\Rightarrow$ Required probability $= P(A)\,P(B)$ [$\because$ A and B are independent events]

$\Rightarrow$ Required probability $= 0.98\times 0.95 = 0.931$

(ii) Required probability

= Probability of occurrence of exactly one of A and B

$= P(A)+P(B)-2P(A\cap B)$

$= P(A)+P(B)-2P(A)\,P(B)$ [$\because$ A and B are independent events]

$= 0.98+0.95-2\times 0.98\times 0.95 = 0.068$

EXAMPLE 32 *A town has two fire extinguishing engines functioning independently. The probability of availability of each engine, when needed, is 0.95. What is the probability that*

(i) *neither of them is available when needed?*
(ii) *an engine is available when needed?*
(iii) *exactly one engine is available when needed?*

SOLUTION Let A denote the event that first engine is available when needed and B, the event that second engine is available when needed. Then,

$P(A) = P(B) = 0.95$

(i) Required probability $= P(\overline{A}\cap\overline{B})$

$\Rightarrow$ Required probability $= P(\overline{A})\,P(\overline{B})$ [$\because$ A, B are independent]

$\Rightarrow$ Required probability $= (0.05)\times(0.05) = 0.0025$

(ii) Required probability $= P(A\cup B)$

$\Rightarrow$ Required probability $= 1-P(\overline{A})\,P(\overline{B})$

$\Rightarrow$ Required probability $= 1-(0.05)(0.05)$ [$\because$ A, B are independent]

$\Rightarrow$ Required probability $= 0.9975$

(iii) Required probability $= P(A)+P(B)-2P(A\cap B)$

$\Rightarrow$ Required probability $= P(A)+P(B)-2\,P(A)\times P(B)$

$\Rightarrow$ Required probability $= 0.95+0.95-2\times 0.95\times 0.95 = 0.095$

EXAMPLE 33 *A company has estimated that the probabilities of success for three products introduced in the market are* $\frac{1}{3}, \frac{2}{5}$ *and* $\frac{2}{3}$ *respectively. Assuming independence, find*

(i) *the probability that the three products are successful.*
(ii) *the probability that none of the products is successful.*

SOLUTION Consider the following events :

A = First product is successful, and B = Second product is successful,

C = Third product is successful

We have,

$$P(A) = \frac{1}{3}, P(B) = \frac{2}{5} \text{ and } P(C) = \frac{2}{3}$$

(i) Required probability $= P(A \cap B \cap C)$

$\Rightarrow$ Required probability $= P(A)\,P(B)\,P(C)$ [$\because$ A, B, C are independent events]

$\Rightarrow$ Required probability $= \frac{1}{3} \times \frac{2}{5} \times \frac{2}{3} = \frac{4}{45}$

(ii) Required probability $= P(\overline{A} \cap \overline{B} \cap \overline{C})$

$\Rightarrow$ Required probability $= P(\overline{A})\,P(\overline{B})\,P(\overline{C})$ [$\because$ A, B, C are independent events]

$\Rightarrow$ Required probability $= \frac{2}{3} \times \frac{3}{5} \times \frac{1}{3} = \frac{2}{15}$

EXAMPLE 34 *A can hit a target 4 times in 5 shots, B 3 times in 4 shots, and C 2 times in 3 shots. Calculate the probability that*

(i) *A, B, C all may hit* (ii) *B, C may hit and A may lose .*
(iii) *any two of A, B and C will hit the target* **[CBSE 2005]**
(iv) *non of them will hit the target.* **[CBSE 2005]**

SOLUTION Consider the following events:

E = A hits the target, F = B hits the target and G = C hits the target

We have, $P(E) = \frac{4}{5}, P(F) = \frac{3}{4}$ and $P(G) = \frac{2}{3}$

(i) We have,

Required probability $= P(E \cap F \cap G)$

$\Rightarrow$ Required probability $= P(E)\,P(F)\,P(G)$ [$\because$ A, B, C are independent events]

$\Rightarrow$ Required probability $= \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{2}{5}$

(ii) We have,

Required probability $= P(\overline{E} \cap F \cap G)$

$\Rightarrow$ Required probability $= P(\overline{E})\,P(F)\,P(G)$ [$\because$ A, B, C are independent events]

$\Rightarrow$ Required probability $= \left(1 - \frac{4}{5}\right) \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{10}$

(iii) We have,

Required probability $= P\left((E \cap F \cap \overline{G}) \cup (\overline{E} \cap F \cap G) \cup (E \cap \overline{F} \cap G)\right)$

$\Rightarrow$ Required probability $= P(E \cap F \cap \overline{G}) + P(\overline{E} \cap F \cap G) + P(E \cap \overline{F} \cap G)$

$\Rightarrow$ Required probability $= P(E)\,P(F)\,P(\overline{G}) + P(\overline{E})\,P(F)\,P(G)\,P(G) + P(E)\,P(\overline{F})\,P(G)$

$\Rightarrow$ Required probability $= \frac{4}{5}\times\frac{3}{4}\times\frac{1}{3}+\frac{1}{5}\times\frac{3}{4}\times\frac{2}{3}+\frac{4}{5}\times\frac{1}{4}\times\frac{2}{3} = \frac{13}{30}$

(iv) Required probability $= P(\overline{E}\cap\overline{F}\cap\overline{G}) = P(\overline{E})\,P(\overline{F})\,P(\overline{G}) = \frac{1}{5}\times\frac{1}{4}\times\frac{1}{3} = \frac{1}{60}$

EXAMPLE 35 *A combination lock on a suitcase has 3 wheels each labelled with nine digits from 1 to 9. If an opening combination is a particular sequence of three digits with no repeats, what is the probability of a person guessing the right combination?*

SOLUTION Let A_i ; $i = 1, 2, 3$ be the event that the digit on ith wheel occupies the correct position. Then,

Required probability $= P(A_1 \cap A_2 \cap A_3)$

$\Rightarrow$ Required probability $= P(A_1)\,P(A_2/A_1)\,P(A_3/A_1\cap A_2) = \frac{1}{9}\times\frac{1}{8}\times\frac{1}{7} = \frac{1}{504}$.

EXERCISE 30.4

1. A coin is tossed thrice and all the eight outcomes are assumed equally likely. In which of the following cases are the following events A and B are independent?
   (i) A = the first throw results in head, B = the last throw results in tail
   (ii) A = the number of heads is odd, B = the number of tails is odd
   (iii) A = the number of heads is two, B = the last throw results in head
2. Prove that in throwing a pair of dice, the occurrence of the number 4 on the first die is independent of the occurrence of 5 on the second die.
3. A card is drawn from a pack of 52 cards so that each card is equally likely to be selected. In which of the following cases are the events A and B independent?
   (i) A = the card drawn is a king or queen
   B = the card drawn is a queen or jack
   (ii) A = the card drawn is black, B = the card drawn is a king
   (iii) B = the card drawn is a spade, B = the card drawn in an ace
4. A coin is tossed three times. Let the events A, B and C be defined as follows :
   A = first toss is head, B = second toss is head, and
   C = exactly two heads are tossed in a row.
   Check the independence of (i) A and B (ii) B and C and (iii) C and A
5. If A and B be two events such that $P(A) = 1/4$, $P(B) = 1/3$ and $P(A\cup B) = 1/2$, show that A and B are independent events.
6. Given two independent events A and B such that $P(A) = 0.3$ and $P(B) = 0.6$. Find
   (i) $P(A\cap B)$ (ii) $P(A\cap\overline{B})$ (iii) $P(\overline{A}\cap B)$ (iv) $P(\overline{A}\cap\overline{B})$
   (v) $P(A\cup B)$ (vi) $P(A/B)$ (vii) $P(B/A)$
7. If P (not B) $= 0.65$, $P(A\cup B) = 0.85$, and A and B are independent events, then find $P(A)$.
8. If A and B are two independent events such that $P(\overline{A}\cap B) = 2/15$ and $P(A\cap\overline{B}) = 1/6$, then find $P(B)$.
9. A and B are two independent events. The probability that A and B occur is $1/6$ and the probability that neither of them occurs is $1/3$. Find the probability of occurrence of two events.
10. If A and B are two independent events such that $P(A\cup B) = 0.60$ and $P(A) = 0.2$ find $P(B)$.

11. A die is tossed twice. Find the probability of getting a number greater than 3 on each toss.
12. Given the probability that A can solve a problem is $2/3$ and the probability that B can solve the same problem is $3/5$. Find the probability that none of the two will be able to solve the problem.
13. An unbiased die is tossed twice. Find the probability of getting 4, 5, or 6 on the first toss and 1, 2, 3 or 4 on the second toss.
14. A bag contains 3 red and 2 black balls. One ball is drawn from it at random. Its colour is noted and then it is put back in the bag. A second draw is made and the same procedure is repeated. Find the probability of drawing (i) two red balls, (ii) two black balls, (iii) first red and second black ball.
15. Three cards are drawn with replacement from a well shuffled pack of cards. Find the probability that the cards drawn are king, queen and jack.
16. An article manufactured by a company consists of two parts X and Y. In the process of manufacture of the part X, 9 out of 100 parts may be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of part Y. Calculate the probability that the assembled product will not be defective.
17. The probability that A hits a target is $1/3$ and the probability that B hits it, is $2/5$. What is the probability that the target will be hit, if each one of A and B shoots at the target?
18. An anti-aircraft gun can take a maximum of 4 shots at an enemy plane moving away from it. The probabilities of hitting the plane at the first, second, third and fourth shot are 0.4, 0.3, 0.2 and 0.1 respectively. What is the probability that the gun hits the plane?
19. The odds against a certain event are 5 to 2 and the odds in favour of another event, independent to the former are 6 to 5. Find the probability that (a) at least one of the events will occur, and (b) none of the events will occur.
20. One card is drawn at random from a pack of well-shuffled deck of 52 cards. In which of the following cases are the events A and B are independent:
    (i) $A =$ The card drawn is a spade, $B =$ The card drawn is an ace
    (ii) $A =$ The card drawn is black, $B =$ The card drawn is a king [NCERT]
    (iii) $A =$ The card drawn is a king or a queen, $B =$ The card drawn is a queen or a jack.
21. A coin is tossed thrice. In which of the following cases are the events A and B independent?
    (i) $A =$ The first throw results in head, $B =$ The last throw results in tail.
    (ii) $A =$ The number of heads is two, $B =$ The last throw results in head
    (iii) $A =$ The number of heads is odd, $B =$ The number of tails is odd.
22. A die is thrown thrice. Find the probability of getting an odd number at least once.
23. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that
    (i) both balls are red
    (ii) first ball is black and second is red.
    (iii) one of them is black and other is red. [NCERT]
24. An urn contains 4 red and 7 black balls. Two balls are drawn at random with replacement. Find the probability of getting (i) 2 red balls (ii) 2 blue balls (iii) one red and one blue ball. [CBSE 2007]

ANSWERS

1. (i) (ii) 3. (ii) and (iii)

4. (i) independent (ii) dependent (iii) independent

6. (i) 0.18 (ii) 0.12 (iii) 0.42
(iv) 0.28 (v) 0.72 (vi) 0.3 (vii) 0.6

7. 0.77 8. $\frac{1}{6}$ or $\frac{4}{5}$ 9. $P(A)=\frac{1}{3}$, $P(B)=\frac{1}{2}$ or $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{3}$

10. 0.5 11. $\frac{1}{4}$ 12. $\frac{2}{15}$ 13. $\frac{1}{3}$ 14. (i) $\frac{9}{25}$ (ii) $\frac{4}{25}$ (iii) $\frac{6}{25}$

15. $\frac{6}{2197}$ 16. 0.8645 17. $\frac{3}{5}$ 18. 0.696 19. (a) $\frac{52}{77}$ (b) $\frac{25}{77}$ 20. (i), (ii) 21. (i)

22. $\frac{7}{8}$ 23. (i) $\frac{16}{81}$ (ii) $\frac{20}{81}$ (iii) $\frac{40}{81}$ 24. (i) $\frac{16}{121}$ (ii) $\frac{49}{121}$ (iii) $\frac{56}{121}$

HINTS TO SELECTED PROBLEMS

2. Here, $n(S) = 36$, $A = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$
and $B = \{(1,5), (2,5), (3,5), (4,5), (5,5), (6,5)\}$.
So, $A \cap B = \{(4,5)\}$. Now, show that $P(A \cap B) = P(A)\, P(B)$

12. Required probability $= P(\overline{A} \cap \overline{B}) = P(\overline{A})\, P(\overline{B})$

16. Let, $A =$ Part X is not defective, $B =$ Part Y is not defective,
Required probability $= P(A \cap B) = P(A)\, P(B) = \frac{91}{100} \times \frac{95}{100}$

17. Required probability $= P(A \cup B) = 1 - P(\overline{A})\, P(\overline{B})$

18. The gun hits the plane, if it hits the plane in at least one shot.
$\therefore$ Required probability $= 1 - (1-0.4)(1-0.3)(1-0.2)(1-0.1)$

19. Let A and B be the events. Then, $P(A) = \frac{2}{2+5}$ and $P(B) = \frac{6}{6+5}$.
(a) So, required probability $= P(A \cup B) = 1 - P(\overline{A})\, P(\overline{B})$
(b) Required probability $= P(\overline{A} \cap \overline{B}) = P(\overline{A})\, P(\overline{B})$.

22. Let A_i denote the event of getting an odd number in i^{th} throw, $i = 1, 2, 3$. Then,

$$P(A_i) = \frac{3}{6} = \frac{1}{2};\ i = 1, 2, 3.$$

$\Rightarrow$ Required probability $= P(A_1 \cup A_2 \cup A_3)$

$\Rightarrow$ Required probability $= 1 - P(\overline{A}_1)\, P(\overline{A}_2)\, P(\overline{A}_3)$

$\Rightarrow$ Required probability $= 1 - \left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = \frac{7}{8}$

30.7 MORE ON THEOREMS OF PROBABILITY

In the previous sections, we have discussed those problems based on addition and multiplication on theorems which require the use of only one of the two theorems. In this section, we will discuss problems based upon the use of both the theorems. Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *A bag contains 4 white and 2 black balls. Another contains 3 white and 5 black balls. If one ball is drawn from each bag, find the probability that*
(i) *both are white;* (ii) *both are black*
(iii) *one is white and one is black.*

SOLUTION Consider the following events :

W_1 = Drawing a white ball from first bag.

W_2 = Drawing a white ball from second bag.

B_1 = Drawing a black ball from first bag.

B_2 = Drawing a black ball from second bag.

Clearly, $P(W_1) = 4/6$, $P(B_1) = 2/6$, $P(W_2) = 3/8$ and $P(B_2) = 5/8$.

(i) P (both balls are white)

$= P$ [(white ball from 1st bag) and (white ball from 2nd bag)]

$= P(W_1 \cap W_2)$

$= P(W_1)\,P(W_2)$ [$\because$ W_1 and W_2 are independent events)]

$$= \frac{4}{6} \times \frac{3}{8} = \frac{1}{4}$$

(ii) P (both balls are black)

$= P$ [(black ball from 1st bag) and (black ball from 2nd bag)]

$= P(B_1 \cap B_2)$

$= P(B_1)\,P(B_2)$ [$\because$ B_1 and B_2 are independent events]

$$= \frac{2}{6} \times \frac{5}{8} = \frac{5}{24}$$

(iii) P (one white ball and one black ball)

$= P$ [(black from 1st and white from 2nd) or (white from 1st and black from 2nd)]

$= P[(B_1 \cap W_2) \cup (W_1 \cap B_2)]$

$= P(B_1 \cap W_2) + P(W_1 \cap B_2)$ [By add. theorem for mutually exclusive events]

$= P(B_1)\,P(W_2) + P(W_1)\,P(B_2)$ [$\because$ B_1 and W_2; B_2 and W_1 are pairs of independent events]

$$= \frac{2}{6} \times \frac{3}{8} + \frac{4}{6} \times \frac{5}{8} = \frac{13}{24}$$

EXAMPLE 2 *A box contains 3 red and 5 blue balls. Two balls are drawn one by one at a time at random without replacement. Find the probability of getting 1 red and 1 blue ball.*

SOLUTION Consider the following events :

R_1 = Getting a red ball in first draw, R_2 = Getting a red ball in second draw,

B_1 = Getting a blue ball in first draw, B_2 = Getting a blue ball in second draw.

Now, P (one red and one blue ball)

$= P$ [(red ball in first draw and blue ball in second draw) or (blue ball in first draw and red ball in second draw)]

$= P[(R_1 \cap B_2) \cup (B_1 \cap R_2)]$

$= P(R_1 \cap B_2) + P(B_1 \cap R_2)$ [by add. Theo. for mutually exclusive events]

$= P(R_1)\, P(B_2/R_1) + P(B_1)\, P(R_2/B_1)$

$$= \frac{3}{8} \times \frac{5}{7} + \frac{5}{8} \times \frac{3}{7} = \frac{15}{28}$$

EXAMPLE 3 *Two cards are drawn from a well shuffled pack of 52 cards without replacement. What is the probability that one is a red queen and the other is a king of black colour ?*

SOLUTION Consider the following events :

R_i = getting a red queen in ith draw ; $i = 1, 2$.

K_i = getting a black king in ith draw ; $i = 1, 2$

Now, Required probability

$= P\left((R_1 \cap K_2) \cup (K_1 \cap R_2)\right)$

$= P(R_1 \cap K_2) + P(K_1 \cap R_2)$

$= P(R_1)\, P(K_2/R_1) + P(K_1)\, P(R_2/K_1)$

$$= \frac{{}^2C_1}{{}^{52}C_1} \times \frac{{}^2C_1}{{}^{51}C_1} + \frac{{}^2C_1}{{}^{52}C_1} \times \frac{{}^2C_1}{{}^{51}C_1}$$

$$= \left(\frac{2}{52} \times \frac{2}{51}\right) + \left(\frac{2}{52} \times \frac{2}{51}\right) = \frac{2}{663}$$

EXAMPLE 4 *Two cards are drawn without replacement from a well shuffled pack of 52 cards. Find the probability that one is a spade and other is a queen of red colour.*

SOLUTION Consider the following events :

S_i = getting a spade card in ith draw; $i = 1, 2$

Q_i = getting a red queen in ith draw; $i = 1, 2$

Now, Required probability

$= P\left((S_1 \cap Q_2) \cup (Q_1 \cap S_2)\right)$

$= P(S_1 \cap Q_2) + P(Q_1 \cap S_2)$

$= P(S_1)\, P(Q_2/S_1) + P(Q_1)\, P(S_2/Q_1)$ [By add. Theorem]

$$= \frac{{}^{13}C_1}{{}^{52}C_1} \times \frac{{}^2C_1}{{}^{51}C_1} + \frac{{}^2C_1}{{}^{52}C_1} \times \frac{{}^{13}C_1}{{}^{51}C_1} = 2\left(\frac{13}{\underset{4}{52}} \times \frac{2}{51}\right) = \boxed{\frac{1}{51}}$$

EXAMPLE 5 *A bag contains 5 white and 3 black balls. Four balls are successively drawn out without replacement. What is the probability that they are alternately of different colours?*

SOLUTION Let W_i denote the event of drawing a white ball in ith drawn and B_i denote the event of drawing a black ball in ith draw, where $i = 1, 2, 3, 4$. Then,

Required probability

$= P[(W_1 \cap B_2 \cap W_3 \cap B_4) \cup (B_1 \cap W_2 \cap B_3 \cap W_4)]$

$= P(W_1 \cap B_2 \cap W_3 \cap B_4) + P(B_1 \cap W_2 \cap B_3 \cap W_4)$ [By add. Theorem]

$= P(W_1)\,P(B_2/W_1)\,P(W_3/W_1 \cap B_2)\,P(B_4/W_1 \cap B_2 \cap W_3)$
$+ P(B_1)\,P(W_2/B_1)\,P(B_3/B_1 \cap W_2)\,P(W_4/B_1 \cap W_2 \cap B_3)$

[By Multiplication Theorem]

$$= \frac{5}{8} \times \frac{3}{7} \times \frac{4}{6} \times \frac{2}{5} + \frac{3}{8} \times \frac{5}{7} \times \frac{2}{6} \times \frac{4}{5} = \frac{1}{7}$$

EXAMPLE 6 *Cards are numbered 1 to 25. Two cards are drawn one after the other. Find the probability that the number on one card is multiple of 7 on the other it is a multiple of 11.*

SOLUTION Two cards can be drawn in the following mutually exclusive ways :

(i) First card bears a multiple of 7 and second bears a multiple of 11

(ii) First card bears a multiple of 11 and second bears a multiple of 7.

Thus, if we define the following events :

A_1 = First card drawn bears a multiple of 7,

A_2 = Second card drawn bears a multiple of 7,

B_1 = First card drawn bears a multiple of 11,

B_2 = Second card drawn bears a multiple of 11.

Then, Required probability

$= P[(A_1 \cap B_2) \cup (B_1 \cap A_2)]$

$= P(A_1 \cap B_2) + P(B_1 \cap A_2)$ [By add. Theorem]

$= P(A_1)\,P(B_2/A_1) + P(B_1)\,P(A_2/B_1)$ ...(i)

There are three multiples of 7 viz. 7, 14, 21 and 2 multiples of 11 viz. 11, 22 between 1 and 25.

$\therefore \quad P(A_1) = \frac{3}{25},\ P(B_2/A_1) = \frac{2}{24},\ P(B_1) = \frac{2}{25},\ P(A_2/B_1) = \frac{3}{24}$

Substituting these values in (i), we have

$$\text{Required probability} = \frac{3}{25} \times \frac{2}{24} + \frac{2}{25} \times \frac{3}{24} = \frac{1}{50}$$

EXAMPLE 7 *Bag A contains 4 red and 5 black balls and bag B contains 3 red and 7 black balls. One ball is drawn from bag A and two from bag B. Find the probability that out of 3 balls drawn, two are black and one is red.*

SOLUTION Two black and one red ball can be drawn in two mutually exclusive ways :

(I) Drawing one black ball from bag A and two balls from bag B out of which one is black and other is red

(II) Drawing one red ball from bag A two black balls from bag B.

Thus, if we define the following events :

E_1 = Drawing a black ball from bag A

E_2 = Drawing one red and one black ball from bag B

E_3 = Drawing one red ball from bag A

E_4 = Drawing two black balls from bag B.

Then, $P(E_1) = \frac{5}{9},\ P(E_2) = \frac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2} = \frac{21}{45},\ P(E_3) = \frac{4}{9},\ P(E_4) = \frac{{}^7C_2}{{}^{10}C_2} = \frac{7}{15}$

Now, P (Two black balls and one red ball)

$= P$ [(1 black from bag A and one red and one black from bag B) or (one red from bag A and 2 black from bag B)]

$= P\,(\text{I or II})$

$= P\,(\text{I} \cup \text{II})$

$= P\,[(E_1 \cap E_2) \cup (E_3 \cap E_4)]$

$= P\,(E_1 \cap E_2) + P\,(E_3 \cap E_4)$ [By add. Theorem]

$= P\,(E_1)\,P\,(E_2) + P\,(E_3)\,P\,(E_4)$ [By mult. Theorem]

$$= \frac{5}{9} \times \frac{21}{45} + \frac{4}{9} \times \frac{7}{15} = \frac{63}{135} = \frac{7}{15}$$

EXAMPLE 8 *Three groups of children contain 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys respectively. One child is selected at random from each group. Find the chance that the three selected comprise one girl and 2 boys.*

SOLUTION One girl and 2 boys can be selected in the following mutually exclusive ways:

	group 1	group 2	group 3
(I)	girl	boy	boy
(II)	boy	girl	boy
(III)	boy	boy	girl

Thus, if we define G_1, G_2, G_3 as the events of selecting a girl from first, second and third group respectively and B_1, B_2, B_3 as the events of selecting a boy from first, second and third group respectively. Then $B_1, B_2, B_3, G_1, G_2, G_3$ are independent events such that

$$P(G_1) = \frac{3}{4},\ P(G_2) = \frac{2}{4},\ P(G_3) = \frac{1}{4},\ P\,(B_1) = \frac{1}{4},\ P(B_2) = \frac{2}{4},\ P\,(B_3) = \frac{3}{4}$$

Now,

P (1 girl and 2 boys)

$=$ (I or II or III)

$= P\,(\text{I} \cup \text{II} \cup \text{III})$

$= P\,[(G_1 \cap B_2 \cap B_3) \cup (B_1 \cap G_2 \cap B_3) \cup (B_1 \cap B_2 \cap G_3)]$

$= P\,(G_1 \cap B_2 \cap B_3) + P\,(B_1 \cap G_2 \cap B_3) + P\,(B_1 \cap B_2 \cap G_3)$

$= P\,(G_1)\,P\,(B_2)\,P\,(B_3) + P\,(B_1)\,P\,(G_2)\,P\,(B_3) + P\,(B_1)\,P\,(B_2)\,P(G_3)$

$$= \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}$$

EXAMPLE 9 *The probabilities of A, B, C solving a problem are 1/3, 2/7 and 3/8 respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them can solve it.*

SOLUTION Let E_1, E_2, E_3 be the events that the problem is solved by A, B, C respectively. Then,

$$P\,(E_1) = \frac{1}{3},\ P\,(E_2) = \frac{2}{7} \text{ and } P\,(E_3) = \frac{3}{8}.$$

Exactly one of A, B and C can solve the problem in the following mutually exclusive ways :

(I) A solves and B and C do not solve i.e. $E_1 \cap \overline{E}_2 \cap \overline{E}_3$

(II) B solves and A and C do not solve i.e. $\overline{E}_1 \cap E_2 \cap \overline{E}_3$

(III) C solves and A and B do not solve i.e. $\overline{E}_1 \cap \overline{E}_2 \cap E_3$

$\therefore$ Required probabilty

$$= P\text{ (I or II or III)}$$

$$= P[(E_1 \cap \overline{E}_2 \cap \overline{E}_3) \cup (\overline{E}_1 \cap E_2 \cap \overline{E}_3) \cup (\overline{E}_1 \cap \overline{E}_2 \cap E_3]$$

$$= P(E_1 \cap \overline{E}_2 \cap \overline{E}_3) + P(\overline{E}_1 \cap E_2 \cap \overline{E}_3) + P(\overline{E}_1 \cap \overline{E}_2 \cap E_3)$$

$$= P(E_1)\, P(\overline{E}_2)\, P(\overline{E}_3) + P(\overline{E}_1)\, P(E_2)\, P(\overline{E}_3) + P(\overline{E}_1)\, P(\overline{E}_2)\, P(E_3)$$

$$= \frac{1}{3}\left(1-\frac{2}{7}\right)\left(1-\frac{3}{8}\right)+\left(1-\frac{1}{3}\right)\left(\frac{2}{7}\right)\left(1-\frac{3}{8}\right)+\left(1-\frac{1}{3}\right)\left(1-\frac{2}{7}\right)\left(\frac{3}{8}\right)$$

$$= \frac{1}{3}\times\frac{5}{7}\times\frac{5}{8}+\frac{2}{3}\times\frac{2}{7}\times\frac{5}{8}+\frac{2}{3}\times\frac{5}{7}\times\frac{3}{8} = \frac{25}{168}+\frac{5}{42}+\frac{5}{28} = \frac{25}{56}$$

EXAMPLE 10 *Three critics review a book. Odds in favour of the book are 5 : 2, 4 : 3 and 3 : 4 respectively for three critics. Find the probability that the majority are in favour of the book.*

SOLUTION Let A, B, C denote the events that the book will be reviewed favourably by the first, the second and the third critic respectively. Then A, B, C are independent events and we are given that

$$P(A) = \frac{5}{5+2} = \frac{5}{7},\ P(B) = \frac{4}{4+3} = \frac{4}{7} \text{ and } P(C) = \frac{3}{3+4} = \frac{3}{7}.$$

The book will be favourably reviewed by the majority of the reviewers if at least two (out of three) review it favourably. This happens in any one of the following mutually exclusive ways :

(I) 1st favours, 2nd favours and third does not favour i.e. $A \cap B \cap \overline{C}$

(II) 1st favours, 2nd does not favour and third favours i.e. $A \cap \overline{B} \cap C$

(III) 1st does not favour, 2nd favours and third favours i.e. $\overline{A} \cap B \cap C$

(IV) 1st favours, 2nd favours and third also favours i.e. $A \cap B \cap C$

So, required probabilty

$$= P(\text{I} \cup \text{II} \cup \text{III} \cup \text{IV})$$

$$= P(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C) \cup (A \cap B \cap C)$$

$$= P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) + P(A \cap B \cap C)$$

[$\because$ four events are mutually exclusive]

$$= P(A)\, P(B)\, P(\overline{C}) + P(A)\, P(\overline{B})\, P(C) + P(\overline{A})\, P(B)\, P(C) + P(A)\, P(B)\, P(C)$$

[$\because$ A, B and C are independent]

$$= \frac{5}{7}\times\frac{4}{7}\times\frac{4}{7}+\frac{5}{7}\times\frac{3}{7}\times\frac{3}{7}+\frac{2}{7}\times\frac{4}{7}\times\frac{3}{7}+\frac{5}{7}\times\frac{4}{7}\times\frac{3}{7}$$

$$= \frac{80+45+24+60}{343} = \frac{209}{343}$$

EXAMPLE 11 *A, B and C shot to hit a target. If A hits the target 4 times in 5 trials; B hits it 3 times in 4 trials and C hits 2 times in 3 trials; what is the probability that the target is hit by a least 2 persons?*

SOLUTION Let E_1, E_2, E_3 be the events that A hits the target, B hits the target and C hits the target respectively. Then, E_1, E_2, E_3 are independent events such that

$$P(E_1) = \frac{4}{5},\ P(E_2) = \frac{3}{4} \text{ and } P(E_3) = \frac{2}{3}.$$

The target is hit by at least two persons in the following mutually exclusive ways :

(I) A hits, B hits and C does not hit i.e. $E_1 \cap E_2 \cap \overline{E}_3$

(II) A hits, B does not hit and C hits i.e. $E_1 \cap \overline{E}_2 \cap E_3$

(III) A does not hit, B hits and C hits i.e. $\overline{E}_1 \cap E_2 \cap E_3$

(IV) A hits, B hits and C hits i.e. $E_1 \cap E_2 \cap E_3$

So, required probability

$$= P(\text{I} \cup \text{II} \cup \text{III} \cup \text{IV})$$

$$= P[(E_1 \cap E_2 \cap \overline{E}_3) \cup (E_1 \cap \overline{E}_2 \cap E_3) \cup (\overline{E}_1 \cap E_2 \cap E_3) \cup (E_1 \cap E_2 \cap E_3)]$$

$$= P(E_1 \cap E_2 \cap \overline{E}_3) + P(E_1 \cap \overline{E}_2 \cap E_3) + P(\overline{E}_1 \cap \overline{E}_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

[$\because$ Four events are mutually exclusive]

$$= P(E_1)\,P(E_2)\,P(\overline{E}_3) + P(E_1)\,P(\overline{E}_2)\,P(E_3)$$
$$+ P(\overline{E}_1)\,P(E_2)\,P(E_3) + P(E_1)\,P(E_2)\,P(E_3) \qquad [\because E_1, E_2 \text{ and } E_3 \text{ are indep.}]$$

$$= \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} + \frac{4}{5} \times \frac{1}{4} \times \frac{2}{3} + \frac{1}{5} \times \frac{3}{4} \times \frac{2}{3} + \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3}$$

$$= \frac{1}{5} + \frac{2}{15} + \frac{1}{10} + \frac{2}{5} = \frac{5}{6}$$

EXAMPLE 12 *A speaks truth in 60% of the cases and B in 90% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact?* **[CBSE 2003]**

SOLUTION Let E be the event that A speaks truth and F be the event that B speaks truth. Then, E and F are independent events such that

$$P(E) = \frac{60}{100} = \frac{3}{5} \text{ and } P(F) = \frac{90}{100} = \frac{9}{10}$$

A and B will contradict each other in narrating the same fact in the following mutually exclusive ways :

(I) A speaks truth and B tells a lie i.e. $E \cap \overline{F}$

(II) A tells a lie and B speaks truth i.e. $\overline{E} \cap F$

$\therefore$ $P(A \text{ and } B \text{ contradict each other})$

$$= P(\text{I or II}) = P(\text{I} \cup \text{II})$$

$$= P[(E \cap \overline{F}) \cup (\overline{E} \cap F)]$$

$$= P(E \cap \overline{F}) + P(\overline{E} \cap F) \qquad [\because E \cap \overline{F} \text{ and } \overline{E} \cap F \text{ are mutually exclusive}]$$

$$= P(E)\,P(\overline{F}) + P(\overline{E})\,P(F) \qquad [\because E \text{ and } F \text{ are in dep.}]$$

$$= \frac{3}{5} \times \left(1 - \frac{9}{10}\right) + \left(1 - \frac{3}{5}\right) \times \frac{9}{10} = \frac{3}{5} \times \frac{1}{10} + \frac{2}{5} \times \frac{9}{10} = \frac{42}{100}$$

Hence, in 42% cases A and B are likely to contradict each other.

EXAMPLE 13 *The odds against a husband who is 45 years old, living till he is 70 are 7 : 5 and the odds against his wife who is now 36, living till she is 61 are 5 : 3. Find the probability that*

(i) *the couple will be alive 25 years hence,*

(ii) *exactly one of them will be alive 25 years hence,*

(iii) *none of them will be alive 25 years hence,*

(iv) *at least one of them will be alive 25 years hence.*

SOLUTION Let A be the event that the husband will be alive 25 years hence and B be the event that the wife will be alive 25 years hence. Then, A and B are independent events such that

$$P(A) = \frac{5}{7+5} = \frac{5}{12} \text{ and } P(B) = \frac{3}{5+3} = \frac{3}{8}.$$

(i) P (couple will be alive 25 years hence)

$= P(A \text{ and } B) = P(A \cap B)$

$= P(A)\,P(B)$ [$\because$ A and B are independent events]

$= \dfrac{5}{12} \times \dfrac{3}{8} = \dfrac{5}{32}$

(ii) Exactly one of them will be alive 25 years hence in two mutually exclusive ways:

(I) Husband will be alive 25 years hence and wife will not i.e. $A \cap \overline{B}$

(II) Wife will be alive 25 years hence and husband will not i.e. $\overline{A} \cap B$

$\therefore$ P (Exactly one will be alive 25 years hence)

$= P$ (I or II)

$= P(A \cap \overline{B}) \cup (\overline{A} \cap B)$

$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$ [$\because$ $A \cap \overline{B}$ and $\overline{A} \cap B$ are mutually exclusive]

$= P(A)\,P(\overline{B}) + P(\overline{A})\,P(B)$ [$\because$ A, B are independent events]

$= \dfrac{5}{12} \times \left(1 - \dfrac{3}{8}\right) + \left(1 - \dfrac{5}{12}\right) \times \dfrac{3}{8}$

$= \dfrac{5}{12} \times \dfrac{5}{8} + \dfrac{7}{12} \times \dfrac{3}{8} = \dfrac{46}{96} = \dfrac{23}{48}$

(iii) P (None of them will be alive 25 years hence)

$= P(\overline{A} \cap \overline{B})$

$= P(\overline{A})\,P(\overline{B})$ [$\because$ A, B are independent events]

$= \left(1 - \dfrac{5}{12}\right) \times \left(1 - \dfrac{3}{8}\right) = \dfrac{7}{12} \times \dfrac{5}{8} = \dfrac{35}{96}$

(iv) P (At least one of them will be alive 25 years hence)

$= 1 - P(\overline{A})\,P(\overline{B})$ [$\because$ $A, B,$ are independent events]

$= 1 - \left(1 - \dfrac{5}{12}\right)\left(1 - \dfrac{3}{8}\right) = 1 - \dfrac{35}{96} = \dfrac{61}{96}$

EXAMPLE 14 *A bag contains 3 white, 3 black and 2 red balls. One by one, three balls are drawn without replacing them. Find the probability that the third ball is red.*

SOLUTION Let O_i be the event of drawing a ball other than a red ball in ith draw and Ri be the event of drawing a red ball in ith draw $(1 \leq i \leq 3)$.

A red ball can be drawn in third draw in the following mutually exclusive ways :

(I) First draw gives an other colour ball, second draw gives an other colour ball and the third draw gives a red ball i.e. $O_1 \cap O_2 \cap R_3$.

(II) First draw gives a red ball, second draw gives other colour ball and the third draw gives a red ball i.e. $R_1 \cap O_2 \cap R_3$

(III) First draw gives an other colour ball, second draw gives a red ball and the third draw gives a red ball i.e. $O_1 \cap R_2 \cap R_3$

$\therefore$ P (Third ball is red)

$= P$ (I or II or III)

$= P(\text{I} \cup \text{II} \cup \text{III})$

$= P[(O_1 \cap O_2 \cap R_3) \cup (R_1 \cap O_2 \cap R_3) \cup (O_1 \cap R_2 \cap R_3)]$

$= P(O_1 \cap O_2 \cap R_3) + P(R_1 \cap O_2 \cap R_3) + P(O_1 \cap R_2 \cap R_3)$

[$\because$ Events are mutually exclusive]

$$= P(O_1)\,P(O_2/O_1)\,P(R_3/O_1 \cap O_2) + P(R_1)\,P(O_2/R_1)\,P(R_3/R_1 \cap O_2) + P(O_1)\,P(R_2/O_1)\,P(R_3/O_1 \cap R_2)$$

$$= \frac{6}{8}\times\frac{5}{7}\times\frac{2}{6}+\frac{2}{8}\times\frac{6}{7}\times\frac{1}{6}+\frac{6}{8}\times\frac{2}{7}\times\frac{1}{6}$$

$$= \frac{5}{28}+\frac{1}{28}+\frac{1}{28} = \frac{1}{4}$$

EXAMPLE 15 *The probability of student A passing an examination is 3/7 and of student B passing is 5/7. Assuming the two events "A passes", "B passes", as independent, find the probability of :*

(i) *Only A passing the examination*

(ii) *Only one of them passing the examination*

SOLUTION Consider the following events :

$E_1 = A$ passes the examination, $E_1 = B$ passes the examination.

We have, $P(E_1) = \frac{3}{7}$ and $P(E_2) = \frac{5}{7}$

(i) Required probability

$= P(E_1 \cap \overline{E}_2)$

$= P(E_1)\,P(\overline{E}_2)$ $\left[\begin{array}{l}\because E_1 \text{ and } E_2 \text{ are independent}\\ \therefore E_1 \text{ and } E_2 \text{ are also independent}\end{array}\right]$

$= \frac{3}{7}\left(1-\frac{5}{7}\right) = \frac{6}{49}$

(ii) Required probability

$= P[(E_1 \cap \overline{E}_2) \cup (\overline{E}_1 \cap E_2)]$

$= P(E_1 \cap \overline{E}_2) + P(\overline{E}_1 \cap E_2)$ $\left[\begin{array}{l}\because E_1 \cap \overline{E}_2 \text{ and } \overline{E}_1 \cap E_2 \text{ are}\\ \text{mutually exclusive events}\end{array}\right]$

$= P(E_1)\,P(\overline{E}_2) + P(\overline{E}_1)\,P(E_2)$ $\left[\begin{array}{l}\because E_1 \text{ and } \overline{E}_2;\ \overline{E}_1 \text{ and } E_2 \text{ are}\\ \text{are independent events}\end{array}\right]$

$= \frac{3}{7}\left(1-\frac{5}{7}\right)+\left(1-\frac{3}{7}\right)\frac{5}{7} = \frac{26}{49}$

EXAMPLE 16 *There are three urns A, B and C. Urn A contains 4 white balls and 5 blue balls. Urn B contains 4 white balls and 3 blue balls. Urn C contains 2 white balls and 4 blue balls. One ball is drawn from each of these urns. What is the probability that out of these three balls drawn, two are white balls and one is a blue ball?*

SOLUTION Consider the following events:

E_1 = Ball drawn from urn A is white,

E_2 = Ball drawn from urn B is white,

E_3 = Ball drawn from urn C is white

Then, $P(E_1) = \frac{4}{9}$, $P(E_2) = \frac{4}{7}$ and $P(E_3) = \frac{2}{6} = \frac{1}{3}$

$\therefore$ $P(\overline{E}_1)$ = Ball drawn from urn A is blue $= 1 - P(E_1) = 1 - \frac{4}{9} = \frac{5}{9}$

$P(\overline{E}_2)$ = Ball drawn from urn B is blue $= 1 - P(E2) = 1 - \frac{4}{7} = \frac{3}{7}$

and, $P(\overline{E}_3)$ = Ball drawn from urn C is blue $= 1 - P(E_3) = 1 - \frac{1}{3} = \frac{2}{3}$

Now, two white balls and one blue ball can be drawn in the following mutually exclusive ways:

(I) White from urn$A$, white from urn B and blue from urn C i.e. $E_1 \cap E_2 \cap \overline{E}_3$

(II) White from urn A, blue from urn B and white from urn C i.e. $E_1 \cap \overline{E}_2 \cap E_3$

(III) Blue from urn A, white from urn B and white from urn C i.e. $\overline{E}_1 \cap E_2 \cap E_3$

$\therefore$ Required probability

$= P(\text{I}) + P(\text{II}) + P(\text{III})$

$= P(E_1 \cap E_2 \cap \overline{E}_3) + P(E_1 \cap \overline{E}_2 \cap E_3) + P(\overline{E}_1 \cap E_2 \cap E_3)$

$= P(E_1)\, P(E_2)\, P(\overline{E}_3) + P(E_1)\, P(\overline{E}_2)\, P(E_3) + P(\overline{E}_1)\, P(E_2)\, P(E_3)$

[$\because$ E_1, E_2, E_3 are independent events]

$$= \frac{4}{9} \times \frac{4}{7} \times \frac{2}{3} + \frac{4}{9} \times \frac{3}{7} \times \frac{1}{3} + \frac{5}{9} \times \frac{4}{7} \times \frac{1}{3} = \frac{64}{189}.$$

EXAMPLE 17 *A certain team wins with probability 0.7, loses with probability 0.2 and ties with probability 0.1 the team plays three games. Find the probability that the team wins at least two of the games, but not lose.*

SOLUTION Let W_i, L_i and D_i ; $i = 1, 2, 3$ denote respectively the events that the team wins, loses and ties the ith game. Then,

$P(W_i) = 0.7, P(L_i) = 0.2$ and $P(D_i) = 0.1; i = 1, 2, 3$

Now, Required probability

$= P$ (Team wins at least two games and does not lose any game)

$= [(W_1 \cap W_2 \cap D_3) \cup (W_1 \cap D_2 \cap W_3) \cup (D_1 \cap W_2 \cap W_3) \cup (W_1 \cap W_2 \cap W_3)]$

$= P(W_1 \cap W_2 \cap D_3) + P(W_1 \cap D_2 \cap W_3) + P(D_1 \cap W_2 \cap W_3) + P(W_1 \cap W_2 \cap W_3)$

$= P(W_1)\, P(W_2)\, P(D_3) + P(W_1)\, P(D_2)\, P(W_3) + P(D_1)\, P(W_2)\, P(W_3)$
$+ P(W_1)\, P(W_2)\, P(W_3)$

$= (0.7)(0.7) \times 0.1 + (0.7) \times (0.1) \times (0.7) + (0.1) \times (0.7) \times (0.7) + (0.7)(0.7)(0.7)$

$= (0.049) \times 3 + 0.343 = 0.49$

EXAMPLE 18 *A clerk was asked to mail three report cards to three students. He addresses three envelopes but unfortunately paid no attention to which report card be put in which envelope. What is the probability that exactly one of the students received his or her own card?*

SOLUTION Consider the following events :

A = First report card is put in the correct envelope.

B = Second report card is put in the correct envelope.

C = Third report card is put in the correct envelope.

We have,

$$P(A) = P(B) = P(C) = \frac{1}{3}.$$

Required probability

$= P$ [Exactly one of the report cards is put in the correct envelope]

$= P(A) + P(B) + P(C) - 2[P(A \cap B) + P(B \cap C) + P(C \cap A)] + 3P(A \cap B \cap C)$

$= 3P(A) - 2[P(A)P(B/A) + P(B)P(C/B) + P(A)P(C/A)]$

$+ 3P(A)P(B/A)P(C/A \cap B)$

$= 3 \times \frac{1}{3} - 2\left[\frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}\right] + 3 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{2} \times 1$

$= 1 - 2 \times \frac{1}{2} + \frac{1}{2} = \frac{1}{2}$

ALITER Required probability

$= P[(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)]$

$= P(A \cap \overline{B} \cap \overline{C}) + (\overline{A} \cap B \cap \overline{C}) + P(\overline{A} \cap \overline{B} \cap C)]$

$= P(A)P(\overline{B}/A)P(\overline{C}/A \cap \overline{B}) + P(B)P(\overline{A}/B)P\overline{(C/B \cap \overline{A})} + P(C)P\overline{(A/C)}P(\overline{B}/\overline{A} \cap C)$

$= \frac{1}{3} \times \frac{1}{2} \times 1 + \frac{1}{3} \times \frac{1}{2} \times 1 + \frac{1}{3} \times \frac{1}{2} \times 1$

$= \frac{3}{6} = \frac{1}{2}.$

EXAMPLE 19 *Neelam is taking up subjects Mathematics, Physics and Chemistry. She estimates that her probabilities of receiving grade A in these courses are 0.2, 0.3 and 0.9 respectively. If the grades can be regarded as independent events, find the probabilities that she receives.*
(i) *All A's* (ii) *No A's* (iii) *Exactly two A's*

SOLUTION Consider the following events :

E = Neelam receives grade A in Mathematics

F = Neelam receives grade A in Physics

G = Neelam receives grade A in Chemistry.

Then, $P(E) = 0.2$, $P(F) = 0.3$ and $P(G) = 0.9$

(i) Required probability

$= P(E \cap F \cap G)$

$= P(E)P(F)P(G)$ [$\because$ E, F, G, are independent events]

$= 0.2 \times 0.3 \times 0.9 = 0.054$

(ii) Required probability

$= P(\overline{E} \cap \overline{F} \cap \overline{G})$

$= P(\overline{E})P(\overline{F})P(\overline{G})$ [$\because$ E, F, G, are independent events]

$= 0.8 \times 0.7 \times 0.1 = 0.056$

(iii) Required probability

$= P[(E \cap F \cap \overline{G}) \cup (\overline{E} \cap F \cap G) \cup (E \cap \overline{F} \cap G)]$

$= P(E \cap F \cap \overline{G}) + P(\overline{E} \cap F \cap G) + P(E \cap \overline{F} \cap G)$

$$= P(E)\,P(F)\,P(\overline{G}) + P(\overline{E})\,P(F)\,P(G) + P(E)\,P(\overline{F})\,P(G)$$

$$= 0.2 \times 0.3 \times 0.1 + 0.8 \times 0.3 \times 0.9 + 0.2 \times 0.7 \times 0.9$$

$$= 0.006 + 0.216 + 0.126 = 0.348.$$

EXAMPLE 20 *A doctor claims that 60% of the patients he examines are allergic to some type of weed. What is the probability that (i) exactly 3 of his next 4 patients are allergic to weeds? (ii) none of his next 4 patients is allergic to weeds?*

SOLUTION Consider the following events :

A = First patient is allergic to weeds

B = Second patient is allergic to weeds

C = Third patient is allergic to weeds

D = Fourth patient is allergic to weeds

Clearly, A, B, C, D are independent events such that

$$P(A) = P(B) = P(C) = P(D) = \frac{60}{100} = \frac{3}{5}$$

(i) Required probability

$$= P[(A \cap B \cap C \cap \overline{D})] \cup (A \cap B \cap \overline{C} \cap D) \cup (A \cap \overline{B} \cap C \cap D) \cup (\overline{A} \cap B \cap C \cap D$$

$$= P(A \cap B \cap C \cap \overline{D}) + P(A \cap B \cap \overline{C} \cap D) + P(A \cap \overline{B} \cap C \cap D) + P(\overline{A} \cap B \cap C \cap$$

$$= P(A)\,P(B)\,P(C)\,P(\overline{D}) + P(A)\,P(B)\,P(\overline{C})\,P(D) + P(A)\,P(\overline{B})\,P(C)\,P(D) + P(\overline{A})\,P(B)\,P(C)\,P(D)$$

$$= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} + \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5} + \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{216}{625}$$

(ii) Required probability

$$= P(\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D})$$

$$= P(\overline{A})\,P(\overline{B})\,P(\overline{C})\,P(\overline{D}) = \left(\frac{2}{5}\right)^4 = \frac{16}{625}$$

EXAMPLE 21 *If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assuming that the individual entries of the determinant are chosen independently, each value being assumed with probability $\frac{1}{2}$).*

SOLUTION Let the given determinant be $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, where $a_{ij} = 0$ or 1; $i, j = 1, 2$

We observe that $\Delta \leq 0$, if $a_{11} = 0$ or $a_{22} = 0$. Therefore,

$$\text{neither } a_{11} = 0 \text{ nor } a_{22} = 0 \Rightarrow a_{11} = a_{22} = 1$$

Also, when $a_{11} = a_{22} = 1$, we observe that $\Delta = 0$, if $a_{12} = a_{21} = 1$.

Thus. we must have

$$a_{11} = a_{22} = 1 \text{ and } a_{12} \neq 1, a_{21} \neq 1$$

So, we have the following possibilities:

$$a_{11} = a_{22} = 1,\ a_{12} = 1,\ a_{21} = 0$$

$a_{11} = a_{22} = 1,\ a_{12} = 0,\ a_{21} = 1$

$a_{11} = a_{22} = 1,\ a_{12} = 0,\ a_{21} = 0$

$\therefore$ Required probability $= P(a_{11} = a_{22} = 1, a_{12} = 1, a_{21} = 0)$

$$+ P(a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = 1)$$

$$+ P(a_{11} = a_{22} = 1, a_{12} = a_{21} = 0)$$

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$

$$= \frac{3}{16}$$

EXAMPLE 22 *An electric system has open-closed switches S_1, S_2 and S_3 as shown in Fig. 30.8. The switches operate independently of one another and the current will flow from A to B either if S_1 is closed or if both S_2 and S_3 are closed. If $P(S_1) = P(S_2) = P(S_3) = \frac{1}{2}$, find the probability that the circuit will work.*

SOLUTION Required probability

$$= P\left(S_1 \cup (S_2 \cap S_3)\right)$$

$$= P(S_1) + P(S_2 \cap S_3) - P\left(S_1 \cap (S_2 \cap S_3)\right)$$

$$= P(S_1) + P(S_2 \cap S_3) - P(S_1 \cap S_2 \cap S_3)$$

$$= P(S_1) + P(S_2)\,P(S_3) - P(S_1)\,P(S_2)\,P(S_3)$$

$$= \frac{1}{2}+\frac{1}{2}\times\frac{1}{2}-\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$

$$= \frac{1}{2}+\frac{1}{4}-\frac{1}{8}$$

$$= \frac{5}{8}$$

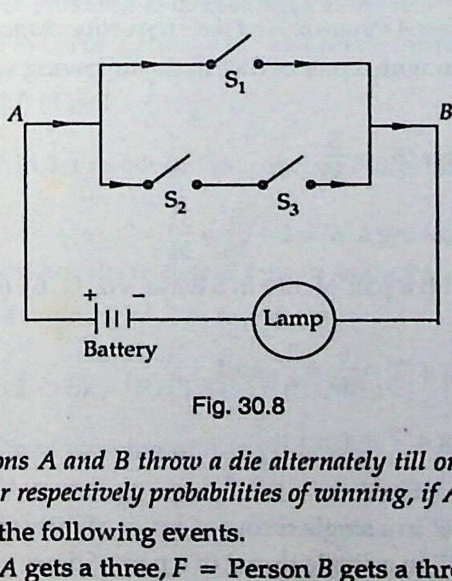

Fig. 30.8

EXAMPLE 23 *Two persons A and B throw a die alternately till one of them gets a 'three' and wins the game. Find their respectively probabilities of winning, if A begins.*

SOLUTION We define the following events.

E = Person A gets a three, F = Person B gets a three. Clearly,

$$P(E)=\frac{1}{6},\ P(F)=\frac{1}{6},\ P(\bar{E})=1-\frac{1}{6}=\frac{5}{6} \text{ and } P(\bar{F})=1-\frac{1}{6}=\frac{5}{6}$$

A wins if he throws a 'three' in 1st or 3rd or 5th ... throws.

His probability of throwing a 'three' in first throw $= P(E)=\frac{1}{6}$

A will get third throw if he fails in first and B fails in second throw.

$\therefore$ Probability of winning of A in third throw

$$= P(\bar{E}\cap\bar{F}\cap E)=P(\bar{E})\,P(\bar{F})\,P(E)=\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\left(\frac{5}{6}\right)^2\times\frac{1}{6}$$

Similarly, we have

Probability of winning of A in fifth throw

$$= P(\bar{E}\cap\bar{F}\cap\bar{E}\cap\bar{F}\cap E)$$
$$= P(\bar{E})\,P(\bar{F})\,P(\bar{E})\,P(\bar{F})\,P(E)$$
$$= (P(\bar{E}))^2\,(P(\bar{F}))^2\,P(E)=\left(\frac{5}{6}\right)^4\times\frac{1}{6}$$

and so on.

Hence, probability of winning of A

$$= P[E\cup(\bar{E}\cap\bar{F}\cap E)\cup(\bar{E}\cap\bar{F}\cap\bar{E}\cap\bar{F}\cap E)\cup \ldots]$$
$$= P(E)+P(\bar{E}\cap\bar{F}\cap E)+P(\bar{E}\cap\bar{F}\cap\bar{E}\cap\bar{F}\cap E)+\ldots$$
$$= \frac{1}{6}+\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}+\left(\frac{5}{6}\right)^4\cdot\frac{1}{6}+\ldots$$
$$= \frac{1/6}{1-(5/6)^2}=\frac{6}{11} \qquad \left[\because\ a+ar+ar^2+\ldots=\frac{a}{1-r}\right]$$

Thus, probability of winning of B = $1-$ Probability of winning of A = $1-\frac{6}{11}=\frac{5}{11}$

EXAMPLE 24 *A and B throw alternately a pair of dice. A wins if he throws 6 before B throws 7 and B wins if the throws 7 before A throws 6. Find their respective chance of winning, if A begins.*

SOLUTION 6 can be thrown with a pair of dice in the following ways: (1, 5), (5, 1), (4, 2), (2, 4), (3, 3).

So, probability of throwing a '6' $=\frac{5}{36}$

and, probability of not throwing a '6' $=1-\frac{5}{36}=\frac{31}{36}$.

Now, 7 can be thrown with a pair of dice in 6 ways, viz. (1, 6), (6, 1), (2, 5), (5, 2), (4, 3), (3, 4).

So, probability of throwing a '7' $=\frac{6}{36}=\frac{1}{6}$ and

probability of not throwing a '7' $=1-\frac{1}{6}=\frac{5}{6}$

Let E and F be two events defined as:

E = throwing a '6' in a single throw of a pair of dice, and

F = throwing a '7' in a single throw of a pair of dice

Then, $P(E) = \frac{5}{36}$, $P(\overline{E}) = \frac{31}{36}$, $P(F) = \frac{1}{6}$ and $P(\overline{F}) = \frac{5}{6}$

A wins if he throws '6' in 1st or 3rd or 5th ... throws.

Probability of A throwing a '6' in first throw $= P(E) = \frac{5}{36}$

A will get third if he fails in first and B fails in second throw.

$\therefore$ Probability of A throwing a '6' in third throw

$$P(\overline{E} \cap \overline{F} \cap E) = P(\overline{E})\, P(\overline{F})\, P(E) = \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36}$$

Similarly, probability of A throwing a '6' in fifth throw

$$= P(\overline{E} \cap \overline{F} \cap \overline{E} \cap \overline{F} \cap E)$$

$$= P(\overline{E})\, P(\overline{F})\, P(\overline{E})\, P(\overline{F})\, P(E)$$

$$= \left(\frac{31}{36}\right)^2 \times \left(\frac{5}{6}\right)^2 \times \frac{5}{36} \text{ and so on}$$

Hence, probability of winning of A

$$= P(E \cup (\overline{E} \cap \overline{F} \cap E) \cup (\overline{E} \cap \overline{F} \cap \overline{E} \cap \overline{F} \cap E) \cup \ldots)$$

$$= P(E) + P(\overline{E} \cap \overline{F} \cap E) + P(\overline{E} \cap \overline{F} \cap \overline{E} \cap \overline{F} \cap E) + \ldots$$

$$= \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right) \times \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)^2 \times \frac{5}{36} + \ldots$$

$$= \frac{5/36}{1 - (31/36) \times (5/6)} = \frac{30}{61}.$$

Thus, probability of winning of $B = 1 - \frac{30}{61} = \frac{31}{61}$.

EXAMPLE 25 *Three persons A, B, C throw a die in succession till one gets a 'six' and wins the game. Find their respective probabilities of winning, if A begins.*

SOLUTION Let E be the event of 'getting a six' in a single throw of an unbiased die. Then,

$$P(E) = \frac{1}{6} \text{ and } P(\overline{E}) = 1 - \frac{1}{6} = \frac{5}{6}.$$

A wins if he gets a 'six' in 1st or 4th or 7th ... throw. His probability of getting a 'six' in first throw $= P(E) = \frac{1}{6}$.

A will get fourth throw if he fails in first, B fails in second and C fails in third throw.

$\therefore$ Probability of winning of A in fourth throw

$$= P(\overline{E} \cap \overline{E} \cap \overline{E} \cap E) = P(\overline{E})\, P(\overline{E})\, P(\overline{E})\, P(E) = \left(\frac{5}{6}\right)^3 \times \frac{1}{6}$$

Similarly, the probability of winning of A in 7th throw

$$= \left(\frac{5}{3}\right)^6 \times \frac{1}{6} \text{ and so on.}$$

Hence, probability of winning of A

$$= \frac{1}{6} + \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^6 \cdot \frac{1}{6} + \ldots = \frac{1/6}{1-(5/6)^3} = \frac{36}{91}$$

B wins if he gets a 'six' in 2nd or 5th or 8th or ... throw.

$\therefore$ Probability of winning of $B = \left(\frac{5}{6}\right)\frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \left(\frac{5}{6}\right)^7 \frac{1}{6} \ldots$

$$= \frac{\left(\frac{5}{6}\right)\frac{1}{6}}{1-\left(\frac{5}{6}\right)^3} = \frac{30}{91}$$

Hence, probability of winning of $C = 1 - \left(\frac{36}{91} + \frac{30}{91}\right) = \frac{25}{91}$

EXERCISE 30.7

1. A bag contains 6 black and 3 white balls. Another bag contains 5 black and 4 white balls. If one ball is drawn from each bag, find the probability that these two balls are of the same colour.
2. A bag contains 3 red and 5 black balls and a second bag contains 6 red and 4 black balls. A ball is drawn from each bag. Find the probability that one is red and the other is black.
3. A bag contains 4 white and 5 black balls and another bag contains 3 white and 4 black balls. A ball is taken out from the first bag and without seeing its colour is put in the second bag. A ball is taken out from the latter. Find the probability that the ball drawn is white.
4. One bag contains 4 white and 5 black balls. Another bag contains 6 white and 7 black balls. A ball is transferred from first bag to the second bag and then a ball is drawn from the second bag. Find the probability that the ball drawn is white.
5. A speaks truth in 75% and B in 80% of the cases. In what percentage of cases are they likely to contradict each other in narrating the same incident?
6. Kamal and Monica appeared for an interview for two vacancies. The probability of Kamal's selection is 1/3 and that of Monika's selection is 1/5. Find the probability that
   (i) both of them will be selected (ii) none of them will be selected
   (iii) at least one of them will be selected (iv) only one of them will be selected.
7. A bag contains 3 white, 4 red and 5 black balls. Two balls are drawn one after the other, without replacement. What is the probability that one is white and the other is black?
8. A bag contains 8 red and 6 green balls. Three balls are drawn one after another without replacement. Find the probability that at least two balls drawn are green.
9. Arun and Tarun appeared for an interview for two vacancies. The probability of Arun's selection is 1/4 and that of Tarun's rejection is 2/3. Find the probability that at least and of them will be selected.
10. An urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two are drawn from first urn and put into the second urn and then a ball is drawn from the latter. Find the probability that it is a white ball.

11. Two cards are drawn from a well shuffled pack of 52 cards, one after another without replacement. Find the probability that one of these is red card and the other a black card?

12. Tickets are numbered from 1 to 10. Two tickets are drawn one after the other at random. Find the probability that the number on one of the tickets is a multiple of 5 and on the other a multiple of 4.

13. In a family, the husband tells a lie in 30% cases and the wife in 35% cases. Find the probability that both contradict each other on the same fact.

14. A husband and wife appear in an interview for two vacancies in the same post. The probability of husband's selection is 1/7 and that of wife's selection is 1/5. What is the probability that
    (a) both of them will be selected?
    (b) only one of them will be selected?
    (c) none of them will be selected?

15. A bag contains 7 white, 5 black and 4 red balls. Four balls are drawn without replacement. Find the probability that at least three balls are black.

16. A, B, and C are independent witness of an event which is known to have occurred. A speaks the truth three times out of four, B four times out of five and C five times out of six. What is the probability that the occurrence will be reported truthfully by majority of three witnesses?

17. A bag contains 4 white balls and 2 black balls. Another contains 3 white balls and 5 black balls. If one ball is drawn from each bag, find the probability that
    (i) both are white
    (ii) both are black
    (iii) one is white and one is black

18. A bag contains 4 white, 7 black and 5 red balls. 4 balls are drawn with replacement. What is the probability that at least two are white?

19. Three cards are drawn with replacement from a well shuffled pack of cards. Find the probability that the cards are a king, a queen and a jack.

20. A bag contains 4 red and 5 black balls, a second bag contains 3 red and 7 black balls. One ball is drawn at random from each bag; find the probability that the (i) balls are of different colours (ii) balls are of the same colour.

21. A can hit a target 3 times in 6 shots, B : 2 times in 6 shots and C : 4 times in 4 shots. They fix a volley. What is the probability that at least 2 shots hit?

22. The probability of student A passing an examination is 2/9 and of student B passing is 5/9. Assuming the two events : 'A passes', 'B passes as independent, find the probability of : (i) only A passing the examination (ii) only one of them passing the examination.

23. There are three urns A, B, and C. Urn A contains 4 red balls and 3 black balls. Urn B contains 5 red balls and 4 black balls. Urn C contains 4 red and 4 black balls. One balls is drawn from each of these urns. What is the probability that 3 balls drawn consist of 2 red balls and a black ball?

24. A bag contains 6 red and 8 black balls and another bag contains 8 red and 6 black balls. A ball is drawn from the first bag and without noticing its colour is put in the second bag. A ball is drawn from the second bag. Find the probability that the ball drawn is red in colour.

25. A and B take turns in throwing two dice, the first to throw 9 being awarded the prize. Show that their chance of winning are in the ratio 9 : 8.

26. A, B and C in order toss a coin. The one to throw a head wins. What are their respective chances of winning assuming that the game may continue indefinitely?

27. Three persons A, B, C throw a die in succession till one gets a 'six' and wins the game. Find their respective probabilities of winning.

28. A and B take turns in throwing two dice, the first to throw 10 being awarded the prize, show that if A has the first throw, their chance of winning are in the ratio 12 : 11.

29. There are 3 red and 5 black balls in bag 'A'; and 2 red and 3 black balls in bag 'B'. One ball is drawn from bag 'A' and two from bag 'B'. Find the probability that out of the 3 balls drawn one is red and 2 are black.

30. Fatima and John appear in an interview for two vacancies in the same post. The probability of Fatima's selection is $\frac{1}{7}$ and that of John's selection is $\frac{1}{5}$. What is the probability that
    (i) both of them will be selected?
    (ii) only one of them will be selected?
    (iii) none of them will be selected?

31. A bag contains 8 marbles of which 3 are blue and 5 are red. One marble is drawn at random, its colour is noted and the marble is replaced in the bag. A marble is again drawn from the bag and its colour is noted. Find the probability that the marble will be
    (i) blue followed by red.
    (ii) blue and red in any order.
    (iii) of the same colour.

32. An urn contains 7 red and 4 blue balls. Two balls are drawn at random with replacement. Find the probability of getting.
    (i) 2 red balls
    (ii) 2 blue balls
    (iii) One red and one blue ball.

33. A card is drawn from a well-shuffled deck of 52 cards. The outcome is noted, the card is replaced and the deck reshuffled. Another card is then drawn from the deck.
    (i) What is the probability that both the cards are of the same suit?
    (ii) What is the probability that the first card is an ace and the second card is a red queen?

34. Two balls are drawn at random with replacment from a box containing 10 black and 8 red balls. find the probability that (i) both the balls are red. (ii) first ball is black and second is red. (iii) one of them is black and other is red. **[CBSE 2005]**

35. Two cards are drawn successively without replacement from a well-shuffled deck of cards. Find the probability of exactly one ace.

36. A and B toss a coin alternately till one of them gets a head and wins the game. If A starts the game, find the probability that B will win the game.

37. X is taking up subjects – Mathematics, Physics and Chemistry in the examination. His probabilities of getting grade A in these subjects are 0.2, 0.3 and 0.5 respectively. Find the probability that he gets
    (i) Grade A in all subjects (ii) Grade A in no subject
    (iii) Grade A in two subjects. **[CBSE 2005]**

38. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among 100 students, what is the probability that: (i) you both enter the same section? (ii) you both enter the different section? **[CBSE 2008]**

ANSWERS

1. $\frac{14}{27}$	2. $\frac{21}{40}$	3. $\frac{31}{32}$	4. $\frac{29}{63}$	5. 35%
6. (i) $\frac{1}{15}$	(ii) $\frac{8}{15}$	(iii) $\frac{7}{15}$	(iv) $\frac{2}{5}$	7. $\frac{5}{22}$
8. $\frac{5}{13}$	9. $\frac{1}{2}$	10. $\frac{59}{130}$	11. $\frac{26}{51}$	12. $\frac{4}{45}$
13. 0.44	14. (i) $\frac{1}{35}$	(ii) $\frac{2}{7}$	(iii) $\frac{24}{35}$	15. $\frac{23}{364}$
16. $\frac{107}{120}$	17. (i) $\frac{1}{4}$	(ii) $\frac{5}{24}$	(iii) $\frac{13}{24}$	18. $\frac{67}{256}$
19. $\frac{6}{2197}$	20. (i) $\frac{43}{90}$	(ii) $\frac{47}{90}$	21. $\frac{2}{3}$	
22. (i) 8/81	(ii) 43/81	23. $\frac{17}{42}$	24. $\frac{59}{105}$	26. $\frac{4}{7}, \frac{2}{7}, \frac{1}{7}$
27. $\frac{36}{91}, \frac{30}{91}, \frac{25}{91}$	29. $\frac{39}{80}$	30. (i) $\frac{1}{35}$	(ii) $\frac{2}{7}$	(iii) $\frac{24}{35}$
31. (i) $\frac{15}{64}$	(ii) $\frac{15}{32}$	(iii) $\frac{17}{32}$	32. (i) $\frac{49}{121}$	(ii) $\frac{16}{121}$ (iii) $\frac{56}{121}$
33. (i) $\frac{1}{4}$	(ii) $\frac{1}{338}$	34. (i) $\frac{16}{81}$	(ii) $\frac{20}{81}$	(iii) $\frac{40}{81}$
35. $\frac{32}{221}$	36. $\frac{1}{3}$	37. (i) 0.03	(ii) 0.28	(iii) 0.22
38. (i) $\frac{13}{25}$	(ii) $\frac{12}{25}$			

HINTS TO SELECTED PROBLEMS

1. Let consider the following events :

B_i = getting a black ball from i^{th} bag

W_i = getting a white ball from i^{th} bag

$\frac{8}{18} \times \frac{7}{10} = 56$

$\frac{10}{18} \times \frac{8}{18}$

So, required probability $= P\left((B_1 \cap B_2)\right) \cup \left((W_1 \cap W_2)\right)$

$= P(B_1 \cap B_2) + P(W_1 \cap W_2)$

$= P(B_1)\, P(B_2) + P(W_1)\, P(W_2)$

6. Consider the following events : K = Kamal is selected M = Monica is selected. Then,

$P(K) = 1/3$, $P(M) = 1/5$.

$0.8 \times 0.7 \times 0.5$

40×7

(i) Required probability $= P(K \cap M) = P(K)\, P(M)$

(ii) Required probability $= P(\overline{K} \cap \overline{M}) = P(\overline{K})\, P(\overline{M})$

(iii) Required probability $= 1 - P(\overline{K})\, P(\overline{M})$

(iv) Required probability $= P(K \cap \overline{M}) + P(\overline{K} \cap M) = P(K)\, P(\overline{M}) + P(\overline{K})\, P(M)$

9. Required probability $= 1 - P(\overline{A})\, P(\overline{T}) = 1 - \left(1 - \frac{1}{4}\right)\left(\frac{2}{3}\right)$

11. Required probability $= P[(R \text{ and } B) \text{ or } (B \text{ and } R)] = P(R \cap B) + P(B \cap R)$

$= P(R)\, P(B/R) + P(B)\, P(R/B) = \dfrac{26}{52} \times \dfrac{26}{51} + \dfrac{26}{52} \times \dfrac{26}{51} = \dfrac{26}{51}$

12. Let, A_i = getting a multiple of 5 in ith draw, B_i = getting a multiple of 4 in i^{th} draw; $i = 1, 2$.

Required probability $= P[(A_1 \cap B_2) \cup P(B_1 \cap A_2)]$

$= P(A_1 \cap B_2) + P(B_1 \cap A_2)$

$= P(A_1)\, P(B_2/A_1) + P(B_1)\, P(A_2/B_1)$

$= \dfrac{2}{10} \times \dfrac{2}{9} + \dfrac{2}{10} \times \dfrac{2}{9}$

20. (i) Required probability $P\Big((R_1 \cap B_2) \cup (B_1 \cap R_2)\Big)$

$= P(R_1) \cap P(B_2) + P(R_2) \cap P(B_1) = P(R_1)\, P(B_2) + P(R_2)\, P(B_1)$

$= \dfrac{4}{9} \times \dfrac{7}{10} + \dfrac{5}{9} \times \dfrac{3}{10}$

(ii) Required probability $= P\Big((R_1 \cap R_2) \cup (B_1 \cap B_2)\Big)$

$= P(R_1 \cap R_2) + P(B_1 \cap B_2) = P(R_1)\, P(R_2) + P(B_1)\, P(B_2)$

$= \dfrac{4}{9} \times \dfrac{3}{10} + \dfrac{5}{9} \times \dfrac{7}{10}$

21. Required probability

$= P[(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C) \cup (A \cap B \cap C)]$

$= P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) + P(A \cap B \cap C)$

$= P(A)\, P(B)\, P(\overline{C}) + P(A)\, P(\overline{B})\, P(C) + P(\overline{A})\, P(B)\, P(C) + P(A)\, P(B)\, P(C)$

30.8 THE LAW OF TOTAL PROBABILITY

THEOREM *(Law of Total Probability) Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then*

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/E_n).$$

PROOF Since $E_1, E_2, \ldots, E_n$ are n mutually exclusive and exhaustive events. Therefore, $S = E_1 \cup E_2 \cup E_3 \ldots \cup E_n$, where $E_i \cap E_j = \phi$ for $i \neq j$.

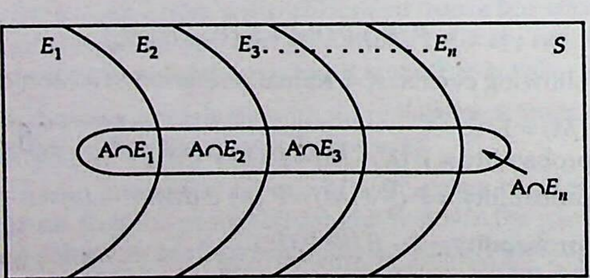

Fig. 30.9

We have,

$$A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_n)$$

$$\therefore \quad P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) \ldots + P(A \cap E_n)$$

[By addition theorem]

But,

$$P(A \cap E_i) = P(E_i)\, P(A/E_i) \quad \text{for } i = 1, 2, \ldots, n$$

Hence,

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/E_n)$$

Q.E.D.

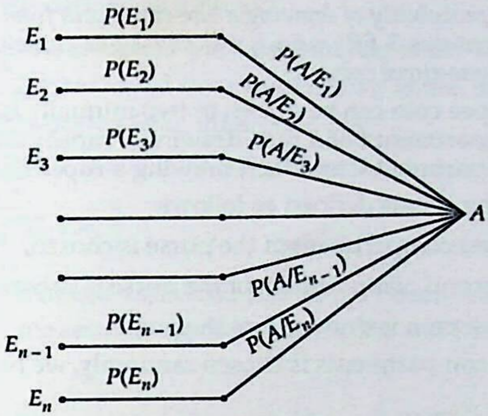

Fig. 30.10

The law of total probability as stated and proved above say that if an event A can occur in n mutually exclusive ways, then the probability of occurrence of A is the sum of the probabilities of various ways as shown in the following tree diagram.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *A bag contains 4 red and 3 black balls. A second bag contains 2 red and 4 black balls. One bag is selected at random. From the selected bag, one ball is drawn. Find the probability that the ball drawn is red.*

SOLUTION A red ball can be drawn in two mutually exclusive ways.

(I) Selecting bag I and then drawing a red ball from it.
(II) Selecting bag II and then drawing a red ball from it.

Let E_1, E_2 and A denote the events defined as follows:

E_1 = Selecting bag I,

E_2 = Selecting bag II,

A = Drawing a red ball

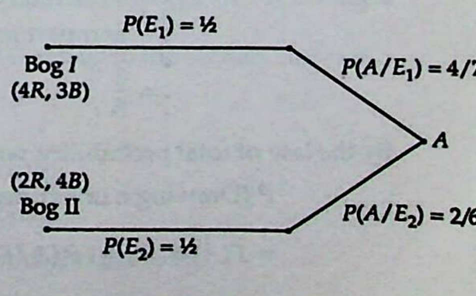

Fig. 30.11

Since one of the two bags is selected randomly.

$$\therefore \quad P(E_1) = \frac{1}{2} \text{ and } P(E_2) = \frac{1}{2}.$$

Now, $P(A/E_1)$ = Probability of drawing a red ball when the first bag has been chosen.

$$= \frac{4}{7} \quad \left[\because \text{First bag contains 4 red and 3 black balls}\right]$$

and, $P(A/E_2)$ = Probability of drawing a red ball when the second bag has been selected

$$= \frac{2}{6} \qquad [\because \text{ Second bag contains 2 red and 4 black balls}]$$

Using the law of total probability, we have

$$P(\text{red ball}) = P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$$

$$= \frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{2}{6} = \frac{19}{42}$$

EXAMPLE 2 *Find the probability of drawing a one-rupee coin from a purse with two compartments one of which contains 3 fifty-paise coins and 2 one-rupee coins and other contains 2 fifty-paise coins and 3 one-rupee coins.*

SOLUTION A one rupee coin can be drawn in two mutually exclusive ways :

(I) Selecting compartment I and then drawing a rupee coin from it.

(II) Selecting compartment II and then drawing a rupee coin from it.

Let E_1, E_2 and A be the events defined as follows:

E_1 = the first compartment of the purse is chosen,

E_2 = the second compartment of the purse is chosen,

A = a rupee coin is drawn from the purse.

Since one of the two compartments is chosen randomly, we have

$$P(E_1) = \frac{1}{2} = P(E_2)$$

Also,

$P(A/E_1)$ = Probability drawing a rupee coin given that the first compartment of the purse is chosen

$$= \frac{2}{5} \qquad \left[\because \begin{array}{l}\text{First compartment contains 3 fifth} \\ \text{paise coins and 2 one rupee coins}\end{array}\right]$$

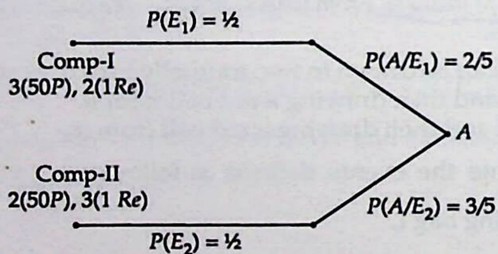

Fig. 30.12

and, $P(A/E_2)$ = Probability of drawing a rupee coin given that the second compartment of the purse is chosen

$$= \frac{3}{5} \qquad \left[\because \begin{array}{l}\text{Second compartment contains 2 fifth} \\ \text{paise coins and 3 one rupee coins}\end{array}\right]$$

By the law of total probability, we have

P (Drawing a one rupee coin)

$$= P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{3}{5} = \frac{1}{2}.$$

EXAMPLE 3 *One bag contains 4 white and 5 black balls. Another bag contains 6 white and 7 black balls. A ball is transferred form first bag to the second bag and then a ball is drawn from the second bag. Find the probability that the ball drawn is white.*

SOLUTION A white ball can be drawn from the second bag in two mutually exclusive ways:

(I) By transferring a white ball from first bag to the second bag and then drawing a white ball from it.

(II) By transferring a black ball from first bag to the second bag and then drawing a white ball from it.

Let E_1, E_2 and A be the events defined as follows :

E_1 = a white ball is transferred from the first bag to the second bag

E_2 = a black ball is transferred from the first bag to the second bag

A = a white ball is drawn from the second bag

Since the first bag contains 4 white and 5 black balls, we have

$$P(E_1) = \frac{4}{9} \text{ and } P(E_2) = \frac{5}{9}$$

If E_1 has already occurred, that is a white ball has already been transferred from first bag to the second bag, then the second bag contains 7 white and 7 black balls.

So, $$P(A/E_1) = \frac{7}{14}$$

If E_2 has already occurred, that is a black ball has been transferred from first bag to the second bag, then the second bag contains 6 white and 8 black balls. So,

$$P(A/E_2) = \frac{6}{14}$$

By the law of total probability, we have

P (Getting a white ball)

$$= P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) = \frac{4}{9}\times\frac{7}{14}+\frac{5}{9}\times\frac{6}{14} = \frac{58}{126} = \frac{29}{63}$$

EXAMPLE 4 *There are two bags. The first bag contains 5 white and 3 black balls and the second bag contains 3 white and 5 black balls. Two balls are drawn at random from the first bag and are put into the second bag without noticing their colours. Then two balls are drawn from the second bag. Find the probability that the balls are white and black.*

SOLUTION A white and a black ball can be drawn from the second bag in the following mutually exclusive ways :

(I) By transferring 2 black balls from first bag to the second bag and then drawing a white and a black ball from it.

(II) By transferring 2 white balls from first bag to the second bag and then drawing a white and a black ball from it.

(III) By transferring one white and one black ball from first bag to the second bag and then drawing a white and a black ball from it.

Let E_1, E_2, E_3 and A be the events as defined below :

E_1 = two black balls are drawn from the first bag,

E_2 = two white balls are drawn from the first bag,

E_3 = one white and one black ball is drawn from the first bag,

A = two balls drawn from the second bag are white and black.

We have, $P(E_1) = \dfrac{{}^3C_2}{{}^8C_2} = \dfrac{3}{28}$, $P(E_2) = \dfrac{{}^5C_2}{{}^8C_2} = \dfrac{5}{14}$, and $P(E_3) = \dfrac{{}^3C_1 \times {}^5C_1}{{}^8C_2} = \dfrac{15}{28}$

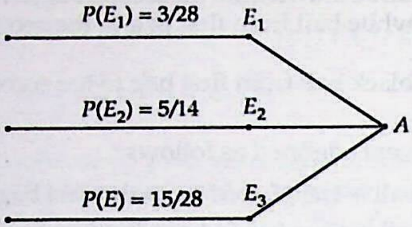

Fig. 30.13

If E_1 has already occurred, that is, if two black balls have been transferred from the first bag to the second bag, then the second bag will contain 3 white and 7 black balls, therefore the probability of drawing a white and a black ball from the second bag is $\dfrac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2}$

$$\therefore \qquad P(A/E_1) = \frac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2} = \frac{7}{15}$$

Similarly, we have

$$P(A/E_2) = \frac{{}^5C_1 \times {}^5C_1}{{}^{10}C_2} = \frac{5}{9} \text{ and } P(A/E_3) = \frac{{}^4C_1 \times {}^6C_1}{{}^{10}C_2} = \frac{8}{15}$$

By the law of total probability, we have

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)$$

$$= \frac{3}{28} \times \frac{7}{15} + \frac{5}{14} \times \frac{5}{9} + \frac{15}{28} \times \frac{8}{15} = \frac{673}{1260}$$

EXAMPLE 5 *A bag contains 6 red and 5 blue balls and another bag contains 5 red and 8 blue balls. A ball is drawn from the first bag and without noticing its colour is put in the second bag. A ball is then drawn from the second bag. Find the probability that the ball drawn is blue in colour.*

SOLUTION A blue colour ball can be drawn from the second bag in the following mutually exclusive ways:

(I) By transferring a blue ball from first bag to the second bag and then drawing a blue ball from the second bag.

(II) By transferring a red ball from first bag to the second bag and then drawing a blue ball from the second bag.

Let E_1, E_2 and A be the events defined as follows:

E_1 = ball drawn from first bag is blue

E_2 = ball drawn from first bag is red

A = ball drawn from the second bag is blue

Since first bag contains 6 red and 5 blue balls, we have

$$P(E_1) = \frac{5}{11} \text{ and } P(E_2) = \frac{6}{11}$$

If E_1 has already occurred, that is, if a blue ball is transferred from the first bag to the second bag, then the second bag contains 5 red and 9 blue balls, therefore the probability of drawing a blue ball from the second bag is $\frac{9}{14}$.

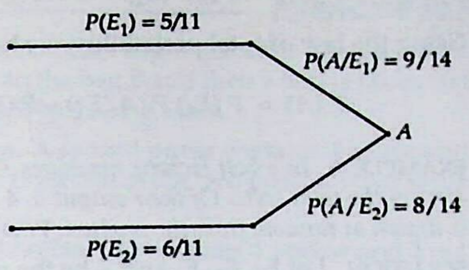

Fig. 30.14

$\therefore \quad P(A/E_1) = \frac{9}{14}$

Similarly, we have $P(A/E_2) = \frac{8}{14}$

By the law of total probability, we have

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) = \frac{5}{11} \times \frac{9}{14} + \frac{6}{11} \times \frac{8}{14} = \frac{93}{154}.$$

EXAMPLE 6 *There are two bags, one of which contains 3 black and 4 white, balls, while the other contains 4 black and 3 white balls. A fair die is cast, if the face 1 or 3 turns up, a ball is taken from the first bag, and if any other face turns up a ball is chosen from the second bag. Find the probability of choosing a black ball.*

SOLUTION Let E_1, E_2 and A be the events defined as follows :

E_1 = The die shows 1 or 3,

E_2 = The die shows 2, 4, 5 or 6,

and, A = The ball drawn is black.

We have, $P(E_1) = \frac{2}{6} = \frac{1}{3}$, $P(E_2) = \frac{4}{6} = \frac{2}{3}$.

If E_1 occurs, then the first bag is chosen and the probability of drawing a black ball from it is $\frac{3}{7}$.

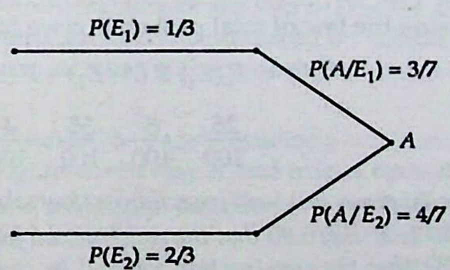

Fig. 30.15

$\therefore \quad P(A/E_1) = \frac{3}{7}$

If E_2 occurs, then the second bag is chosen and the probability of drawing a black ball from it is $\frac{4}{7}$.

$\therefore \quad P(A/E_2) = \frac{4}{7}$

By the law of total probability, we have

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) = \frac{1}{3} \times \frac{3}{7} + \frac{2}{3} \times \frac{4}{7} = \frac{11}{21}.$$

EXAMPLE 7 *Two thirds of the students in a class are boys and the rest girls. It is known that the probability of a girl getting a first class is 0.25 and that of a boy getting a first class is 0.28. Find the probability that a student chosen at random will get first class marks in the subject.*

SOLUTION Let E_1, E_2 and A be the events defined as follows :

E_1 = a boy is chosen from the class,

E_2 = a girl is chosen from the class,

and, A = the students gets first class marks.

Then, $P(E_1) = 2/3, P(E_2) = 1/3, P(A/E_1) = 0.28$ and $P(A/E_2) = 0.25$

Using the law of total probability, we have

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) = \frac{2}{3} \times 0.28 + \frac{1}{3} \times 0.25 = 0.27$$

EXAMPLE 8 *In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product. What is the probability that the bolt drawn is defective?*

SOLUTION Let E_1, E_2, E_3 and A be the events defined as follows :

E_1 = the bolts is manufactured by machine A;

E_2 = the bolt is manufactured by machine B;

E_3 = the bolt is manufactured by machine C,

and, A = the bolt is defective. Then,

$$P(E_1) = \frac{25}{100} = \frac{1}{4},\ P(E_2) = \frac{35}{100}\ P(E_3) = \frac{40}{100}.$$

$P(A/E_1)$ = Probability that the bolt drawn is defective given the condition that it is manufactured by machine A

$= 5/100$

Similarly, we have

$$P(A/E_2) = \frac{4}{100} \text{ and } P(A/E_3) = \frac{2}{100}$$

Using the law of total probability, we have

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)$$

$$= \frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100} = 0.0345$$

EXAMPLE 9 *A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.*

SOLUTION Let A be the event that the construction job will be completed on time, E_1 be the event that there will be a strike and E_2 be the event that there will be no strike.

We have,

$$P(E_1) = 0.65,\ P(E_2) = 1 - P(E_1) = 1 - 0.65 = 0.35$$

$$P(A/E_1) = 0.32 \text{ and } P(A/E_2) = 0.80$$

By total probability theorem, we have

Required probability $= P(A)$

$\Rightarrow$ Required probability $= P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$

$\Rightarrow$ Required probability $= 0.65 \times 0.32 + 0.35 \times 0.80$

$\Rightarrow$ Required probability $= 0.208 + 0.28 = 0.488$

EXERCISE 30.8

1. A bag A contains 5 white and 6 black balls. Another bag B contains 4 white and 3 black balls. A ball is transferred from bag A to the bag B and then a ball is taken out of the second bag. Find the probability of this ball being black.
2. A purse contains 2 silver and 4 copper coins. A second purse contains 4 silver and 3 copper coins. If a coin is pulled at random from one of the two purses, what is the probability that it is a silver coin?
3. One bag contains 4 yellow and 5 red balls. Another bag contains 6 yellow and 3 red balls. A ball is transferred from the first bag to the second bag and then a ball is drawn from the second bag. Find the probability that ball drawn is yellow. [CBSE 2002]
4. A bag contains 3 white and 2 black balls and another bag contains 2 white and 4 black balls. One bag is chosen at random. From the selected bag, one ball is drawn. Find the probability that the ball drawn is white.
5. The contents of three bags I, II and III are as follows :
   Bag I : 1 white, 2 black and 3 red balls,
   Bag II : 2 white, 1 black and 1 red ball, and
   Bag III : 4 white, 5 black and 3 red balls.
   A bag is chosen at random and two balls are drawn. What is the probability that the balls are white and red?
6. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the sum of the numbers obtained is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3, 4, ..., 12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8?
7. A factory has two machines A and B. Past records show that the machine A produced 60% of the items of output and machine B produced 40% of the items. Further 2% of the items produced by machine A were defective and 1% produced by machine B were defective. If an item is drawn at random, what is the probability that it is defective?
8. The bag A contains 8 white and 7 black balls while the bag B contains 5 white and 4 black balls. One ball is randomly picked up from the bag A and mixed up with the balls in bag B. Then a ball is randomly drawn out from it. Find the probability that ball drawn is white. [CBSE 2007]

ANSWERS

1. $\frac{39}{88}$ 2. $\frac{19}{42}$ 3. $\frac{29}{45}$ 4. $\frac{7}{15}$ 5. $\frac{118}{495}$ 6. $\frac{193}{792}$ 7. 0.016 8. $\frac{83}{150}$

HINTS TO SELECTED PROBLEMS

2. Consider the following events :
   E_1 = Selecting first purse,
   E_2 = selecting second purse,
   A = coin drawn is silver coin.
   We have, $P(E_1) = P(E_2) = \frac{1}{2}, P(A/E_1) = \frac{2}{6}, P(A/E_2) = \frac{4}{7}$.
   Required probability = $P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2)$.
4. Let E_1 = selecting first bag,
   E_2 = selecting second bag,
   A = ball drawn is white.

Then, $P(E_1) = P(E_2) = 1/2$, $P(A/E_1) = 3/5$, $P(A/E_2) = 2/6$.

Required probability $= P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$

5. Let E_1 = bag I is selected, E_2 = bag II is selected, E_3 = bag III is selected and, A = two balls drawn from the chosen bag are white and red.

   Then, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$,

   $$P(A/E_1) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2},\ P(A/E_2) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} \text{ and } P(A/E_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2}.$$

   $\therefore$ Required probbability

   $= P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)$

6. Let E_1 = the coin shows a head, E_1 = the coin shows a tail,

   A = the noted number is 7 or 8.

   Then, $P(E_1) = 1/2$, $P(E_2) = 1/2$, $P(A/E_1) = 11/36$ and $P(A/E_2) = 2/11$.

   Required probability $= P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$

30.9 BAYE'S THEOREM

THEOREM *(Baye's Theorem) Let S be the sample space and let $E_1, E_2 \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then*

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum\limits_{i=1}^{n} P(E_i)\,P(A/E_i)},\ i = 1, 2, \ldots, n$$

PROOF Since $E_1, E_2, \ldots, E_n$ are n mutually exclusive and exhaustive events, we have

$$S = E_1 \cup E_2 \cup \ldots \cup E_n, \text{ where } E_i \cap E_j = \phi \text{ for } i \neq j$$

$$\Rightarrow \quad A = A \cap S$$

$$\Rightarrow \quad A = (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)$$

$$\Rightarrow \quad P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n) \qquad \text{[By add. theorem]}$$

$$\Rightarrow \quad P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$

$$\Rightarrow \quad P(A) = \sum_{i=1}^{n} P(E_i)\,P(A/E_i) \qquad [\because P(A \cap E_i) = P(E_i)\,P(A/E_i)] \qquad \text{...(i)}$$

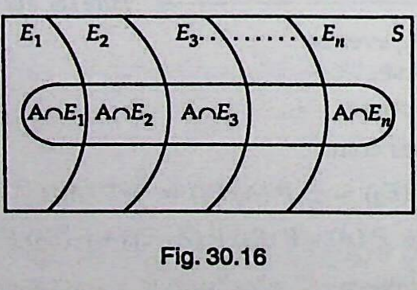

Fig. 30.16

Now, using multiplication theorem of probability, we have

$$P(A \cap E_i) = P(A)\,P(E_i/A) \quad \text{for } i = 1, 2, \ldots, n$$

$$\Rightarrow \quad P(E_i/A) = \frac{P(A \cap E_i)}{P(A)}$$

$$\Rightarrow \quad P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{P(A)} \qquad [\because P(A \cap E_i) = P(E_i)\, P(A/E_i)]$$

$$\Rightarrow \quad P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)} \qquad \text{[Using (i)]}$$

Hence, $$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)},\ i = 1, 2, \ldots, n$$

Q.E.D.

NOTE 1 *The events $E_1, E_2, \ldots, E_n$ are usually referred to as 'hypothesis' and the probabilities $P(E_1), P(E_2), \ldots, P(E_n)$ are known as the 'priori' probabilities as they exist before we obtain any information from the experiment.*

NOTE 2 *The probabilities $P(A/E_i)$; $= 1, 2, \ldots, n$ are called the 'likelyhood probabilities' as they tell us how likely the event A under consideration occurs, given each and every priori probabilities.*

NOTE 3 *The probabilities $P(E_i/A)$; $i = 1, 2, \ldots, n$ are called the 'posterior probabilities' as they are determined after the results of the experiment are known.*

The significance of Baye's theorem may be understood in the following manner :

An experiment can be performed in n mutually exclusive and exhaustive ways $E_1, E_2, \ldots, E_n$. The probability $P(E_i)$ of the occurrence of event E_i; $i = 1, 2, \ldots, n$ is known. The experiment is performed and we are told that the event A has occurred. With this information the probability $P(E_i)$ is changed to $P(E_i/A)$. Baye's theorem enables us to evaluate $P(E_i/A)$ if all the $P(E_i)$ (priori probabilities) and $P(A/E_i)$ (likelyhood probabilities) are known as explained in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product. If the bolt drawn is found to be defective, what is the probability that it is manufactured by the machine B?* **[NCERT, CBSE 2008]**

SOLUTION Let E_1, E_2, E_3 and A be the events defined as follows :

E_1 = the bolt is manufactured by machine A,

E_2 = the bolt is manufactured by machine B,

E_3 = the bolt is manufactured by machine C.

A = the bolt is defective.

Then, $P(E_1)$ = Probability that the bolt drawn is manufactured by machine A
$= 25/100$,

$P(E_2)$ = Probability that the bolt drawn is manufactured by machine B
$= 35/100$,

$P(E_3)$ = Probability that the bolt drawn is manufactured by machine C
$= 40/100$.

$P(A/E_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A
$= 5/100$

Similarly, we have $P(A/E_2) = \dfrac{4}{100}$ and $P(A/E_3) = \dfrac{2}{100}$

Now,

Required probability
= Probability that the bolt is manufactured by machine B given that the bolt drawn is defective
$= P(E_2/A)$

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{35}{100}\times\frac{4}{100}}{\frac{25}{100}\times\frac{5}{100}+\frac{35}{100}\times\frac{4}{100}+\frac{40}{100}\times\frac{2}{100}} = \frac{140}{125+140+80} = \frac{140}{345} = \frac{28}{69}$$

EXAMPLE 2 *Three urns contain 6 red, 4 black; 4 red, 6 black, and 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the first urn.*

SOLUTION Let E_1, E_2, E_3 and A be the events defined as follows :

E_1 = urn first is chosen, E_2 = urn second is chosen,

E_3 = urn third is chosen, and A = ball drawn is red.

Since there are three urns and one of the three urns is chosen at random, therefore

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}.$$

If E_1 has already occurred, then urn first has been chosen which contains 6 red and 4 black balls. The probability of drawing a red ball from it is 6/10.

So, $P(A/E_1) = \dfrac{6}{10}$.

Similarly, we have $P(A/E_2) = \dfrac{4}{10}$ and $P(A/E_3) = \dfrac{5}{10}$.

We are required to find $P(E_1/A)$, i.e. given that the ball drawn is red, what is the probability that it is drawn from the first urn.

By Baye's theorem, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times\frac{6}{10}}{\frac{1}{3}\times\frac{6}{10}+\frac{1}{3}\times\frac{4}{10}+\frac{1}{3}\times\frac{5}{10}} = \frac{6}{15} = \frac{2}{5}$$

EXAMPLE 3 *A company has two plants to manufacture scooters. Plant I manufactures 70% of the scooters and Plant II manufactures 30%. At Plant I, 80% of the scooters are rated as of standard quality and at Plant II, 90% of the scooters are rated as of standard quality. A scooter is chosen at random and is found to be of standard quality. What is the probability that it has come from Plant II?* **[CBSE 2000, 2004, 2005]**

SOLUTION Let E_1, E_2 and A be the following events.

E_1 = Plant I is chosen,

E_2 = Plant II is chosen, and A = Scooter is of standard quality.

Then, $P(E_1) = \frac{70}{100}$, $P(E_2) = \frac{30}{100}$, $P(A/E_1) = \frac{80}{100}$ and $P(A/E_2) = \frac{90}{100}$.

We are required to find $P(E_2/A)$.

By Baye's theorem, we have

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$\Rightarrow \quad P(E_2/A) = \frac{\frac{30}{100}\times\frac{90}{100}}{\frac{70}{100}\times\frac{80}{100}+\frac{30}{100}\times\frac{90}{100}} = \frac{27}{56+27} = \frac{27}{83}$$

EXAMPLE 4 *An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter driver, car driver and a truck is 0.01, 0.03 and 0.15 respectively. One of the insured person meets with an accident. What is the probability that he is a scooter driver?* **[CBSE 2000, 2002, 2008]**

SOLUTION Let E_1, E_2, E_3 and A be the events defined as follows :

E_1 = person chosen is a scooter driver,

E_2 = person chosen is a car driver,

E_3 = person chosen is a truck driver, and

A = person meets with an accident.

Since there are 12000 persons, therefore

$$P(E_1) = \frac{2000}{12000} = \frac{1}{6} \text{ and } P(E_2) = \frac{4000}{12000} = \frac{1}{3} \text{ and } P(E_2) = \frac{6000}{12000} = \frac{1}{2}$$

It is given that $P(A/E_1)$ = Probability that a person meets with an accident given that he is a scooter driver = 0.01.

Similarly, we have $P(A/E_2) = 0.03$ and $P(A/E_3) = 0.15$.

We are required to find $P(E_1/A)$, i.e. given that the person meets with an accident, what is the probability that he was a scooter driver.

By Baye's rule, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$\Rightarrow \quad P(E_1/A) = \frac{\frac{1}{6}\times 0.01}{\frac{1}{6}\times 0.01+\frac{1}{3}\times 0.03+\frac{1}{2}\times 0.15} = \frac{1}{1+6+45} = \frac{1}{52}$$

EXAMPLE 5 *Urn A contains 2 white, 1 black and 3 red balls, urn B contains 3 white, 2 black and 4 red balls and urn C contains 4 white, 3 black and 2 red balls. One urn is chosen at random and 2 balls are drawn at random from the urn. If the chosen balls happen to be red and black, what is the probability that both balls come from urn B?*

SOLUTION Let E_1, E_2, E_3 and A denote the following events.

E_1 = urn A is chosen, E_2 = urn B is chosen, E_3 = urn C is chosen, and A = two balls drawn at random are red and black. Since one of the urns is chosen at random, therefore

$$P(E_1) = \frac{1}{3} = P(E_2) = P(E_3)$$

If E_1 has already occurred, then urn A has been chosen. The urn A contains 2 white, 1 black and 3 red balls. Therefore, the probability of drawing a red and a black ball is $\frac{{}^3C_1 \times {}^1C_1}{{}^6C_2}$.

i.e. $$P(A/E_1) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{3}{15} = \frac{1}{5}$$

Similarly, we have $P(A/E_2) = \frac{{}^4C_1 \times {}^2C_1}{{}^9C_2} = \frac{2}{9}$, and $P(A/E_3) = \frac{{}^2C_1 \times {}^3C_1}{{}^9C_2} = \frac{1}{6}$

We are required to find $P(E_2/A)$. By Baye's theorem, we have

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$\Rightarrow \quad P(E_2/A) = \frac{\frac{1}{3}\times\frac{2}{9}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{2}{9}+\frac{1}{3}\times\frac{1}{6}} = \frac{\frac{2}{9}}{\frac{1}{5}+\frac{2}{9}+\frac{1}{6}} = \frac{20}{53}$$

EXAMPLE 6 *There are 3 bags, each containing 5 white balls and 3 black balls. Also there are 2 bags, each containing 2 white balls and 4 black balls. A white ball is drawn at random. Find the probability that this white ball is from a bag of the first group.*

SOLUTION Let E_1, E_2 and A be the events defined as follows :

E_1 = selecting a bag from the first group,

E_2 = selecting a bag from the second group, and

A = ball drawn is white

Since there are 5 bags out of which 3 bags belong to first group and 2 bags to second group. Therefore,

$\therefore \quad P(E_1) = \frac{3}{5}$, $P(E_2) = \frac{2}{5}$

If E_1 has already occurred, then a bag from the first group is chosen. The bag chosen contains 5 white balls and 3 black balls. Therefore the probability of drawing a white ball from it is 5/8.

$\therefore \quad P(A/E_1) = 5/8$

Similarly, we have $P(A/E_2) = 2/6 = 1/3$.

We have to find $P(E_1/A)$, i.e. given that the ball drawn is white, what is the probability that it is drawn from a bag of the first group. By Baye's theorem, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{3}{5}\times\frac{5}{8}}{\frac{3}{5}\times\frac{5}{8}+\frac{2}{5}\times\frac{1}{3}} = \frac{45}{61}$$

EXAMPLE 7 *A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.* **[NCERT, CBSE 2005]**

SOLUTION Let E_1, E_2 and A be the events defined as follows :

E_1 = six occurs, E_2 = six does not occur, and A = the man reports that it is a six.

We have, $P(E_1) = \frac{1}{6}$, $P(E_2) = \frac{5}{6}$.

Now, $P(A/E_1)$ = Probability that the man reports that there is a six on the die given that six has occurred on the die

= Probability the man speaks truth $= 3/4$

and, $P(A/E_2)$ = Probability that the man reports that there is a six on the die given that six has not occurred on the die

= Probability that the man does not speak truth $= 1 - \frac{3}{4} = \frac{1}{4}$

We have to find $P(E_1/A)$ i.e., the probability that there is six on the die given that the man has reported that there is six.

By Baye's theorem, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4}} = \frac{3}{8}$$

EXAMPLE 8 *In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct, given that he copied it, is 1/8. Find the probability that he knew the answer to the question, given that he correctly answered it.* **[NCERT]**

SOLUTION Let E_1, E_2, E_3 and A be the events defined as follows :

E_1 = the examinee guesses the answer, E_2 = the examinee copies the answer, E_3 = the examinee knows the answer, and A = the examinee answers correctly.

We have, $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{6}$.

Since E_1, E_2, E_3 are mutually exclusive and exhaustive events.

$$\therefore \quad P(E_1) + P(E_2) + P(E_3) = 1 \Rightarrow P(E_3) = 1 - (P(E_1) + P(E_2)) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}.$$

If E_1 has already occurred, then the examinee guesses. Since there are four choices out of which only one is correct, therefore the probability that he answers correctly given that he has made a guess is $1/4$ i.e. $P(A/E_1) = 1/4$. It is given that $P(A/E_2) = 1/8$, and

$P(A/E_3)$ = Probability that he answers correctly given that he knew the answer

$\Rightarrow P(A/E_3) = 1$

By Baye's Theorem, we have

Required probability

$= P(E_3/A)$

$$= \frac{P(E_3)\,P(A/E_3)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{1}{3} \times \frac{1}{4} + \frac{1}{6} \times \frac{1}{8} + \frac{1}{2} \times 1} = \frac{24}{29}$$

EXAMPLE 9 *A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart.* **[CBSE 2000, 2010]**

SOLUTION Let E_1, E_2, E_3, E_4 and A be the events as defined below :

E_1 = the missing card is a heart card,

E_2 = the missing card is a spade card,

E_3 = the missing card is a club card,

E_4 = the missing card is a diamond card, and

A = Drawing two heart cards from the remaining cards.

Then,

$$P(E_1) = \frac{13}{52} = \frac{1}{4}, P(E_2) = \frac{13}{52} = \frac{1}{4}, P(E_3) = \frac{13}{52} = \frac{1}{4}, P(E_4) = \frac{13}{52} = \frac{1}{4}.$$

$P(A/E_1)$ = Probability of drawing two heart cards given that one heart card is missing

$12 \times 11 \times 2$

$\Rightarrow \quad P(A/E_1) = \dfrac{{}^{12}C_2}{{}^{51}C_2}$

$P(A/E_2)$ = Probability of drawing two heart cards given that one spade card is missing

$\Rightarrow \quad P(A/E_2) = \dfrac{{}^{13}C_2}{{}^{51}C_2}$

Similarly, we have $P(A/E_3) = \dfrac{{}^{13}C_2}{{}^{51}C_2}$ and $P(A/E_4) = \dfrac{{}^{13}C_2}{{}^{51}C_2}$

By Baye's Theorem, we have

Required probability

$= P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3) + P(E_4)\,P(A/E_4)}$$

$$= \frac{\frac{1}{4} \times \frac{{}^{12}C_2}{{}^{51}C_2}}{\frac{1}{4} \times \frac{{}^{12}C_2}{{}^{51}C_2} + \frac{1}{4} \times \frac{{}^{13}C_2}{{}^{51}C_2} + \frac{1}{4} \times \frac{{}^{13}C_2}{{}^{51}C_2} + \frac{1}{4} \times \frac{{}^{13}C_2}{{}^{51}C_2}} = \frac{{}^{12}C_2}{{}^{12}C_2 + {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2}$$

$$= \frac{66}{66 + 78 + 78 + 78} = \frac{11}{50}$$

EXAMPLE 10 *A letter is known to have come either from TATANAGAR or CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter has come from (i) Calcutta (ii) Tatanagar?*

SOLUTION Let E_1 be the event that the letter came from Calcutta and E_2 be the event that the letter came from Tatanagar. Let A denote the event that two consecutive letters visible on the envelope are TA.

Since the letters have come either from Calcutta or Tatanagar. Therefore,

$$P(E_1) = \frac{1}{2} = P(E_2)$$

If E_1 has occurred, then it means that the letter came from Calcutta. In the word CALCUTTA there are 8 letters in which TA occurs in the end. Considering TA as one letter there are seven letters out of which one can be in 7 ways. Therefore,

$$P(A/E_1) = \frac{1}{7}$$

If E_2 has occurred, then the letter came from Tatanagar. In the word TATANAGAR there are 9 letters in which TA occurs twice. Considering one of the two TA's as one letter there are 8 letters. Therefore,

$$P(A/E_2) = \frac{2}{8}$$

By Baye's theorem, we have

(i) $$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)P(A/E_2)} = \frac{\frac{1}{2}\times\frac{1}{7}}{\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{2}{8}} = \frac{4}{11}$$

(ii) $$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{1}{2}\times\frac{2}{8}}{\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{2}{8}} = \frac{7}{11}$$

EXAMPLE 11 *A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively* $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ *and* $\frac{2}{5}$. *The probability that he will be late are* $\frac{1}{4}, \frac{1}{3}$ *and* $\frac{1}{12}$, *if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train ?* **[NCERT]**

SOLUTION Let E_1, E_2, E_3, E_4 be the events that the doctor comes by train, bus, scooter and other means of transport respectively. Then,

$$P(E_1) = \frac{3}{10}, P(E_2) = \frac{1}{5}, P(E_3) = \frac{1}{10} \text{ and } (P(E_4) = \frac{2}{5} \qquad \text{[Given]}$$

Let A be the event that the doctor visits the patient late. Then,

$P(A/E_1)$ = Probability that the doctor will be late if he comes by train

$\Rightarrow \quad P(A/E_1) = \frac{1}{4}$ [Given]

$P(A/E_2)$ = Probability that the doctor will be late if he comes by bus

$\Rightarrow \quad P(A/E_2) = \frac{1}{3}$ [Given]

$P(A/E_3)$ = Probability that the doctor will be late if he comes by scootor

$\Rightarrow \quad P(A/E_3) = \frac{1}{12}$ [Given]

$P(A/E_4)$ = Probability that the doctor will be late if he comes by other means of transport

$\Rightarrow \quad P(A/E_4) = 0$ [Given]

We have to find $P(E_1/A)$.

By Baye's theorem, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3) + P(E_4)\,P(A/E_4)}$$

$$\Rightarrow \quad P(E_1/A) = \frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0} = \frac{3}{40}\times\frac{120}{18} = \frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$.

EXAMPLE 12 *Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets a 1, 2, 3, or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head; what is the probability that she threw a 1, 2, 3, or 4 with the die?* **[NCERT]**

SOLUTION Consider the following events:

E_1 = Getting 5 or 6 in a single throw of a die.

E_2 = Getting 1, 2, 3, or 4 in a single throw of a die.

A = Getting exactly one head.

We have,

$$P(E_1) = \frac{2}{6} = \frac{1}{3},\; P(E_2) = \frac{4}{6} = \frac{2}{3}$$

$P(A/E_1)$ = Probability of getting exactly one head when a coin is tossed three times

$$\Rightarrow \quad P(A/E_1) = {}^3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

$P(A/E_2)$ = Probability of getting exactly one head when a coin is tossed once only

$$\Rightarrow \quad P(A/E_2) = \frac{1}{2}$$

Now, Required probability

$= P(E_2/A)$

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{2}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{3}{8}+\frac{1}{2}\times\frac{2}{3}} = \frac{8}{11}$$

EXAMPLE 13 *Let d_1, d_2, d_3 be three mutually exclusive diseases. Let $S = \{s_1, s_2, s_3 ..., s_6\}$ be the set of observable symptoms of these diseases. For example, s_1 is the shortness of breath, s_2 is loss of weight, s_3 is fatigue etc. Suppose a random sample of 10,000 patients contains 3200 patients with disease d_1, 3500 with disease d_2 and 3300 with disease d_3. Also, 3100 patients with disease d_1, 3300 with disease d_2 and 3000 with disease d_3 show the symptom S. Knowing that the patient has symptom S, the doctor wishes to determine the patient's illness. On the basis of this information, what should the doctor conclude?*

SOLUTION Let E_i denote the event that the patient has disease d_i ; $i = 1, 2, 3$ and A be the event that the patient has symptom S. Then,

$$P(E_1) = \frac{3200}{10000} = \frac{32}{100}, P(E_2) = \frac{3500}{10000} = \frac{35}{100}, P(E_3) = \frac{3300}{10000} = \frac{33}{100}$$

$$P(A \cap E_1) = \frac{3100}{10000} = \frac{31}{100}, P(A \cap E_2) = \frac{3300}{10000} = \frac{33}{100}$$

and, $$P(A \cap E_3) = \frac{30000}{10000} = \frac{30}{100}$$

$$\therefore \quad P(A/E_1) = \frac{P(A \cap E_1)}{P(E_1)} \Rightarrow P(A/E_1) = \frac{\frac{31}{100}}{\frac{32}{100}} = \frac{31}{32}$$

$$P(A/E_2) = \frac{P(A \cap E_2)}{P(E_2)} \Rightarrow P(A/E_2) = \frac{\frac{33}{100}}{\frac{35}{100}} = \frac{33}{35}$$

$$P(A/E_3) = \frac{P(A \cap E_3)}{P(E_3)} \Rightarrow P(A/E_3) = \frac{\frac{30}{100}}{\frac{33}{100}} = \frac{30}{33}$$

Using Baye's theorem, we have

$$P(E_1/A) = \frac{P(E_1)\, P(A/E_1)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)}$$

$$\Rightarrow \quad P(E_1/A) = \frac{\frac{32}{100} \times \frac{31}{32}}{\frac{32}{100} \times \frac{31}{32} + \frac{35}{100} \times \frac{33}{35} + \frac{33}{100} \times \frac{30}{33}} = \frac{31}{94}$$

$$P(E_2/A) = \frac{P(E_2)\, P(A/E_2)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)}$$

$$\Rightarrow \quad P(E_2/A) = \frac{\frac{35}{100} \times \frac{33}{35}}{\frac{32}{100} \times \frac{31}{32} + \frac{35}{100} \times \frac{33}{35} + \frac{33}{100} \times \frac{30}{33}} = \frac{33}{94}$$

and,

$$P(E_3/A) = \frac{P(E_3)\,(A/E_3)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)}$$

$$\Rightarrow \quad P(E_3/A) = \frac{\frac{33}{100} \times \frac{30}{33}}{\frac{32}{100} \times \frac{31}{32} + \frac{35}{100} \times \frac{33}{35} + \frac{33}{100} \times \frac{30}{33}} = \frac{30}{94}$$

Clearly, $P(E_3/A) < P(E_1/A) < P(E_2/A)$ i.e. $P(E_2/A)$ is largest. Thus, the doctor should conclude that the patient is most likely to have disease d_2.

EXAMPLE 14 *Given three identical boxes I, II and III, each containing two coins. In box I both coins are gold coins, in box II both are silver coins and in box III there is one gold and one sliver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?* **[NCERT]**

SOLUTION Consider the following events:

E_1 = Box I is choosen, E_2 = Box II is choosen, E_3 = Box III is choosen.

A = The coin drawn is of gold.

We have,

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$P(A/E_1)$ = Probability of drawing a gold coin from box I

$$\Rightarrow \quad P(A/E_1) = \frac{2}{2} = 1$$

$P(A/E_2)$ = Probability of drawing a gold coin from box II

$$\Rightarrow \quad P(A/E_2) = 0$$

$P(A/E_3)$ = Probability of drawing a gold coin from box III

$$\Rightarrow \quad P(A/E_3) = \frac{1}{2}$$

Now,

Probability that the other coin in the box is of gold

= Probability that gold coin is drawn from the box I

= $P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{2}{3}$$

EXAMPLE 15 *Suppose that the reliability of a HIV test is specifed as follows:*

Of people having HIV, 90% of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV–ive but 1% are diagnosed as showning HIV+ive. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV +ive. What is the probability that the person actually has HIV? **[NCERT]**

SOLUTION Consider the following events:

E_1 = The person selected is actually having HIV

E_2 = The person selected is not having HIV

A = The person's HIV test is diagnosed as + ive.

We have,

$$P(E_1) = 0.1\% = \frac{0.1}{100} = 0.001,\ P(E_2) = 1 - P(E_1) = 1 - 0.001 = 0.999$$

$P(A/E_1)$ = Probability that the person tested as HIV +ive given that he/she is actually having HIV.

$\Rightarrow \quad P(A/E_1) = \frac{90}{100} = 0.9$

and,

$P(A/E_2)$ = Probability that the person tested as HIV +ive given that he/she is actually not having HIV

$\Rightarrow \quad P(A/E_2) = \frac{1}{100} = 0.01$

Now,

Required probability

$= P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{0.001 \times 0.9}{0.001 \times 0.9 + 0.999 \times 0.01} = \frac{90}{1089}$$

EXAMPLE 16 *Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn in found to be red in colour. Find the probability that the transferred ball is black.*

SOLUTION Consider the following events:

E_1 = Ball transferred from Bag I to Bag II is red

E_2 = Ball transferred from Bag I to Bag II is black

A = Ball drawn from Bag II is red in colour.

We have,

$$P(E_1) = \frac{3}{7},\ P(E_2) = \frac{4}{7},\ P(A/E_1) = \frac{5}{10} = \frac{1}{2} \text{ and } P(A/E_2) = \frac{4}{10} = \frac{2}{5}$$

Required probability

$= P(E_2/A)$

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{4}{7} \times \frac{2}{5}}{\frac{3}{7} \times \frac{1}{2} + \frac{4}{7} \times \frac{2}{5}} = \frac{16}{31}$$

EXAMPLE 17 *Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and femals.* **[NCERT]**

SOLUTION Consider the following events:

E_1 = Person selected is male, E_2 = Person selected is female.

A = Person selected is grey haired.

We have,

$$P(E_1) = P(E_2) = \frac{1}{2},\ P(A/E_1) = \frac{5}{100} \text{ and } P(A/E_2) = \frac{1}{400}$$

$\therefore$ Required probability $= P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{1}{2} \times \frac{5}{100}}{\frac{1}{2} \times \frac{5}{100} + \frac{1}{2} \times \frac{1}{400}} = \frac{20}{21}$$

EXAMPLE 18 *A bag contains 4 balls. Two balls are drawn at random and are found to be white. What is the probability that all balls are white?* **[CBSE 2010]**

SOLUTION Since two balls drawn are white. So, we have the following prossibilities:

(i) The bag contains two white balls and 2 balls of other colour.

(ii) The bag contains 3 white balls and one ball of other colour.

(iii) The bag contains all white balls.

Consider the following events:

E_1 = There are two white and two other colour balls in the bag

E_2 = There are three white and one other colour ball in the bag

E_3 = There are all white balls in the bag

A = Drawing 2 white balls from the bag

Clearly,

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P(A/E_1) = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{6},\ P(A/E_2) = \frac{{}^3C_2}{{}^4C_2} = \frac{1}{2},\ P(A/E_3) = \frac{{}^4C_2}{{}^4C_2} = 1$$

$\therefore$ Required probability

$$= P(E_3/A) = \frac{P(E_3)\,P(A/E_3)}{\sum_{i=1}^{3} P(E_i)\,P(A/E_i)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 1} = \frac{3}{5}$$

EXERCISE 30.11

1. The contents of urns I, II, III are as follows :
   Urn I : 1 white, 2 black and 3 red balls
   Urn II : 2 white, 1 black and 1 red balls
   Urn III : 4 white, 5 black and 3 red balls.
   One urn is chosen at random and two balls are drawn. They happen to be white and red. What is the probability that they come from Urns I, II, III? **[CBSE 2003]**
2. A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag B. **[CBSE 2007, 2010]**
3. Three urns contain 2 white and 3 black balls; 3 white and 2 black balls and 4 white and 1 black ball respectively. One ball is drawn from an urn chosen at random and it was found to be white. Find the probability that it was drawn from the first urn.
4. The contents of three urns are as follows :
   Urn 1 : 7 white, 3 black balls,
   Urn 2 : 4 white, 6 black balls, and
   Urn 3 : 2 white, 8 black balls.

One of these urns is chosen at random with probabilities 0.20, 0.60 and 0.20 respectively. From the chosen urn two balls are drawn at random without replacement. If both these balls are white, what is the probability that these came from urn 3?

5. By examining the chest X-ray, probability that T.B is detected when a person is actually suffering is 0.99. The probability that the doctor diagnoses incorrectly that a person has T.B. on the basis of X-ray is 0.001. In a certain city 1 in 1000 persons suffers from T.B. A person is selected at random is diagnosed to have T.B. What is the chance that he actually has T.B.?
6. Two groups are competing for the positions of the Board of Directors of a Corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group. **[NCERT, CBSE 2009]**
7. Suppose 5 men out of 100 and 25 women out of 1000 are good orators. An orator is chosen at random. Find the probability that a male person is selected. Assume that there are equal number of men and women.
8. A letter is known to have come either from LONDON or CLIFTON. On the envelope just two consecutive letters ON are visible. What is the probability that the letter has come from (i) LONDON (ii) CLIFTON ?
9. In a class, 5% of the boys and 10% of the girls have an IQ of more than 150. In this class, 60% of the students are boys. If a student is selected at random and found to have an IQ of more than 150, find the probability that the student is a boy.
10. A factory has three machines X, Y and Z producing 1000, 2000 and 3000 bolts per day respectively. The machine X produces 1% defective bolts, Y produces 1.5% and Z produces 2% defective bolts. At the end of a day, a bolt is drawn at random and is found to be defective. What is the probability that this defective bolt has been produced by machine X? **[CBSE 2002]**
11. An insurance company issued 3000 scooters, 4000 cars and 5000 trucks. The probabilities of the accident involving a scooter, a car and a truck are 0.02, 0.03 and 0.04 respectively. One of the insured vehicles meet with an accident. Find the probability that it is a (a) scooter (ii) car (iii) truck. **[NCERT, CBSE 2001C]**
12. Suppose we have four boxes A, B, C, D containing coloured marbles as given below:

Box	*Marble Colour*		
	Red	*White*	*Black*
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

One of the boxes has been selected at random and a single marble drawn from it. If the marble is red, what is the probability that it was drawn from box A ? box B ? box C ? **[NCERT]**

13. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, whereas the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B on the job

for 30% of the time and C on the job for 20% of the time. A defective item is produced. What is the probability that it was produced by A ? [NCERT]

14. A test for detection of a particular disease is not fool proof. The test will correctly detect the disease 90% of the time, but will incorrectly detect the disease 1% of the time. For a large population of which an estimated 0.2% have the disease, a person is selected at random, given the test, and told that he has the disease. What are the chances that the person actually have the disease?

15. Let d_1, d_2, d_3 be three mutually exclusive diseases. Let S be the set of observable symptoms of these diseases. A doctor has the following information from a random sample of 5000 patients:
1800 had disease d_1, 2100 has disease d_2 and the others had disease d_3.
1500 patients with disease d_1, 1200 patients with disease d_2 and 900 patients with disease d_3 showed the symptom.
Which of the diseases is the patient most likely to have?

16. In a factory, machine A produces 30% of the total output, machine B produces 25% and the machine C produces the remaining output. If defective items produced by machines A, B and C are 1%, 1.2%, 2% respectively. Three machines working together produce 10000 items in a day. An item is drawn at random from a day's output and found to be defective. Find the probability that it was produced by machine B ?

17. A company has two plants to manufacture bicycles. The first plant manufactures 60% of the bicycles and the second plant 40%. Out of that 80% of the bicycles are rated of standard quality at the first plant and 90% of standard quality at the second plant. A bicycle is picked up at random and found to be standard quality. Find the probability that it comes from the second plant. [CBSE 2003]

18. Three urns A, B and C contain 6 red and 4 white; 2 red and 6 white; and 1 red and 5 white balls respectively. An urn is chosen at random and a ball is drawn. If the ball drawn is found to be red, find the probability that the ball was drawn from urn A. [CBSE 2004]

19. A is known to speak truth 3 times out of 5 times. He throws a die and reports that it is 1. Find the probability that it is actually 1. [CBSE 2004]

20. A factory has three machines A, B and C, which produce 100, 200 and 300 items of a particular type daily. The machines produce 2%, 3% and 5% defective items respectively. One day when the production was over, an item was picked up randomly and it was found to be defective. Find the probability that it was produced by machine A. [CBSE 2004]

21. A bag contains 1 white and 6 red balls, and a second bag contains 4 white and 3 red balls. One of the bags is picked up at random and a ball is randomly drawn from it, and is found to be white in colour. Find the probability that the drawn ball was from the first bag [CBSE 2005]

22. A speaks the truth 8 times out of 10 times. A die is tossed. He reports that it was 5. What is the probability that it was actually 5? [CBSE 2005]

23. For A, B and C the chances of being selected as the manager of a firm are in the ratio 4:1:2 respectively. The respective probabilities for them to introduce a radical change in marketing startegy are 0.3, 0.8 and 0.5. If the change does take place, find the probability that it is due to the appointment of B or C. [CBSE 2005]

24. An insurance company insured 2000 scooters and 3000 motorcycles. The probability of an accident involving a scooter is 0.01 and that of a motorcycle is 0.02. An insured

vehicle met with an accident. Find the probability that the accidented vehicle was a motorcycle. [CBSE 2005]

25. In answering a question on a multiple choice test a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that a student knows the answer given that he answered it correctly?

26. There are three coins. One is two headed coin, another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is choosen at random and tossed, it shows heads, what is the probability that it was the two headed coin? [NCERT, CBSE 2009]

27. A laboratory blood test is 99% effective in detecting a certain disease when it is, infact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1% of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive? [NCERT]

28. Coloured balls are distributed in four boxes as shown in the following table:

Box	Colour			
	Black	White	Red	Blue
I	3	4	5	6
II	2	2	2	2
III	1	2	3	1
IV	4	3	1	5

A box is selected at random and then a ball is randomly drawn from the selected box. The colour of the ball is black, what is the probability that ball drawn is from the box III.

29. If a machine is correctly set up it produces 90% acceptable items. If it is incorrectly set up it produces only 40% acceptable items. Past experience shows that 80% of the setups are correctly done. If after a certain set up, the machine produces 2 acceptable items, find the probability that the machine is correctly set up. [NCERT]

30. Suppose we have four boxes A, B, C and D containing coloured marbles as given below:

Box	*Marble Colour*		
	Red	*White*	*Black*
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from the box A? box B? Box C?

31. Assume that the chances of a patient having a heart attack is 40%. It is also assumed that meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options and patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga? [NCERT]

ANSWERS

1. $\frac{1}{5}, \frac{1}{3}, \frac{4}{15}$ 2. $\frac{25}{52}$ 3. $\frac{2}{9}$ 4. $\frac{1}{40}$ 5. $\frac{110}{221}$ 6. $\frac{2}{9}$

7. $\frac{2}{3}$ 8. (i) $\frac{12}{17}$ (ii) $\frac{5}{17}$ 9. $\frac{3}{7}$ 10. 0.1 11. (i) $\frac{3}{19}$ (ii) $\frac{6}{19}$ (iii) $\frac{10}{19}$

12. $\frac{1}{15}, \frac{2}{5}, \frac{8}{15}$ 13. $\frac{5}{34}$ 14. 0.15 15. d_1 16. 0.2 17. $\frac{3}{7}$ 18. $\frac{36}{61}$

19. $\frac{3}{13}$ 20. $\frac{2}{23}$ 21. $\frac{1}{5}$ 22. $\frac{4}{9}$ 23. $\frac{3}{5}$ 24. $\frac{3}{4}$ 25. $\frac{12}{13}$ 26. $\frac{4}{9}$

27. $\frac{198}{1197}$ 28. 0.165 29. 0.95 30. $\frac{1}{15}, \frac{2}{5}, \frac{8}{15}$ 31. $\frac{14}{29}$

HINTS TO SELECTED PROBLEMS

1. Let E_1 = Urn I is chosen, E_2 = Urn II is chosen, E_3 = Urn III is chosen, and A = two balls drawn are white and red. Then,

$$P(E_1) = 1/3 = P(E_2) = P(E_3),\ P(A/E_1) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2},\ P(A/E_2) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2},$$

$$P(A/E_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2}$$

2. Let E_1 = bag A is chosen, E_2 = bag B is chosen, and A = ball drawn is red. Then, $P(E_1) = P(E_2) = 1/2$, $P(A/E_1) = 3/5$ and $P(A/E_2) = 5/9$.

5. Let E_1 = The person selected is suffering from T.B

E_2 = The person selected is not suffering from T.B

A = the doctor diagnoses correctly.

Then, $P(E_1) = 1/1000$, $P(E_2) = 999/1000$, $P(A/E_1) = 0.99$ and $P(A/E_2) = 0.001$

Required prob. = $P(E_1/A) = \dfrac{P(E_1)\,P(A/E_1)}{P(E)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$

7. Let E_1 = Person chosen is a man, E_2 = Person chosen is a woman, and A = Person is a good orator. Then,

$P(E_1) = P(E_2) = \frac{1}{2}$, $P(A/E_1) = \frac{5}{100}$ and $P(A/E_2) = \frac{25}{1000}$

Now, find $P(E_1/A)$ by using Bay's theorem.

14. Let E_1 = The person selected has disease

E_2 = The person selected does not have disease

A = Test is positive.

We have,

$P(E_1) = 0.002, P(E_2) = 0.998, P(A/E_1) = 0.90, P(A/E_2) = 0.01.$

Required probability $= P(E_1/A) = \dfrac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$

15. Let E_i denote the event that the patient has disease d_i ; $i = 1, 2, 3$ and A be the event that the patient showed the symptom S. Then,

$$P(E_1) = \frac{1800}{5000}, P(E_2) = \frac{2100}{5000}, P(E_3) = \frac{1100}{5000}$$

$$P(A/E_1) = \frac{P(A \cap E_1)}{P(E_1)} = \frac{\frac{1500}{5000}}{\frac{1800}{5000}} = \frac{15}{18}, P(A/E_2) = \frac{12}{21} \text{ and } P(A/F_3) = \frac{9}{11}.$$

Compute $P(E_1/A)$, $P(E_2/A)$ and $P(E_3/A)$ and find the greatest of these.

16. Let E_1, E_2, E_3 be the events;

E_1 = Item is produced by machine A

E_2 = Item is produced by machine B

E_3 = Item is produced by machine C

and, A be the event that the item is defective.

We have, $P(E_1) = \dfrac{30}{100}, P(E_2) = \dfrac{25}{100}, P(E_3) = \dfrac{45}{100}, P(A/E_1) = \dfrac{1}{100},$

$P(A/E_2) = \dfrac{1.2}{100}$ and $P(A/E_3) = \dfrac{2}{100}.$

Required probability

$$= P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}.$$

25. Consider the following events:

E_1 = Student knows the answer

E_2 = Student guesses the answer

A = Student answers correctly.

We have,

$$P(E_1) = \frac{3}{4}, P(E_2) = \frac{1}{4}, P(A/E_2) = \frac{1}{4}, P(A/E_1) = 1$$

Required probability $= P(E_1/A) = \dfrac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$

$$\Rightarrow \quad \text{Required probability} = \frac{\frac{3}{4} \times 1}{\frac{3}{4} \times 1 + \frac{1}{4} \times \frac{1}{4}} = \frac{12}{13}$$

26. Consider the following evenets:

E_1 = Selecting two headed coin

E_2 = Selecting biased coin

E_3 = Selecting unbiased coin

A = Getting head on the coin.

We have, $P(E_1) = P(E_2) = P(E_3) = \dfrac{1}{3}$

$P(A/E_1) = 1,\ P(A/E_2) = \frac{75}{100} = \frac{3}{4},\ P(A/E_3) = \frac{1}{2}$

Required probability $= P(E_1/A)$

27. Consider the following events:

E_1 = Person selected has the disease

E_2 = Person selected does not have the disease

A = Test result is positive

We have,

$$P(E_1) = \frac{1}{1000},\ P(E_2) = \frac{999}{1000},\ P(A/E_1) = \frac{99}{100} \text{ and } P(A/E_2) = \frac{5}{1000}$$

Required probability $= P(E_1/A) = \dfrac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$

29. Let A be the event that the machine produces 2 acceptable items. Let E_1 represent the event of correct setup and E_2 represent the event of incorrect setup.

We have,

$$P(E_1) = 0.8,\ P(E_2) = 0.2,\ P(A/E_1) = 0.9 \times 0.9,\ P(A/E_2) = 0.4 \times 0.4$$

Required probability $= P(E_1/A) = \dfrac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$

31. Consider the following events:

E_1 = The patient follows a course of meditation and yoga

E_2 = The patient takes a certain drug

A = The patient suffers a heart attack.

We have,

$$P(E_1) = \frac{1}{2},\ P(E_2) = \frac{1}{2},\ P(A/E_1) = \frac{70}{100} \times \frac{40}{100} \text{ and } P(A/E_2) = \frac{75}{100} \times \frac{40}{100}$$

Required probability $= P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{70}{100} \times \frac{40}{100}}{\frac{1}{2} \times \frac{70}{100} \times \frac{40}{100} + \frac{1}{2} \times \frac{75}{100} \times \frac{40}{100}} = \frac{14}{29}$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. A four digit number is formed using the digits 1, 2, 3, 5 with no repetitions. Write the probability that the number is divisible by 5.
2. When three dice are thrown, write the probability of getting 4 or 5 on each of the dice simultaneously.
3. Three digit numbers are formed with the digits 0, 2, 4, 6 and 8. Write the probability of forming a three digit number with the same digits.
4. A ordinary cube has four plane faces, one face marked 2 and another face marked 3, find the probability of getting a total of 7 in 5 throws.

5. Three numbers are chosen from 1 to 20. Find the probability that they are consecutive.
6. 6 boys and 6 girls sit in a row at random. Find the probability that all the girls sit together.
7. If A and B are two independent events such that $P(A) = 0.3$ and $P(A \cup \overline{B}) = 0.8$. Find $P(B)$.
8. An unbiased die with face marked 1, 2, 3, 4, 5, 6 is rolled four times. Out of 4 face values obtained, find the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5.
9. If A and B are two events write the expression for the probability of occurrence of exactly one of two events.
10. Write the probability that a number selected at random from the set of first 100 natural numbers is a cube.
11. In a competition A, B and C are participating. The probability that A wins is twice that of B, the probability that B wins is twice that of C. Find the probability that A losses.
12. If A, B, C are mutually exclusive and exhaustive events associated to a random experiment, then write the value of $P(A) + P(B) + P(C)$.
13. If two events A and B are such that $P(\overline{A}) = 0.3$, $P(B) = 0.4$ and $P(A \cap \overline{B}) = 0.5$, find $P(B/\overline{A} \cap \overline{B})$.
14. If A and B are two independent events, then write $P(A \cap \overline{B})$ in terms of $P(A)$ and $P(B)$.
15. If $P(A) = 0.3$, $P(B) = 0.6$, $P(B/A) = 0.5$, find $P(A \cup B)$.

ANSWERS

1. $\frac{1}{4}$ 2. $\frac{1}{27}$ 3. $\frac{1}{25}$ 4. $\frac{5}{6^4}$ 5. $\frac{18}{^{20}C_3}$

6. $\frac{1}{132}$ 7. $\frac{2}{7}$ 8. $\frac{16}{81}$ 9. $P(A) + P(B) - 2P(A \cap B)$ 10. $\frac{1}{25}$

11. $\frac{3}{7}$ 12. 1 13. $\frac{1}{4}$ 15. 0.75

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If one ball is drawn at random from each of three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, then the probability that 2 white and 1 black balls will be drawn is

   (a) $\frac{13}{32}$ (b) $\frac{1}{4}$ (c) $\frac{1}{32}$ (d) $\frac{3}{16}$

2. A and B draw two cards each, one after another, from a pack of well-shuffled pack of 52 cards. The probability that all the four cards drawn are of the same suit is

   (a) $\frac{44}{85 \times 49}$ (b) $\frac{11}{85 \times 49}$

   (c) $\frac{13 \times 24}{17 \times 25 \times 49}$ (d) none of these

3. A and B are two events such that $P(A) = 0.25$ and $P(B) = 0.50$. The probability of both happening together is 0.14. The probability of both A and B not happening is

(a) 0.39 (b) 0.25 (c) 0.11 (d) none of these.

4. The probabilities of a student getting I, II and III division in an examination are $\frac{1}{10}$, $\frac{3}{5}$ and $\frac{1}{4}$ respectively. The probability that the student fails in the examination is

(a) $\frac{197}{200}$ (b) $\frac{27}{100}$ (c) $\frac{83}{100}$ (d) none of these

5. India play two matches each with West Indies and Australia. In any match the probabilities of India getting 0,1 and 2 points are 0.45, 0.05 and 0.50 respectively. Assuming that the outcomes are independent, the probability of India getting at least 7 points is

(a) 0.0875 (b) 1/16 (c) 0.1125 (d) none of these

6. Three faces of an ordinary dice are yellow, two faces are red and one face is blue. The dice is rolled 3 times. The probability that yellow red and blue face appear in the first second and third throws respectively, is

(a) $\frac{1}{36}$ (b) $\frac{1}{6}$ (c) $\frac{1}{30}$ (d) none of these

7. The probability that a leap year will have 53 Fridays or 53 Saturdays is

(a) $\frac{2}{7}$ (b) $\frac{3}{7}$ (c) $\frac{4}{7}$ (d) $\frac{1}{7}$

8. A person write 4 letters and addresses 4 envelopes. If the letters are placed in the envelopes at random, then the probability that all letters are not placed in the right envelopes, is

(a) $\frac{1}{4}$ (b) $\frac{11}{24}$ (c) $\frac{15}{24}$ (d) $\frac{23}{24}$

9. A speaks truth in 75% cases and B speaks truth in 80% cases. Probability that they contradict each other in a statement, is

(a) $\frac{7}{20}$ (b) $\frac{13}{20}$ (c) $\frac{3}{5}$ (d) $\frac{2}{5}$

10. Three integers are chosen at random from the first 20 integers. The probability that their product is even is

(a) $\frac{2}{19}$ (b) $\frac{3}{29}$ (c) $\frac{17}{19}$ (d) $\frac{4}{19}$

11. Out of 30 consecutive integers, 2 are chosen at random. The probability that their sum is odd, is

(a) $\frac{14}{29}$ (b) $\frac{16}{29}$ (c) $\frac{15}{29}$ (d) $\frac{10}{29}$

12. A bag contains 5 black balls, 4 white balls and 3 red balls. If a ball is selected randomwise, the probability that it is black or red ball is

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{5}{12}$ (d) $\frac{2}{3}$

13. Two dice are thrown simultaneously. The probability of getting a pair of aces is

(a) $\frac{1}{36}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) none of these

14. An urn contains 9 balls two of which are red, three blue and four black. Three balls are drawn at random. The probability that they are of the same colour is

(a) $\frac{5}{84}$ (b) $\frac{3}{9}$ (c) $\frac{3}{7}$ (d) $\frac{7}{17}$

15. A coin is tossed three times. If events A and B are defined as A = Two heads come, B = Last should be head. Then, A and B are

(a) independent (b) dependent

(c) both (d) none of these

16. Five persons entered the lift cabin on the ground floor of an 8 floor house. Suppose that each of them independently and with equal probability can leave the cabin at any floor beginning with the first, then the probability of all 5 persons leaving at different floor is

(a) $\frac{^7P_5}{7^5}$ (b) $\frac{7^5}{^7P_5}$ (c) $\frac{6}{^6P_5}$ (d) $\frac{^5P_5}{5^5}$

17. A box contains 10 good articles and 6 with defects. One item is drawn at random. The probability that it is either good or has a defect is

(a) $\frac{64}{64}$ (b) $\frac{49}{64}$ (c) $\frac{40}{64}$ (d) $\frac{24}{64}$

18. A box contains 6 nails and 10 nuts. Half of the nails and half of the nuts are rusted. If one item is chosen at random, the probability that it is rusted or is a nail is

(a) $\frac{3}{16}$ (b) $\frac{5}{16}$ (c) $\frac{11}{16}$ (d) $\frac{14}{16}$

19. A bag contains 5 brown and 4 white socks. A man pulls out two socks. The probability that these are of the same colour is

(a) $\frac{5}{108}$ (b) $\frac{18}{108}$ (c) $\frac{30}{108}$ (d) $\frac{48}{108}$

20. If S is the sample space and $P(A) = \frac{1}{3} P(B)$ and $S = A \cup B$, where A and B are two mutually exclusive events, then $P(A) =$

(a) 1/4 (b) 1/2 (c) 3/4 (d) 3/8

21. If A and B are two events, then $P(\overline{A} \cap B) =$

(a) $P(\overline{A})\, P(\overline{B})$ (b) $1 - P(A) - P(B)$

(c) $P(A) + P(B) - P(A \cap B)$ (d) $P(B) - P(A \cap B$

22. If $P(A \cup B) = 0.8$ and $P(A \cap B) = 0.3$, then $P(\overline{A}) + P(\overline{B}) =$

(a) 0.3 (b) 0.5 (c) 0.7 (d) 0.9

23. A bag X contains 2 white and 3 black balls and another bag Y contains 4 white and 2 black balls. One bag is selected at random and a ball is drawn from it. Then, the probabil hosen to be white is

(a) 2/15 (b) 7/15 (c) 8/15 (d) 14/15

24. Two persons A and B take turns in throwing a pair of dice. The first person to throw 9 from both dice will be awarded the prize. If A throws first, then the probability that B wins the game is

(a) 9/17 (b) 8/17 (c) 8/9 (d) 1/9

25. The probability that in a year of 22nd century chosen at random, there will be 53 Sundays is

(a) 3/28 (b) 2/28 (c) 7/28 (d) 5/28

26. From a set of 100 cards numbered 1 to 100, one card is drawn at random. The probability that the number obtained on the card is divisible by 6 or 8 but not by 24 is

(a) 6/25 (b) 1/4 (c) 1/6 (d) 2/5 (e) 4/5

ANSWERS

1. (a)	**2.** (a)	**3.** (a)	**4.** (b)	**5.** (a)	**6.** (a)	**7.** (b)	**8.** (d)
9. (a)	**10.** (c)	**11.** (c)	**12.** (d)	**13.** (a)	**14.** (a)	**15.** (b)	**16.** (a)
17. (a)	**18.** (c)	**19.** (d)	**20.** (a)	**21.** (d)	**22.** (d)	**23.** (c)	**24.** (b)
25. (d)	**26.** (a)						

SUMMARY

1. If a random experiment is performed, then each of its outcomes is known as an elementary event.
2. The set of all possible outcomes of a random experiment is called the sample space assoicated with it and it is generally denoted by S.
   In other words, the set of all elementary events assoicated to a random experiment is called its sample space.
3. A subset of the sample space associated to a random experiment is said to define a compound event if it is disjoint union of single element subsets of the sample space.
4. An event A associated to a random experiment is said to occur if any one of the elementary events associated to it is an outcome.
   Thus, if an elementary event E is an outcome of a random experiment and A is an event such that $E \in A$, then we say that the event A has occurred.
5. Corresponding to every event A associated with a random experiment we define an event "not A" which occurs when and only when A does not occur.
6. Let S be the sample space associated with a random experiment and A be an event associated to the experiment. Then elementary events belonging to A are known as favourable elementary events to the event A.
   In order words, an elementary event E is said to be favourable to an event A if the occurrence of E ensures the happening or occurrence of event.
7. If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happening or occurrence of A is denoted by $P(A)$ and is defined as the ratio $\frac{m}{n}$
   If $P(A) = 1$, then A is called certain event and A is called an impossible event, if $P(A) = 0$.
   The number of elementary events which will ensure the non-occurrence of A i.e. which ensure the occurrence of $\overline{A}$ is $(n-m)$.

   $$\therefore \quad P(\overline{A}) = \frac{n-m}{n} \Rightarrow P(\overline{A}) = 1 - \frac{m}{n} \Rightarrow P(\overline{A}) = 1 - P(A) \Rightarrow P(A) + P(\overline{A}) = 1$$

   The odds in favour of occurrence of the event A are defined by $m : (n-m)$ i.e ; $P(A) : P(\overline{A})$ and the odds against the occurrence of A are defined by $n-m : m$ i.e; $P(\overline{A}) : P(A)$.

8. Two or more events associated to a random experiment are mutually exclusive if the occurrence of one of them prevents or denies the occurrence all others.
It follows from the above definition that two or more events associated to a random experiment are mutually exclusive, if there is no elementary event which is favourable to all the events.
Thus, if two events A and B are mutually exclusive, then.
$$P(A \cap B) = 0.$$
Similarly, if A, B and C are mutually exclusive events, then $P(A \cap B \cap C) = 0.$
Clearly, all elementary events associated to a random experiment are mutually exclusive as no two or more of them can occur together.

9. Two or more events associated to a random experiment are exhaustive if their union is the sample space. i.e. events $A_1, A_2, \ldots, A_n$ associated to a random experiment with sample space S are exhaustive if $A_1 \cup A_2 \ldots \cup A_n = S$.
All elementary events associated to a random experiment form a system of mutually exclusive and exhaustive events.
For any event A associated to a random experiment A and $\overline{A}$ form a pair of exhaustive and mutually exclusive events.

10. Two events A and B associated to a random experiment are independent if the probability of occurrence or non occurrence of A is not affected by the occurrence or non-occurrence of B.
Three or more events are independent if the probability of occurrence or non-occurrence of any one of them is not affected by the occurrence or non-occurrence of others.
Events associated to independent random experiments are always independent.
If A and B are two mutually exclusive events associated to a random experiment, then the occurrence of any one of these two prevents the occurrence of the other i.e.
If A occurs, then $P(B) = 0$.
If B occurs, then $P(A) = 0$.
It follows from this that mutually exclusive events associated to a random experiment are not independent and vice-versa.

11. (i) If A and B are two events associated with a random experiment, then
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
(ii) If A and B are mutually exclusive events, then
$$P(A \cap B) = 0$$
$$\therefore \quad P(A \cup B) = P(A) + P(B)$$

12. (i) If A, B, C are three events associated with a random experiment, then
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$
(ii) If A, B, C are mutually exclusive events, then
$$P(A \cap B) = P(B \cap C) = P(A \cap C) = P(A \cap B \cap C) = 0.$$
$$\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

13. Let A and B be two events associated to a random experiment. Then,
$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$
$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$
$$P((A \cap \overline{B}) \cup (\overline{A} \cap B)) = P(A) + P(B) - 2P(A \cap B)$$

$P(\overline{A} \cap B)$ is known as the probability of occurrence of B only.
$P(A \cap \overline{B})$ is known as the probability of occurrence of A only.
$P((A \cap \overline{B}) \cup (\overline{A} \cap B))$ is known as the probability of occurrence of excatly one of two events A and B.

14. For any two events A and B the probability that exactly one of A, B occurs is given by

$$P(A) + P(B) - 2P(A \cup B) = P(A \cup B) - P(A \cap B).$$

15. If A, B, C are three events, then

(i) *P (Atleast two of A, B, C occur)*

$$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$$

(ii) *P (Exactly two of A, B, C occur)*

$$= P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)$$

(iii) *P (Exactly one of A, B, C occurs)*

$$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) + 3P(A \cap B \cap C).$$

16 (i) Let A and B be two events associated with a random experiment. Then, the probability of occurrence of event A under the condition that B has already occurred and $P(B) \neq 0$, is called the conditional probability and it is denoted by $P(A/B)$.

Thus, we have

$P(A/B) =$ Probability of occurrence of A given that B has already occurred.

Similarly, $P(B/A)$ when $P(A) \neq 0$ is defined as the probability of occurrence of event B when A has already occurred.

In fact, the meanings of symbols $P(A/B)$ and $P(B/A)$ depend on the nature of the events A and B and also on the nature of the random experiment. These two symbols have the following meaning also.

$P(A/B) =$ Probability of occurrence of A when B occurs

OR

$P(A/B) =$ Probability of occurrence of A when B is taken as the sample space

OR

$P(A/B) =$ Probability of occurrence of A with respect to B.

and,

$P(B/A) =$ Probability of occurrence of B when A occurs

OR

$P(A/B) =$ Probability of occurence of B when A is taken as the sample space.

OR

$P(A/B) =$ Probability of occurrence of B with respect to A.

If A and B are independent events associated with a random experiment, then

$P(A/B) = P(A)$ and $P(B/A) = P(B)$.

17. If A and B are two events associated with a random experiment, then

$P(A \cap B) = P(A)\,P(B/A)$, if $P(A) \neq 0$

or, $P(A \cap B) = P(B)\,P(A/B)$, if $P(B) \neq 0$

18. If A and B are independent events, then $P(A/B) = P(A)$ and $P(B/A) = P(B)$.

$\therefore \quad P(A \cap B) = P(A)\,P(B)$.

Also, $P(A \cup B) = 1 - P(\overline{A})\,P(\overline{B})$

19. If $A_1, A_2, ..., A_n$ are n events associated with a random experiment, then

$$P(A_1 \cap A_2 \cap A_3 \ldots \cap A_n)$$

$$= P(A_1)\,P(A_2/A_1)\;P(A_3/A_1 \cap A_2) \ldots\ldots\ldots\ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1}),$$

where $P(A_i/A_1 \cap A_2 \ldots \cap A_{i-1})$ represents the conditional probability of the occurrence of event A_i, given that the events $A_1, A_2, ..., A_{i-1}$ have already occurred.

20. If A and B are independent events associated with a random experiment, then
   (i) $\overline{A}$ and B are independent events
   (ii) A and $\overline{B}$ are independent events
   (iii) $\overline{A}$ and $\overline{B}$ are also independent events.

21. (i) If A and B are independent events associated to a random experiment, then Probability of occurrence of at least one is given by

$$P(A \cup B) = 1 - P(\overline{A})\,P(\overline{B})$$

(ii) If $A_1, A_2, \ldots A_n$ are independent events associated with a random experiment, then
Probability of occurrence of at least one is given by

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - P(\overline{A_1})\,P(\overline{A_2}) \ldots P(\overline{A_n})$$

22. Let S be the sample space and let $E_1, E_2, ..., E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + \ldots + \ldots\ldots + P(E_n)\,P(A/E_n)$$

or, $$P(A) = \sum_{r=1}^{n} P(E_r)\,P(A/E_r)$$

23. Let S be the sample space and let $E_1, E_2 \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)},\; i = 1, 2, \ldots, n$$

The events $E_1, E_2, \ldots, E_n$ are usually referred to as 'hypothesis' and the probabilities $P(E_1), P(E_2), \ldots, P(E_n)$ are known as the 'priori' probabilities as they exist before we obtain any information from the experiment.

The probabilities $P(A/E_i)$; $i = 1, 2, \ldots, n$ are called the 'likelyhood probabilities' as they tell us how likely the event A under consideration occurs, given each and every priori probabilities.

The probabilities $P(E_i/A)$; $i = 1, 2, ..., n$ are called the 'posterior probabilities' as they are determined after the results of the experiment are known.'

19. If $A_1, A_2, \ldots, A_n$ are n events associated with a random experiment, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

where $P(A_i/A_1 \cap A_2 \cap \ldots \cap A_{i-1})$ represents the conditional probability of the occurrence of event A_i, given that the events $A_1, A_2, \ldots, A_{i-1}$ have already occurred.

20. If A and B are independent events associated with a random experiment, then

(i) A and $\bar{B}$ are independent events

(ii) $\bar{A}$ and B are independent events

(iii) $\bar{A}$ and $\bar{B}$ are also independent events.

21. (i) If A and B are two independent events associated to a random experiment, then Probability of occurrence of at least one $= P(A \cup B) = 1 - P(\bar{A})\, P(\bar{B})$.

(ii) If $A_1, A_2, \ldots, A_n$ are independent events associated with a random experiment, then probability of occurrence of at least one is given by

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - P(\bar{A}_1)\, P(\bar{A}_2) \ldots P(\bar{A}_n)$$

22. Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/E_n)$$

or $P(A) = \sum_{i=1}^{n} P(E_i)\, P(A/E_i)$

23. Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}, \quad i = 1, 2, \ldots, n$$

The events $E_1, E_2, \ldots, E_n$ are usually referred to as 'hypothesis' and the probabilities $P(E_1), P(E_2), \ldots, P(E_n)$ are known as the 'priori' probabilities as they exist before we obtain any information from the experiment.

The probabilities $P(A/E_i)$, $i = 1, 2, \ldots, n$ are called the likelihood probabilities as they tell us how likely the event A under consideration occurs, given each and every priori probabilities.

The probabilities $P(E_i/A)$, $i = 1, 2, \ldots, n$ are called the posterior probabilities as they are determined after the results of the experiment are known.

31

MEAN AND VARIANCE OF A RANDOM VARIABLE

31.1 INTRODUCTION

Corresponding to every outcome of a random experiment, we can associate a real number. This correspondence between the elements of the sample space associated to a random experiment and the set of real numbers is defined as a random variable. If a random variable assumes countable number of values, it is called a discrete random variable. Otherwise, it is known as continuous random variable. We shall study these two types of random variables in the following sections.

31.2 DISCRETE RANDOM VARIABLE

DEFINITION *Let S be the sample space associated with a given random experiment. Then, a real valued function X which assigns to each event $w \in S$ to a unique real number X (w) is called a random variable.*

In other words, a random variable is a real valued function having domain as the sample space associated with a random experiment.

Thus, a random variable associated with a given random experiment associates every event to a unique real number as discussed below.

Consider a random experiment of tossing three coins. The sample space of eight possible outcomes of this experiment is given by

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let X be *a* real valued function on S, defined by

$X(w)$ = number of heads in $w \in S$.

Then, X is a random variable such that :

$$X(HHH) = 3,\ X(HHT) = 2,\ X(HTH) = 2,\ X(THH) = 2$$

$$X(HTT) = 1,\ X(THT) = 1,\ X(TTH) = 1, \text{ and } X(TTT) = 0$$

Also, if w denotes the event "getting two heads", then

$$w = \{HTH, THH, HHT\}$$

and, $\quad X(w) = 2$

Similarly, X associates every other compound event to a unique real number.

For the random variable X, we have range $(X) = \{0, 1, 2, 3\}$ and we say that X is a random variable such that it assumes values 0, 1, 2, 3. This random variable can also be described as the number of heads in a single throw of three coins.

Now, consider the random experiment of throwing an unbiased die. Let Y be a real valued function defined on the sample space $S = \{1, 2, 3, 4, 5, 6\}$ associated with the random experiment, defined by

$$Y(w) = \begin{cases} 1, \text{ if the outcome is an even number} \\ -1, \text{ if the outcome is an odd number} \end{cases}$$

Clearly, Y is a random variable such that :

$$Y(1) = -1, Y(2) = 1, Y(3) = -1, Y(4) = 1, Y(5) = -1 \text{ and } Y(6) = 1.$$

Here, range $(Y) = \{-1, 1\}$. Therefore, we say that Y is a random variable such that it assumes values -1 and 1.

ILLUSTRATION 1 *Consider a random experiment of tossing three coins. Let X be a real valued function defined on the sample space*

$$S = \{HHH, HHT, HTH, THH, HTT, TTH, THT, TTT\} \text{ such that}$$

$$X(w) = \text{Number of tails in } w \in S.$$

Then, X is a random variable such that

$$X(HHH) = 0, X(HHT) = 1, X(HTH) = 1, X(THH) = 1, X(HTT) = 2,$$

$$X(THT) = 2, X(TTH) = 2 \text{ and } X(TTT) = 3$$

Clearly, range of X is $\{0, 1, 2, 3\}$

ILLUSTRATION 2 *Consider a random experiment of throwing a six faced die. Let X denote the number on the upper face of the die. Then,*

$$X(1) = 1, X(2) = 2, X(3) = 3, X(4) = 4, X(5) = 5 \text{ and } X(6) = 6$$

Clearly, X is a random variable which assumes values 1, 2, 3, 4, 5, 6 i.e. range of $X = \{1, 2, 3, 4, 5, 6\}$.

ILLUSTRATION 3 *Let there be a bag containing 5 white, 4 red and 3 green balls. Three balls are drawn. If X denotes the number of green balls in the draw. Then, X can assume values 0, 1, 2, 3. Clearly, X is a random variable with its range* $= \{0, 1, 2, 3\}$

ILLUSTRATION 4 *A pair of dice is thrown. If X denotes the sum of the numbers on two dice, then X assumes values 2, 3, 4,..., 12. Clearly, X is a random variable with its range* $\{2, 3, 4, \ldots, 12\}$.

31.3 PROBABILITY DISTRIBUTION

In the previous section, we have defined random variable. Now, consider a random experiment in which three coins are tossed simultaneously (or a coin is tossed three times). Let X be a random variable defined on the sample space

$$S = \{HHH, HTH, THH, HHT, THT, TTH, HTT, TTT\} \text{ such that}$$

$$X(w) = \text{number of heads in } w \in S.$$

Clearly, X assumes value 0, 1, 2, 3.

Now, $\quad P(X = 0) = \text{Probability of getting no head} = P(TTT) = \frac{1}{8}$

$$P(X = 1) = \text{Probability of getting one head}$$

$$= P(HTT \text{ or } THT \text{ or } TTH) = \frac{3}{8}$$

$$P(X = 2) = \text{Probability of getting two heads}$$

$$= P(HHT \text{ or } THH \text{ or } HTH) = \frac{3}{8}$$

and, $\quad P(X = 3) = \text{Probability of getting 3 heads}$

$$= P(HHH) = \frac{1}{8}$$

These values of X and the corresponding probabilities can be exhibited as under :

X :	0	1	2	3
P(X) :	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

This tabular representation of the values of a random variable X and the corresponding probabilities is known as its probability distribution.

The formal definition of the probability distribution of a random variable is as given below.

PROBABILITY DISTRIBUTION If a random variable X takes values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \cdots, p_n$, then

X :	x_1	x_2	x_3	......	x_n
P (X) :	p_1	p_2	p_3	......	p_n

is known as the probability distribution of X.

Thus, a tabular description giving the values of the random variable along with the corresponding probabilities is called its probability distribution.

REMARK 1 *The probability distribution of a random variable X is defined only when we have the various values of the random variable e.g. $x_1, x_2, \ldots, x_n$ together with respective probabilities $p_1, p_2, \cdots, p_n$ satisfying $\sum_{i=1}^{n} p_i = 1$.*

REMARK 2 *If X is a random variable with the probability distribution*

X :	x_1	x_2	......	x_n
P (X) :	p_1	p_2	......	p_n

Then,

$$P(X \le x_i) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_i)$$
$$= p_1 + p_2 + \ldots + p_i$$
$$P(X < x_i) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_{i-1})$$
$$= p_1 + p_2 + \ldots p_{i-1}$$
$$P(X \ge x_i) = P(X = x_i) + P(X = x_{i+1}) + \ldots + P(X = x_n)$$
$$= p_i + p_{i+1} + \ldots + p_n$$
$$P(X > x_i) = P(X = x_{i+1}) + P(X = x_{i+2}) + \ldots + P(X = x_n)$$
$$= p_{i+1} + p_{i+2} + \ldots + p_n$$

Also, $P(X \ge x_i) = 1 - P(X < x_i)$, $P(X > x_i) = 1 - P(X \le x_i)$,

$$P(X \le x_i) = 1 - P(X > x_i) \text{ and } P(X < x_i) = 1 - P(X \ge x_i)$$
$$P(x_i \le X \le x_j) = P(X = x_i) + P(X = x_{i+1}) + \ldots + P(X = x_j)$$
$$P(x_i < X < x_j) = P(X = x_{i+1}) + P(X = x_{i+2}) + \ldots + P(X = x_{j-1})$$

The graphical representation of a probability distribution is as follows :

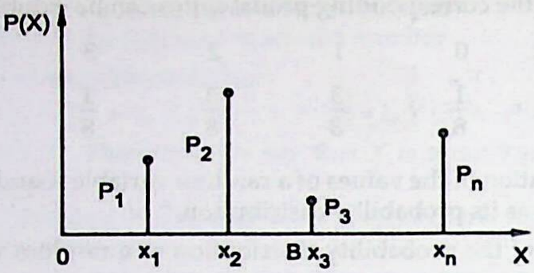

Fig. 31.1

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Determine which of the following can be probability distributions of a random variable X :*

(i)

X:	0	1	2
$P(X)$:	0.4	0.4	0.2

(ii)

X:	0	1	2
$P(X)$:	0.6	0.1	0.2

(iii)

X:	0	1	2	3	4
$P(X)$:	0.1	0.5	0.2	-0.1	0.3

SOLUTION We have,

(i) $P(X=0)+P(X=1)+P(X=2)=0.4+0.4+0.2=1.$

Hence, the given distribution of probabilities is a probability distribution of random variable X.

(ii) $P(X=0)+P(X=1)+P(X=2)=0.6+0.1+0.2=0.9 \neq 1.$

Hence, the given distribution of probabilities is not a probability ditribution.

(iii) We have,

$$P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$$
$$=0.1+0.5+0.2-0.1+0.3=1$$

But, $P(X=3) = 0.1 < 0$

So, the given distribution of probabilities is not a probability distribution.

EXAMPLE 2 *An unbiased die is rolled. If the random variable X is defined as*

$$X(w) = \begin{cases} 1, \text{ if the outcome } w \text{ is an even number} \\ 0, \text{ if the outcome } w \text{ is an odd number.} \end{cases}$$

Find the probability distribution of X.

SOLUTION In a single throw of a die either we get an even number or we get an odd number. Thus, the possible values of the random variable X are 0 and 1.

Now,

$$P(X=0) = \text{Probability of getting an odd number} = \frac{3}{6} = \frac{1}{2}$$

$$P(X=1) = \text{Probability of getting an even number} = \frac{3}{6} = \frac{1}{2}$$

Thus, the probability distribution of the random variable X is given by

X:	0	1
$P(X)$:	$\frac{1}{2}$	$\frac{1}{2}$

EXAMPLE 3 *The random variable X has a probability distribution P (x) of the following form, where k is some number:*

$$P(X=x)=\begin{cases} k & , \text{if } x=0 \\ 2k & , \text{if } x=1 \\ 3k & , \text{if } x=2 \\ 0 & , \text{otherwise} \end{cases}$$

(i) Determine the value of k *(ii) Find $P(X<2)$, $P(X \leq 2)$, $P(X \geq 2)$.*

SOLUTION (i) The probability distribution of X is

$X:$	0	1	2
$P(X):$	k	$2k$	$3k$

The given distribution of probabilities will be a probability distribution, if

$$P(X=0)+P(X=1)+P(X=2)=1$$

$$\Rightarrow \quad k+2k+3k=1 \Rightarrow 6k=1 \Rightarrow k=\frac{1}{6}$$

(ii) $P(X<2)=P(X=0)+P(X=1)=k+2k=3k=\frac{3}{6}=\frac{1}{2}$

(iii) $P(X \leq 2)=P(X=0)+P(X=1)+P(X=2)=k+2k+3k=6k=1$

(iii) $P(X \geq 2)=1-P(X<2)=1-\frac{1}{2}=\frac{1}{2}$

EXAMPLE 4 *Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the value x has the following form, where k is some unknown constant.*

$$P(X=x)=\begin{cases} 0.1 & , \text{if } x=0 \\ kx & , \text{if } x=1 \text{ or } 2 \\ k(5-x) & , \text{if } x=3 \text{ or } 4 \\ 0 & , \text{otherwise} \end{cases}$$

(i) Find the value of k (ii) What is the probability that you study at least two hours? (iii) Exactly two hours? (iv) At most two hours? **[NCERT]**

SOLUTION The probability distribution of X is

$X:$	0	1	2	3	4
$P(X):$	0.1	k	$2k$	$2k$	k

(i) The given distribution is a probability distribution.

$$\therefore \quad P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)=1$$

$$\Rightarrow \quad 0.1+k+2k+2k+k=1$$

$$\Rightarrow \quad 6k=0.9 \Rightarrow k=0.15$$

(ii) Required probability $= P(X \geq 2)$

$$= P(X=2)+P(X=3)+P(X=4)$$

$$= 2k+2k+k=5k=5\times 0.15=0.75$$

(iii) Required probability $= P(X=2)=2k=2\times 0.15=0.3$

(iv) Required probability $= P(X \leq 2)=P(X=0)+P(X=1)+P(X=2)$

$$= 0.1+k+2k=0.1+3k=0.1+3\times 0.15=0.55$$

EXAMPLE 5 *A random variable X has the following probability distribution values of X,*

$X:$	0	1	2	3	4	5	6	7
$P(X):$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find each of the following:

(i) k (ii) $P(X<6)$ (iii) $P(X \geq 6)$ (iv) $P(0<X<5)$

SOLUTION (i) Since the sum of all the probabilities in a probability distribution is always unity. Therefore,

$$P(X=0)+P(X=1)+\ldots\ldots+P(X=7) = 1$$

$\Rightarrow$ $0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$

$\Rightarrow$ $10k^2+9k-1 = 0$

$\Rightarrow$ $(10k-1)(k+1) = 0$

$\Rightarrow$ $10k-1 = 0$ $[\because k \geq 0 \therefore k+1 \neq 0]$

$\Rightarrow$ $k = \dfrac{1}{10}$

(ii) $P(X<6) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$\Rightarrow$ $P(X<6) = 0+k+2k+2k+3k+k^2$

$\Rightarrow$ $P(X<6) = k^2+8k$

$\Rightarrow$ $P(X<6) = \left(\dfrac{1}{10}\right)^2+\dfrac{8}{10}$ $[\because k = 1/10]$

$\Rightarrow$ $P(X<6) = \dfrac{81}{100}$

(iii) $P(X \geq 6) = P(X=6)+P(X=7)$

$\Rightarrow$ $P(X \geq 6) = 2k^2+7k^2+k$

$\Rightarrow$ $P(X \geq 6) = 9k^2+k$

$\Rightarrow$ $P(X \geq 6) = \dfrac{9}{100}+\dfrac{1}{10}$ $[\because k = 1/10]$

$\Rightarrow$ $P(X \geq 6) = \dfrac{19}{100}$

<u>**ALITER**</u> $P(X \geq 6) = 1-P(X<6) = 1-\dfrac{81}{100} = \dfrac{19}{100}.$

(v) $P(0<X<5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$\Rightarrow$ $P(0<X<5) = k+2k+2k+3k$

$\Rightarrow$ $P(0<X<5) = 8k$

$\Rightarrow$ $P(0<X<5) = \dfrac{8}{10}$ $[\because k = 1/10]$

$\Rightarrow$ $P(0<X<5) = \dfrac{4}{5}$

EXAMPLE 6 *A random variable X can take all non-negative integral values and the probability that X takes the value r is proportional to α^r $(0 < \alpha < 1)$. Find $P(X = 0)$.*

SOLUTION We have,

$$P(X = r) \propto \alpha^r \Rightarrow P(X = r) = \lambda\, \alpha^r, r = 0, 1, 2, \ldots$$

Since sum of all the probabilities in a probability distribution is 1.

$$\therefore \quad P(X = 0) + P(X = 1) + P(X = 2) + \ldots = 1$$

$$\Rightarrow \quad \lambda\alpha^0 + \lambda\alpha^1 + \lambda\alpha^2 + \ldots = 1 \qquad [\because P(X = r) = \lambda\, \alpha^r \text{ (given)}]$$

$$\Rightarrow \quad \lambda(1 + \alpha + \alpha^2 + \alpha^3 + \ldots) = 1$$

$$\Rightarrow \quad \lambda \cdot \left(\frac{1}{1-\alpha}\right) = 1$$

$$\Rightarrow \quad \lambda = 1 - \alpha.$$

$$\therefore \quad P(X = r) = (1-\alpha)\,\alpha^r, \quad r = 0, 1, 2, \ldots$$

Hence, $P(X = 0) = (1-\alpha)\,\alpha^0 = (1-\alpha)$.

EXAMPLE 7 *Find the probability distribution of X, the number of heads in two tosses of a coin (or a simultaneous toss of two coins).*

SOLUTION When two coins are tossed, there may be 1 head, 2 heads or no head at all. Thus, the possible values of X are 0, 1, 2.

Now,

$$P(X = 0) = P(\text{getting no head}) = P(TT) = \frac{1}{4}$$

$$P(X = 1) = P(\text{getting one head})$$

$$= P(HT \text{ or } TH) = \frac{2}{4} = \frac{1}{2}$$

$$P(X = 2) = P(\text{getting two heads}) = P(HH) = \frac{1}{4}$$

Thus, the required probability distribution of X is given by

X :	0	1	2
P(X) :	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

EXAMPLE 8 *Three cards are drawn from a pack of 52 playing cards. Find the probability distribution of the number of aces.* **[CBSE 2001]**

SOLUTION Let X denote the number of aces in a sample of 3 cards drawn from a well shuffled pack of 52 playing cards. Since there are four aces in the pack, therefore in the sample of 3 cards drawn either there can be no ace or there can be one ace or two aces or three aces. Thus, X can take values 0, 1, 2, and 3.

Now, $P(X = 0)$ = Probability of getting no ace

$$\Rightarrow \quad P(X = 0) = \text{Probability of getting 3 other cards} = \frac{{}^{48}C_3}{{}^{52}C_3} = \frac{4324}{5525}$$

$$P(X = 1) = \text{Probability of getting one ace and two other cards}$$

$$\Rightarrow \quad P(X = 1) = \frac{{}^{4}C_1 \times {}^{48}C_2}{{}^{52}C_3} = \frac{1128}{5525}$$

$P(X=2)$ = Probability of getting two aces and one other card

$$\Rightarrow \quad P(X=2) = \frac{{}^4C_2 \times {}^{48}C_1}{{}^{52}C_3} = \frac{72}{5525}$$

and, $P(X=3)$ = Probability of getting 3 aces

$$\Rightarrow \quad P(X=3) = \frac{{}^4C_3}{{}^{52}C_3} = \frac{1}{5525}$$

Thus, the probability distribution of random variable X is given by

X :	0	1	2	3
P(X) :	$\frac{4324}{5525}$	$\frac{1128}{5525}$	$\frac{72}{5525}$	$\frac{1}{5525}$

It is to note here that the sum of the probabilities is 1 which is the condition for a distribution to be a probability distribution.

EXAMPLE 9 *An urn contains 4 white and 6 red balls. Four balls are drawn at random from the urn. Find the probability distribution of the number of white balls.*

SOLUTION Let X denote the number of white balls drawn from the urn. Since there are 4 white balls, therefore X can take values 0, 1, 2, 3 and 4.

Now, $P(X=0)$ = Probability of getting no white ball

$\Rightarrow$ $P(X=0)$ = Probability that 4 balls drawn are red $= \frac{{}^6C_4}{{}^{10}C_4} = \frac{1}{14}$,

$P(X=1)$ = Probability of getting one white ball $= \frac{{}^4C_1 \times {}^6C_3}{{}^{10}C_4} = \frac{8}{21}$,

$P(X=2)$ = Probability of getting two white balls $= \frac{{}^4C_2 \times {}^6C_2}{{}^{10}C_4} = \frac{6}{14}$,

$P(X=3)$ = Probability of getting three white balls $= \frac{{}^4C_3 \times {}^6C_1}{{}^{10}C_4} = \frac{4}{35}$,

and, $P(X=4)$ = Probability of getting 4 white balls $= \frac{{}^4C_4}{{}^{10}C_4} = \frac{1}{210}$.

Thus, the probability distribution of X is given by

X :	0	1	2	3	4
P(X) :	$\frac{1}{14}$	$\frac{8}{21}$	$\frac{6}{14}$	$\frac{4}{35}$	$\frac{1}{210}$

EXAMPLE 10 *Four bad oranges are mixed accidently with 16 good oranges. Find the probability distribution of the number of bad oranges in a draw of two oranges.* [CBSE 2002C]

SOLUTION Let X denote the number of bad oranges in a draw of 4 oranges drawn from group of 16 good oranges and 4 bad oranges. Since there are 4 bad oranges in the group, therefore X can take values 0, 1 and 2.

Now, $P(X=0)$ = Probability of getting no bad orange

$\Rightarrow$ $P(X=0)$ = Probability of getting 2 good oranges $= \frac{{}^{16}C_2}{{}^{20}C_2} = \frac{12}{19}$,

$$P(X=1) = \text{Probability of getting one bad orange} = \frac{{}^4C_1 \times {}^{16}C_1}{{}^{20}C_2} = \frac{32}{95},$$

and, $$P(X=2) = \frac{{}^4C_2}{{}^{20}C_2} = \frac{3}{95}$$

Thus, the probability distribution of X is given by

X :	0	1	2
P(X) :	$\frac{12}{19}$	$\frac{32}{95}$	$\frac{3}{95}$

EXAMPLE 11 *An unbiased die is thrown twice. Find the probability distribution of the number of sixes.*

SOLUTION Let X denote the number of times six occurs i.e. the number of sixes. Since the die is thrown twice, X can take values 0, 1 and 2.

Let S_i denote the event that a six occurs on the die in i th throw and F_i denote the event that the six does not occur in the i th throw. Then,

$$\begin{aligned} P(X=0) &= \text{Probability of not getting six in both the throws} \\ &= P(F_1 \text{ and } F_2) = P(F_1 \cap F_2) \\ &= P(F_1)\,P(F_2) \qquad [\because F_1, F_2 \text{ are independent events}] \\ &= \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}, \end{aligned}$$

$$\begin{aligned} P(X=1) &= \text{Probability of getting one six in two throws} \\ &= P[(F_1 \text{ and } S_2) \text{ or } (S_1 \text{ and } F_2)] \\ &= P[(F_1 \cap S_2) \cup (S_1 \cap F_2)] \\ &= P(F_1 \cap S_2) + P(S_1 \cap F_2) \qquad [\text{By add. Theo.}] \\ &= P(F_1)\,P(S_2) + P(S_1)\,P(F_2) \qquad \left[\begin{array}{r}\text{By multiplication theorem} \\ \text{for indep. events}\end{array}\right] \\ &= \frac{5}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{5}{6} = \frac{10}{36} = \frac{5}{18}, \end{aligned}$$

and, $$\begin{aligned} P(X=2) &= \text{Probability of getting sixes in both the throws} \\ &= P(S_1 \cap S_2) \\ &= P(S_1)\,P(S_2) \qquad [\text{By Multiplication Theorem}] \\ &= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \end{aligned}$$

Thus, the probability distribution of X is

X :	0	1	2
P(X) :	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

EXAMPLE 12 *Two cards are drawn successively with replacement from a well-shuffled pack of 52 cards. Find the probability distribution of the number of kings.*

SOLUTION Let X denote the number of kings. Then, X can take values 0, 1 or 2.
Let S_i denote the event of getting a king in the i th draw and F_i denote the event of not getting a king in the i th draw. Then,

$$\begin{aligned}
P(X=0) &= \text{Probability of not getting a king in the two draws}\\
&= P(\text{not a king in 1st draw and not a king in second draw})\\
&= P(F_1 \cap F_2) = P(F_1)\,P(F_2) \qquad [\because F_1 \text{ and } F_2 \text{ are independent}]\\
&= \frac{48}{52}\times\frac{48}{52} = \frac{144}{169},
\end{aligned}$$

$$\begin{aligned}
P(X=1) &= \text{Probability of getting one king in the two draws}\\
&= P((S_1 \cap F_2) \text{ or } (F_1 \cap S_2)) = P(S_1 \cap F_2) + P(F_1 \cap S_2)\\
&= P(S_1)\,P(F_2) + P(F_1)\,P(S_2) \qquad [\text{By multiplication theorem}]\\
&= \frac{4}{52}\times\frac{48}{52} + \frac{48}{52}\times\frac{4}{52} = \frac{24}{169},
\end{aligned}$$

and,
$$\begin{aligned}
P(X=2) &= \text{Probability of getting kings in both the draws}\\
&= P(S_1 \cap S_2) = P(S_1)\,P(S_2) \qquad [\text{By multiplication theorem}]\\
&= \frac{4}{52}\times\frac{4}{52} = \frac{1}{169}
\end{aligned}$$

Thus, the probability distribution of X is

X:	0	1	2
$P(X)$:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

EXAMPLE 13 *A coin is tossed until a head appears or the tail appears 4 times in succession. Find the probability distribution of the number of tosses.*

SOLUTION Let S be the sample space associated with the given random experiment. Then,

$$S = \{H, TH, TTH, TTTH, TTTT\}$$

Let X denote the number of tosses. Then, X can take values 1, 2, 3 and 4

Now,

$$P(X=1) = P(H) = \frac{1}{2},\quad P(X=2) = P(TH) = P(T)\,P(H) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4},$$

$$P(X=3) = P(TTH) = P(T)\,P(T)\,P(H) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8},$$

and, $$P(X=4) = P(TTTH \text{ or } TTTT) = P(TTTH) + P(TTTT)$$

$$\Rightarrow \quad P(X=4) = P(T)\,P(T)\,P(T)\,P(H) + P(T)\,P(T)\,P(T)\,P(T) = \frac{1}{16}+\frac{1}{16} = \frac{1}{8}.$$

Thus, the probability distribution of X is given by

X:	1	2	3	4
$P(X)$:	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

EXAMPLE 14 *An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in a random draw of three balls.*

SOLUTION When three balls are drawn, there may be all red, 2 red, 1 red or no red ball at all. Thus, if X denotes the number of red balls in a random draw of three balls. Then, X can take values 0, 1, 2, 3.

Now,

$$P(X=0) = P(\text{Getting no red ball})$$

$$\Rightarrow \quad P(X=0) = P(\text{Getting three white balls})$$

$$\Rightarrow \quad P(X=0) = \frac{{}^4C_3}{{}^7C_3} = \frac{4\times 3\times 2}{7\times 6\times 5} = \frac{4}{35}$$

$$P(X=1) = P(\text{Getting one red and two white balls}) = \frac{{}^3C_1 \times {}^4C_2}{{}^7C_3} = \frac{18}{35}$$

$$P(X=2) = P(\text{Getting two red and one white ball}) = \frac{{}^3C_2 \times {}^4C_1}{{}^7C_3} = \frac{12}{35}$$

$$P(X=3) = P(\text{Getting three red balls}) = \frac{{}^3C_3}{{}^7C_3} = \frac{1}{35}$$

Thus, the probability distribution of the number of red balls is given by

X:	0	1	2	3
$P(X)$:	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

EXAMPLE 15 *Two cards are drawn without replacement from a well-shuffled deck of 52 cards. Determine the probability distribution of the number of face cards (i.e. Jack, Queen, King and Ace).*

SOLUTION Let X denote the number of face cards in two draws. Then, X can take values 0, 1, 2.

Let F_i denote the event of getting a face card in i^{th} draw.

Then,

$$P(X=0) = \text{Probability of getting no face card}$$

$$\Rightarrow \quad P(X=0) = P(\overline{F_1} \cap \overline{F_2})$$

$$\Rightarrow \quad P(X=0) = P(\overline{F_1})\, P(\overline{F_2}/\overline{F_1}) \qquad \text{[By multiplication theorem]}$$

$$\Rightarrow \quad P(X=0) = \frac{36}{52} \times \frac{35}{51} = \frac{105}{221}$$

$$P(X=1) = \text{Probability of getting one face card and one other card}$$

$$\Rightarrow \quad P(X=1) = P\left((F_1 \cap \overline{F_2}) \cup (\overline{F_1} \cap F_2)\right)$$

$$\Rightarrow \quad P(X=1) = P(F_1 \cap \overline{F_2}) + P(\overline{F_1} \cap F_2) \qquad \text{[By addition theorem]}$$

$$\Rightarrow \quad P(X=1) = P(F_1)\, P(\overline{F_2}/F_1) + P(\overline{F_1})\, P(F_2/\overline{F_1})$$

$$\Rightarrow \quad P(X=1) = \frac{16}{52} \times \frac{36}{51} + \frac{36}{52} \times \frac{16}{51} = \frac{96}{221}$$

$$P(X=2) = \text{Probability of getting both face cards}$$

$$\Rightarrow \quad P(X=2) = P(F_1 \cap F_2)$$

$$\Rightarrow \quad P(X=2) = P(F_1)\, P(F_2/F_1)$$

$$\Rightarrow \quad P(X=2) = \frac{16}{52} \times \frac{15}{51} = \frac{20}{221}$$

Hence, the required probability distribution is

X:	0	1	2
$P(X)$:	$\frac{105}{221}$	$\frac{96}{221}$	$\frac{20}{221}$

EXAMPLE 16 *Find the probability distribution of the number of green balls drawn when 3 balls are drawn, one by one, without replacement from a bag containing 3 green and 5 white balls.*

SOLUTION Let X denote the total number of green balls drawn in three draws without replacement. Clearly, there may be all green, 2 green, 1 green or no green at all. Thus, X can assume values 0, 1, 2, and 3. Let G_i denote the event of getting a green ball in i^{th} draw.

Now,

$$P(X=0) = \text{Probability of getting no green ball in three draws}$$

$$\Rightarrow \quad P(X=0) = P(\overline{G_1} \cap \overline{G_2} \cap \overline{G_3})$$

$$\Rightarrow \quad P(X=0) = P(\overline{G_1})\, P(\overline{G_2}/\overline{G_1})\, P(\overline{G_3}/\overline{G_1} \cap \overline{G_2})$$

$$\Rightarrow \quad P(X=0) = \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{5}{28}.$$

$$P(X=1) = \text{Probability of getting one green ball in three draws}$$

$$\Rightarrow \quad P(X=1) = P\Big((G_1 \cap \overline{G_2} \cap \overline{G_3}) \cup (\overline{G_1} \cap G_2 \cap \overline{G_3}) \cup (\overline{G_1} \cap \overline{G_2} \cap G_3)\Big)$$

$$\Rightarrow \quad P(X=1) = P(G_1 \cap \overline{G_2} \cap \overline{G_3}) + P(\overline{G_1} \cap G_2 \cap \overline{G_3}) + P(\overline{G_1} \cap \overline{G_2} \cap G_3)$$

$$\Rightarrow \quad P(X=1) = P(G_1)\, P(\overline{G_2}/G_1)\, P(\overline{G_3}/G_1 \cap \overline{G_2}) + P(\overline{G_1})\, P(G_2/\overline{G_1})\, P(\overline{G_3}/\overline{G_1} \cap G_2)$$
$$+ P(\overline{G_1})\, P(\overline{G_2}/\overline{G_1})\, P(G_3/\overline{G_1} \cap \overline{G_2})$$

$$\Rightarrow \quad P(X=1) = \frac{3}{8} \times \frac{5}{7} \times \frac{4}{6} + \frac{5}{8} \times \frac{3}{7} \times \frac{4}{6} + \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{15}{28}$$

$$P(X=2) = P\Big((G_1 \cap G_2 \cap \overline{G_3}) \cap (\overline{G_1} \cap G_2 \cap G_3) \cup (G_1 \cap \overline{G_2} \cap G_3)\Big)$$

$$\Rightarrow \quad P(X=2) = P(G_1)\, P(G_2/G_1)\, P(\overline{G_3}/G_1 \cap G_2) + P(\overline{G_1})\, P(G_2/\overline{G_1})\, P(G_3/\overline{G_1} \cap G_2)$$
$$+ P(G_1)\, P(\overline{G_2}/G_1)\, P(G_3/G_1 \cap \overline{G_2})$$

$$\Rightarrow \quad P(X=2) = \frac{3}{8} \times \frac{2}{7} \times \frac{5}{6} + \frac{5}{8} \times \frac{3}{7} \times \frac{2}{6} + \frac{3}{8} \times \frac{5}{7} \times \frac{2}{6} = \frac{15}{56}$$

and,

$$P(X=3) = P(G_1 \cap G_2 \cap G_3) = P(G_1)\, P(G_2/G_1)\, P(G_3/G_1 \cap G_2)$$

$$\Rightarrow \quad P(X=3) = \frac{3}{8} \times \frac{2}{7} \times \frac{1}{6} = \frac{1}{56}$$

Thus, the probability distribution of the number of green balls is given by

X:	0	1	2	3
$P(X)$:	$\frac{5}{28}$	$\frac{15}{28}$	$\frac{15}{56}$	$\frac{1}{56}$

EXAMPLE 17 *From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. If the sample is drawn randomly, find*

(i) *the probability distribution of X* **[CBSE 2010]** (ii) $P(X \leq 1)$

(iii) $P(X < 1)$ (iv) $P(0 < X < 2)$.

SOLUTION (i) Clearly, X can assume values 0, 1, 2, 3 such that

$$P(X = 0) = \text{Probabilty of getting no defective item} = \frac{^7C_4}{^{10}C_4} = \frac{1}{6}$$

$$P(X = 1) = \text{Probability of getting one defective item} = \frac{^3C_1 \times {^7C_3}}{^{10}C_4} = \frac{1}{2}$$

$$P(X = 2) = \text{Probability of getting two defective items} = \frac{^3C_2 \times {^7C_2}}{^{10}C_4} = \frac{3}{10}$$

and, $$P(X = 3) = \text{Probability of getting three defective items} = \frac{^3C_3 \times {^7C_1}}{^{10}C_4} = \frac{1}{30}$$

Hence, the probability distribution of X is

X:	0	1	2	3
P(X):	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

(ii) $P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$

(iii) $P(X < 1) = P(X = 0) = \frac{1}{6}$

(iv) $P(0 < X < 2) = P(X = 1) = \frac{1}{2}$

EXAMPLE 18 *We take 8 identical slips of paper, write the number 0 on one of them, the number 1 on three of the slips, the number 2 on three of the slips and the number 3 on one of the ships. These slips are folded, put in a box and thoroughly mixed. One slip is drawn at random from the box. If X is the random variable denoting the number written on the drawn slip, find the probability distribution of X*

SOLUTION Clearly, X takes values 0, 1, 2, 3 such that

$$P(X = 0) = \text{Probability of getting a slip marked } 0 = \frac{1}{8}$$

$$P(X = 1) = \text{Probability of getting a slip marked } 1 = \frac{3}{8}$$

$$P(X = 2) = \text{Probability of getting a slip marked } 2 = \frac{3}{8}$$

and, $$P(X = 3) = \text{Probability of getting a slip marked } 3 = \frac{1}{8}$$

Hence, the probability distribution of X is

X:	0	1	2	3
P(x):	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

EXAMPLE 19 *A coin is biased so that the head is 3 times as likely to occur as tail. If coin is tossed twice, find the probability distribution for the number of tails.* **[NCERT]**

SOLUTION Let p be the probability of getting a tail in a single toss of a coin. Then, probability of getting a head $= 3p$.

Since "getting head" and "getting tail" are mutually exclusive and exhaustive events in a single toss of a coin.

$$\therefore \quad P(H) + P(T) = 1$$

$$\Rightarrow \quad p + 3p = 1 \Rightarrow p = \frac{1}{4}$$

$$\Rightarrow \quad P(H) = \frac{3}{4} \text{ and } P(T) = \frac{1}{4}$$

Let X denote the number of tails in two tosses of a coin. Then, X can take values 0, 1, 2.

Now,

$$P(X=0) = \text{Probability of getting no tail}$$

$$\Rightarrow \quad P(X=0) = \text{Probability of getting both heads}$$

$$\Rightarrow \quad P(X=0) = P(HH)$$

$$\Rightarrow \quad P(X=0) = P(H) \times P(H) \qquad [\because \text{ two trials are independent}]$$

$$\Rightarrow \quad P(X=0) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$$

$$P(X=1) = \text{Probability of getting one tail and one head.}$$

$$\Rightarrow \quad P(X=1) = P(HT) + P(TH)$$

$$\Rightarrow \quad P(X=1) = P(H)\,P(T) + P(T)\,P(H)$$

$$\Rightarrow \quad P(X=1) = \frac{3}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4} = \frac{3}{8}$$

$$P(X=2) = \text{Probability of getting both tails}$$

$$\Rightarrow \quad P(X=2) = P(TT)$$

$$\Rightarrow \quad P(X=2) = P(T)\,P(T) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

Hence, the probability distribution of X is

X :	0	1	2
P(X) :	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

EXAMPLE 20 *A die is loaded in such a way that an even number is twice likely to occur as an odd number. If the die is tossed twice, find the probability distribution of the random variable X representing the perfect squares in the two tosses.*

SOLUTION Let p be the probability of getting on odd number in a single throw of a die. Then, probability of getting on even number is $2p$.

We have,

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

$$\Rightarrow \quad p + 2p + p + 2p + p + 2p = 1 \qquad [\because \text{ Sum of the probability} = 1]$$

$$\Rightarrow \quad 9p = 1 \Rightarrow p = \frac{1}{9}$$

Now,

Probability of getting a perfect square i.e. 1 or 4 in a single throw of a die

$$= P(1) + P(4) = p + 2p = 3p = \frac{3}{9} = \frac{1}{3}$$

Since X denotes the number of perfect squares in two tosses. Then, X can take values 0, 1, 2 such that

$P(X=0)$ = Probability of not getting perfect squares in both the tosses

$\Rightarrow \quad P(X=0) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

$P(X=1)$ = Probability of getting perfect squares in one of the two tosses

$\Rightarrow \quad P(X=1) = \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{4}{9}$

$P(X=2)$ = Probability of getting perfect squares in both two tosses

$\Rightarrow \quad P(X=2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

Hence, the probability distribution of X is

X :	0	1	2
P(X) :	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

EXERCISE 31.1

1. Which of the following distributions of probabilities of a random variable X are the probability distributions ?

(i)

X :	3	2	1	0	−1
P (X) :	0.3	0.2	0.4	0.1	0.05

(ii)

X :	0	1	2
P (X) :	0.6	0.4	0.2

(iii)

X :	0	1	2	3	4
P (X) :	0.1	0.5	0.2	0.1	0.1

(iv)

X :	0	1	2	3
P (X) :	0.3	0.2	0.4	0.1

2. A random variable X has the following probability distribution

Values of X :	−2	−1	0	1	2	3
P(X) :	0.1	k	0.2	$2k$	0.3	k

Find the value of k.

3. A random variable X has the following probability distribution

Values of X:	0	1	2	3	4	5	6	7	8
P (X) :	a	3a	5a	7a	9a	11a	13a	15a	17a

Determine :

(i) The value of a

(ii) $P(X<3)$, $P(X \geq 3)$, $P(0<X<5)$.

4. The probability distribution function of a random variable X is given by

x_i :	0	1	2
p_i :	$3c^3$	$4c-10c^2$	$5c-1$

where $c>0$

Find : (i) c (ii) $P(X<2)$ (iii) $P(1<X \leq 2)$

5. Let X be a random variable which assumes values x_1, x_2, x_3, x_4 such that $2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4)$. Find the probability distribution of X.
6. A random variable X takes the values 0, 1, 2 and 3 such that:
$P(X = 0) = P(X > 0) = P(X < 0)$; $P(X = -3) = P(X = -2) = P(X = -1)$;
$P(X = 1) = P(X = 2) = P(X = 3)$. Obtain the probability distribution of X.
7. Two cards are drawn from a well shuffled pack of 52 cards. Find the probability distribution of the number of aces.
8. Find the probability distribution of the number of heads, when three coins are tossed.
9. Four cards are drawn simultaneously from a well shuffled pack of 52 playing cards. Find the probability distribution of the number of aces.
10. A bag contains 4 red.and 6 black balls. Three balls are drawn at random. Find the probability distribution of the number of red balls.
11. Five defective mangoes are accidently mixed with 15 good ones. Four mangoes are drawn at random from this lot. Find the probability distribution of the number of defective mangoes.
12. Two dice are thrown together and the number appearing on them noted. X denotes the sum of the two numbers. Assuming that all the 36 outcomes are equally likely, what is the probability distribution of X?
13. A class has 15 students whose ages are 14, 17, 15, 14, 21, 19, 20, 16, 18, 17, 20, 17, 16, 19 and 20 years respectively. One student is selected in such a manner that each has the same chance of being selected and the age X of the selected student is recorded. What is the probability distribution of the random variable X ? [NCERT]
14. Five defective bolts are accidently mixed with twenty good ones. If four bolts are drawn at random from this lot, find the probability distribution of the number of defective bolts.
15. Two cards are drawn successively with replacement from well shuffled pack of 52 cards. Find the probability distribution of the number of aces.
16. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of kings.
17. Two cards are drawn successively without replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of aces. [CBSE 2001]
18. Find the probability distribution of the number of white balls drawn in a random draw of 3 balls without replacement, from a bag containing 4 white and 6 red balls.
19. Find the probability distribution of Y in two throws of two dice, where Y represents the number of times a total of 9 appears.
20. From a lot containing 25 items, 5 of which are defective, 4 are chosen at random. Let X be the number of defectives found. Obtain the probability distribution of X if the items are chosen without replacement.
21. Three cards are drawn successively with replacement from a well-shuffled deck of 52 cards. A random variable X denotes the number of hearts in the three cards drawn. Determine the probability distribution of X.
22. From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs. [NCERT, CBSE 2004]
23. An urn contains 4 red and 3 blue balls. Find the probability distribution of the number of blue balls in a random draw of 3 balls with replacement.

24. Two cards are drawn simultaneously from a well-shuffled deck of 52 cards. Find the probability distribution of the number of successes, when getting a spade is considered 4 success.
25. A fair die is tossed twice. If the number appearing on the top is less than 3, it is a success. Find the probability distribution of number of successes. [CBSE 2004]
26. An urn contains 5 red and 2 black balls. Two balls are randomly selected. Let X represent the number of black balls. What are the possible values of X. Is X a random variable?
27. Let X represent the difference between the number of heads and the number of tails when a coin is tossed 6 times. What are possible values of X?
28. From a lot of 10 bulbs, which includes 3 defectives, a sample of 2 bulbs is drawn at random. Find the probability distribution of the number of defective bulbs. [NCERT, CBSE 2010]

ANSWERS

1. (iii) and (iv) 2. $k = 0.1$ 3. (i) $a = \frac{1}{81}$ (ii) $\frac{1}{9}, \frac{8}{9}$

4. (i) $\frac{1}{3}$ (ii) $\frac{1}{3}$ (iii) $\frac{2}{3}$

5.

X:	x_1	x_2	x_3	x_4
P(X):	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

6.

X:	−3	−2	−1	0	1	2	3
P(X):	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

7.

X:	0	1	2
P(X):	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

8.

X:	0	1	2	3
P(X):	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

9.

X:	0	1	2	3	4
P(X):	$\frac{^{48}C_4}{^{52}C_4}$	$\frac{^4C_1 \times {^{48}C_3}}{^{52}C_4}$	$\frac{^4C_2 \times {^{48}C_2}}{^{52}C_4}$	$\frac{^4C_3 \times {^{48}C_1}}{^{52}C_4}$	$\frac{^4C_4}{^{52}C_4}$

10.

X:	0	1	2	3
P(X):	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

11.

X:	0	1	2	3	4
P(X):	$\frac{91}{323}$	$\frac{455}{969}$	$\frac{70}{323}$	$\frac{10}{323}$	$\frac{1}{969}$

12.

X :	2	3	4	5	6	7	8	9	10	11	12
P(X) :	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

13.

X :	14	15	16	17	18	19	20	21
P(X) :	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

14.

X :	0	1	2	3	4
P(X) :	$\frac{969}{2530}$	$\frac{114}{253}$	$\frac{38}{253}$	$\frac{4}{253}$	$\frac{1}{2530}$

15.

X :	0	1	2
P(X) :	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

16.

X :	0	1	2
P(X) :	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

17.

X :	0	1	2
P(X) :	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

18.

X:	0	1	2	3
P(X)	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

19.

***Y* :**	x_1	x_2	x_3
***P(X)* :**	$\frac{64}{81}$	$\frac{16}{81}$	$\frac{1}{81}$

20. $P(X = r) = \dfrac{{}^5C_r \cdot {}^{20}C_{4-r}}{{}^{25}C_r}$, $r = 0, 1, 2, 3, 4.$

21.

X:-	0	1	2	3
P (X):	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

22.

X:-	0	1	2	3	4
P (X):	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

22.

X:	0	1	2	3	4
P (X):	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

23.

X:	0	1	2	3
P (X):	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$

24.

X:	0	1	2
P (X):	$\frac{19}{34}$	$\frac{13}{34}$	$\frac{2}{34}$

25.

X:	0	1	2
P (X):	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

26. 0, 1, 2. Yes.

27. $-6, -4, -2, 0, 2, 4, 6$

28.

X:	0	1	2
P (X):	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

HINTS TO SELECTED PROBLEMS

5. Let $P(X = x_3) = k$. Then,

$$P(X = x_1) = \frac{k}{2}, P(X = x_2) = \frac{k}{3}, P(X = x_4) = \frac{k}{5}$$

$$\therefore \quad P(X = x_1) + P(X = x_2) + P(X = x_3) + P(X = x_4) = 1$$

$$\Rightarrow \quad \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1 \Rightarrow k = \frac{30}{61}$$

6. Let $P(X = 0) = k$. Then,

$P(X = 0) = P(X > 0) = P(X < 0) \Rightarrow P(X > 0) = k$ and $P(X < 0) = k$.

$$P(X = 0) + P(X < 0) + P(X > 0) = 1 \Rightarrow 3k = 1 \Rightarrow k = \frac{1}{3}$$

$$P(X < 0) = k$$

$$\Rightarrow \quad P(X = -1) + P(X = -2) + P(X = -3) = k$$

$$\Rightarrow \quad 3P(X = -1) = k$$

$$\Rightarrow \quad P(X = -1) = P(X = -2) = P(X = -3) = \frac{k}{3} = \frac{1}{9}$$

Similarly, $P(X > 0) = k \Rightarrow P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{9}$

31.4 MEAN OF A DISCRETE RANDOM VARIABLE

DEFINITION *If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with respective probabilities $p_1, p_2, p_3, \ldots, p_n$, then the mean $\overline{X}$ of X is defined as*

$$\overline{X} = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n \quad \text{or,} \quad \overline{X} = \sum_{i=1}^{n} p_i x_i$$

REMARK 1 *The mean of a random variable X is also known as its mathematical expectation or expected value and is denoted by E (X).*

REMARK 2 *In case of a frequency distribution x_i / f_i; $i = 1, 2, \ldots, n$ the mean $\overline{X}$ is given by*

$$\overline{X} = \frac{1}{N}(f_1 x_1 + f_2 x_2 + \ldots + f_n x_n)$$

$$\Rightarrow \quad \overline{X} = \frac{f_1}{N}x_1 + \frac{f_2}{N}x_2 + \ldots\ldots + \frac{f_n}{N}x_n$$

$$\Rightarrow \quad \overline{X} = p_1x_1 + p_2x_2 + \ldots\ldots + p_nx_n, \quad \text{where } p_i = \frac{f_i}{N}$$

Thus, if we replace $\frac{f_i}{N}$ by p_i in the definition of mean, we obtain the mean of a discrete random variable. Consequently, the term 'mean' is appropriate for the sum $\sum_{i=1}^{n} p_i \; x_i$..

NOTE *The mean of a random variable means the mean of its probabilitiy distribution.*

ILLUSTRATION 1 *In a single throw of a die, if X denotes the number on its upper face. Find the mean of X.*

SOLUTION Clearly, X can take the values 1, 2, 3, 4, 5, 6 each with probabilities $\frac{1}{6}$.

So, the probability distribution of X is as given below :

X :	1	2	3	4	5	6
P(X) :	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\therefore \quad \overline{X} = \frac{1}{6}\times 1 + \frac{1}{6}\times 2 + \frac{1}{6}\times 3 + \frac{1}{6}\times 4 + \frac{1}{6}\times 5 + \frac{1}{6}\times 6$$

$$\Rightarrow \quad \overline{X} = \frac{1}{6}\times(1+2+3+4+5+6) \quad = \frac{1}{6}\times\frac{6\,(6+1)}{2} = \frac{7}{2}$$

ILLUSTRATION 2 *If a pair of dice is thrown and X denotes the sum of the numbers on them. Find the probability distribution of X. Also, find the expectation of X.* **[NCERT]**

SOLUTION In a single throw of a pair of dice the sum of the numbers on them can be 2, 3, 4, ..., 12. So, X can assume values 2, 3, 4, ..., 12. The probability distribution of X is as given below :

X:	2	3	4	5	6	7	8	9	10	11	12
P(X) :	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\therefore \quad E(X) = \frac{1}{36}\times 2 + \frac{2}{36}\times 3 + \frac{3}{36}\times 4 + \frac{4}{36}\times 5 + \frac{5}{36}\times 6 + \frac{6}{36}\times 7$$
$$+ \frac{5}{36}\times 8 + \frac{4}{36}\times 9 + \frac{3}{36}\times 10 + \frac{2}{36}\times 11 + \frac{1}{36}\times 12$$

$$\Rightarrow \quad E(X) = \frac{1}{36}[2+6+12+20+30+42+50+36+30+22+12]$$

$$\Rightarrow \quad E(X) = \frac{252}{36} = 7.$$

ILLUSTRATION 3 *A dealer in refrigerators estimates from his past experience the probabilities of his selling refrigerators in a day. These are as follows :*

No. of refrigerators sold in a day :	0	1	2	3	4	5	6
Probability :	0.03	0.20	0.23	0.25	0.12	0.10	0.07

Find the mean number of refrigerators sold in a day.

SOLUTION Let X denots the number of refrigerators sold in a day. Then, the probability distribution of X is

x_i:	0	1	2	3	4	5	6
p_i:	0.03	0.20	0.23	0.25	0.12	0.10	0.07

$\therefore \quad \overline{X} = 0.03 \times 0 + 0.20 \times 1 + 0.23 \times 2 + 0.25 \times 3 + 0.12 \times 4 + 0.10 \times 5 + 0.07 \times 6$

$\Rightarrow \quad \overline{X} = 0.20 + 0.46 + 0.75 + 0.48 + 0.50 + 0.42 = 2.81.$

ILLUSTRATION 4 *A salesman wants to know the average number of units he sells per sales call. He checks his past sales records and comes up with the following probabilities:*

Sales (in units) :	0	1	2	3	4	5
Probability :	0.15	0.20	0.10	0.05	0.30	0.20

What is the average number of units he sells per sale call ?

SOLUTION Let X denote the number of units. Then, X is a random variable with the following probability distribution

x_i:	0	1	2	3	4	5
p_i:	0.15	0.20	0.10	0.05	0.30	0.20

$\therefore \quad \overline{X} = 0.15 \times 0 + 0.20 \times 1 + 0.10 \times 2 + 0.05 \times 3 + 0.30 \times 4 + 0.20 \times 5$

$\Rightarrow \quad \overline{X} = 0.20 + 0.20 + 0.15 + 1.20 + 1.00 = 2.75$

Thus, the average number of units he would sell per sale call is 2.75.

31.5 VARIANCE OF A DISCRETE RANDOM VARIABLE

DEFINITION *If X is a discrete random variable which assumes values $x_1, x_2, x_3, \dots, x_n$ with the respective probabilities $p_1, p_2, \dots, p_n$, then variance of X is defined as*

$$\text{Var}(X) = p_1(x_1 - \overline{X})^2 + p_2(x_2 - \overline{X})^2 + \dots + p_n(x_n - \overline{X})^2$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i (x_i - \overline{X})^2, \quad \text{where } \overline{X} = \sum_{i=1}^{n} p_i x_i \text{ is the mean of X.}$$

Now,
$$\text{Var}(X) = \sum_{i=1}^{n} p_i (x_i - \overline{X})^2$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i (x_i^2 - 2x_i \overline{X} + \overline{X}^2)$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - 2\overline{X}\left(\sum_{i=1}^{n} p_i x_i\right) + \overline{X}^2\left(\sum_{i=1}^{n} p_i\right)$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - 2\overline{X}.\overline{X} + \overline{X}^2 \qquad \left[\because \sum_{i=1}^{n} p_i = 1\right]$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - 2\overline{X}^2 + \overline{X}^2$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - \overline{X}^2$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2$$

Thus, $\quad \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2.$

or, $\quad \text{Var}(X) = E(X^2) - \{E(X)\}^2$

<u>REMARK 1</u> The variance of a discrete random variable X is also known as its second central moment μ_2.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the mean and variance of the number of heads in the two tosses of a coin.*

SOLUTION Let X denote the number of heads in the two tosses of a coin. Then, X can take values 0, 1 or 2 such that

$$P(X=0) = \text{Probability of getting no head} = P(TT) = \frac{1}{4}$$

$$P(X=1) = \text{Probability of getting one head}$$

$$\Rightarrow \quad P(X=1) = P(HT \text{ or } TH) = \frac{2}{4} = \frac{1}{2}$$

and, $$P(X=2) = \text{Probability of getting both heads} = P(HH) = \frac{1}{4}$$

Thus, the probability distribution of X is as given below :

X :	0	1	2
P(X) :	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Computation of mean and variance

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{1}{4}$	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{2}$	1
		$\Sigma p_i x_i = 1$	$\Sigma p_i x_i^2 = 3/2$

We have,

$$\Sigma p_i x_i = 1 \text{ and } \Sigma p_i x_i^2 = \frac{3}{2}$$

$$\therefore \quad \overline{X} = \text{Mean} = \Sigma p_i x_i = 1$$

and, $$\text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{3}{2} - 1 = \frac{1}{2}$$

Hence, $\quad$ Mean = 1 and Variance $= \frac{1}{2}$

EXAMPLE 2 *Find the mean, variance and standard deviation of the number of heads in a simultaneous toss of three coins.* **[CBSE 2007]**

SOLUTION Let X denote the number of heads in a simultaneous toss of three coins. Then, X can take values 0, 1, 2, 3.

Now, $P(X=0) = P(TTT) = \frac{1}{8}$

$P(X=1) = P(HTT \text{ or } TTH \text{ or } THT) = \frac{3}{8}$

$P(X=2) = P(HHT \text{ or } THH \text{ or } HTH) = \frac{3}{8}$

and, $P(X=3) = P(HHH) = \frac{1}{8}$

Thus, the probability distribution of X is given by

X:	0	1	2	3
P(X):	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Computation of mean and variance

x_i	$p_i = P(X=x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{1}{8}$	0	0
1	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{12}{8}$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{9}{8}$
		$\Sigma p_i x_i = \frac{3}{2}$	$\Sigma p_i x_i^2 = 3$

We have,

$$\Sigma p_i x_i = \frac{3}{2} \text{ and } \Sigma p_i x_i^2 = 3$$

$\therefore$ $\overline{X} = \text{Mean} = \Sigma p_i x_i = \frac{3}{2}$

and, $\text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$

$\therefore$ Standard deviation $= \sqrt{\text{Var}(X)} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = 0.87$

Hence, Mean $= \frac{3}{2}$, Variance $= \frac{3}{4}$ and Standard deviation $= 0.87$

EXAMPLE 3 *Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.* **[NCERT]**

SOLUTION Clearly, X can take values 0, 1 and 2.

We have,

$$P(X=0) = \text{Probability of not getting six on any dice} = \frac{25}{36}$$

$$P(X=1) = \text{Probability of getting one six} = \frac{10}{36}$$

$$P(X=2) = \text{Probability of getting two sixes} = \frac{1}{36}$$

Thus, the probability distribution of X is given by

X :	0	1	2
P(X) :	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

Computation of mean and variance

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{25}{36}$	0	0
1	$\frac{10}{36}$	$\frac{10}{36}$	$\frac{10}{36}$
2	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$
		$\Sigma p_i x_i = \frac{12}{36}$	$\Sigma p_i x_i^2 = \frac{14}{36}$

We have,

$$\Sigma p_i x_i = \frac{12}{36} = \frac{1}{3} \text{ and } \Sigma p_i x_i^2 = \frac{7}{18}$$

$\therefore$ $$E(X) = \Sigma p_i x_i = \frac{1}{3}$$

and, $$\text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{7}{18} - \frac{1}{9} = \frac{5}{18}$$

Hence, $E(X) = \frac{1}{3}$ and $\text{Var}(X) = \frac{5}{18}$.

EXAMPLE 4 *Two numbers are selected at random (with replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find E (X) and Var (X).* **[NCERT]**

SOLUTION We observe that X can take values 2, 3, 4, 5, 6 such that

$P(X=2)$ = Probability that the larger of two numbers is 2

$\Rightarrow$ $P(X=2)$ = Probability of getting 1 in first selection and 2 in second selection or getting 2 in first selection and 1 in second selection

$\Rightarrow$ $$P(X=2) = \frac{1}{6} \times \frac{1}{5} + \frac{1}{6} \times \frac{1}{5} = \frac{2}{30} = \frac{1}{15}$$

$P(X=3)=$ Probability that the larger of two numbers is 3

$\Rightarrow$ $P(X=3)=$ Probability of getting a number less than 3 in first selection and 3 in second selection or getting 3 in first selection and a number less than 3 in second selection.

$$\Rightarrow \quad P(X=3)=\frac{2}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{2}{5}=\frac{4}{30}=\frac{2}{15}$$

$P(X=4)=$ Probability that the larger of two numbers is 4

$$\Rightarrow \quad P(X=4)=\frac{3}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{3}{5}=\frac{6}{30}=\frac{1}{15}$$

$$P(X=5)=\frac{4}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{4}{5}=\frac{8}{30}=\frac{4}{15}$$

$$P(X=6)=\frac{5}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{5}{5}=\frac{10}{30}=\frac{1}{3}$$

Thus, the probability distribution of X is

X:	2	3	4	5	6
P(X):	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{5}$	$\frac{4}{15}$	$\frac{1}{3}$

$$\therefore \quad E(X)=\frac{1}{15}\times 2+\frac{2}{15}\times 3+\frac{1}{5}\times 4+\frac{4}{15}\times 5+\frac{1}{3}\times 6$$

$$\Rightarrow \quad E(X)=\frac{70}{15}=\frac{14}{3}$$

EXAMPLE 5 *In a meeting 70% of the members favour a certain proposal, 30% being opposed. A member is selected at random and let $X=0$ if he opposed, and $X=1$ if he is in favour. Find $E(X)$ and $Var(X)$.* **[NCERT]**

SOLUTION The probability distribution of X is

X:	0	1
P(X):	$\frac{30}{100}$	$\frac{70}{100}$

$$\therefore \quad E(X)=\frac{30}{100}\times 0+\frac{70}{100}\times 1=\frac{7}{10}$$

and, $$E(X^2)=\frac{30}{100}\times 0^2+\frac{70}{100}\times 1^2=\frac{7}{10}$$

$$\therefore \quad \text{Var}(X)=E(X^2)-[E(X)]^2=\frac{7}{10}-\frac{49}{100}=\frac{21}{100}$$

Hence, $E(X)=\frac{7}{10}$ and $\text{Var}(X)=\frac{21}{100}$

EXAMPLE 6 *A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being choosen and the age X of the selected student is recorded. What is the probability distribution of random variable X? Find mean, variance and standard deviation of X.* **[NCERT]**

SOLUTION We observe that X takes values 14, 15, 16, 17, 18, 19, 20 and 21 such that

$$P(X=14)=\frac{2}{15},\ P(X=15)=\frac{1}{15},\ P(X=16)=\frac{2}{15},\ P(X=17)=\frac{3}{15}$$

$$P(X=18)=\frac{1}{15},\ P(X=19)=\frac{2}{15},\ P(X=20)=\frac{3}{15}\ P(X=21)=\frac{1}{15}$$

So, the probability distribution of X is as given below:

X:	14	15	16	17	18	19	20	21
P(X):	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

Computation of mean and variance

x_i	$p_i=P(X=x_i)$	$p_i x_i$	$p_i x_i^2$
14	$\frac{2}{15}$	$\frac{28}{15}$	$\frac{392}{15}$
15	$\frac{1}{15}$	$\frac{15}{15}$	$\frac{225}{15}$
16	$\frac{2}{15}$	$\frac{32}{15}$	$\frac{512}{15}$
17	$\frac{3}{15}$	$\frac{51}{15}$	$\frac{867}{15}$
18	$\frac{1}{15}$	$\frac{18}{15}$	$\frac{324}{15}$
19	$\frac{2}{15}$	$\frac{38}{15}$	$\frac{722}{15}$
20	$\frac{3}{15}$	$\frac{60}{15}$	$\frac{1200}{15}$
21	$\frac{1}{15}$	$\frac{21}{15}$	$\frac{441}{15}$
		$\Sigma p_i x_i=\frac{263}{15}$	$\Sigma p_i x_i^2=\frac{4683}{15}$

We have,

$$\Sigma p_i x_i=\frac{263}{15} \text{ and } \Sigma p_i x_i^2=\frac{4683}{15}$$

$\therefore$ $$\text{Mean}=\Sigma p_i x_i=\frac{263}{15}=17.53$$

$$\text{Variance}=\Sigma p_i x_i^2-(\Sigma p_i x_i)^2$$

$\Rightarrow$ $$\text{Variance}=\frac{4683}{15}-\left(\frac{263}{15}\right)^2=\frac{70245-69169}{225}=\frac{1076}{225}$$

$\therefore$ $$\text{Standard Deviation}=\sqrt{\text{Variance}}=\sqrt{\frac{1076}{225}}=\frac{\sqrt{1076}}{15}=\frac{32.80}{15}=2.186$$

EXAMPLE 7 *Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as 'getting a number greater than 4'. Also, find the mean and variance of the distribution.* **[NCERT]**

SOLUTION Let X denote the number of successes in two tosses of a die. Then, X can take values 0, 1, 2.

Let S_i = Getting a success in i th toss and, F_i = Getting a failure in i th toss.

Then, $P(S_1)$ = Probability of getting a number greater than 4 in first toss

$\Rightarrow \quad P(S_1) = \frac{2}{6} = \frac{1}{3}$

Also, $P(S_2) = \frac{1}{3}$

$\therefore \quad P(F_1) = P(F_2) = \frac{2}{3}$

Now, $P(X=0)$ = Probability of getting no success in two tosses of a die

$\Rightarrow \quad P(X=0) = P(F_1 \cap F_2)$

$\Rightarrow \quad P(X=0) = P(F_1) \times P(F_2)$ [By multiplication theorem]

$\Rightarrow \quad P(X=0) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

$P(X=1)$ = Probability of getting one success in two tosses of a die.

$\Rightarrow \quad P(X=1) = P\{(S_1 \cap F_2) \cup (F_1 \cap S_2)\}$

$\Rightarrow \quad P(X=1) = P(S_1 \cap F_2) + P(F_1 \cap S_2)$

$\Rightarrow \quad P(X=1) = P(S_1)\,P(F_2) + P(F_1)\,P(S_2)$

$\Rightarrow \quad P(X=1) = \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{4}{9}.$

and, $P(X=2)$ = Probability of getting two successes in two tosses of a die

$\Rightarrow \quad P(X=2) = P(S_1 \cap S_2) = P(S_1)\,P(S_2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

Thus, the probability distribution of X is given by

X :	0	1	2
P(X) :	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Computation of mean and variance

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{4}{9}$	0	0
1	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{4}{9}$
2	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$
		$\Sigma p_i x_i = \frac{6}{9}$	$\Sigma p_i x_i^2 = \frac{8}{9}$

We have,

$$\Sigma p_i x_i = \frac{6}{9} = \frac{2}{3} \text{ and } \Sigma p_i x_i^2 = \frac{8}{9}$$

$$\therefore \qquad \overline{X} = \text{Mean} = \Sigma p_i x_i = \frac{2}{3}$$

and, $$\text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{8}{9} - \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Hence, Mean $= \frac{2}{3}$ and Variance $= \frac{4}{9}$.

EXAMPLE 8 *Find the probability distribution of the number of sixes in three tosses of a die. Find also the mean and variance of the distribution.*

SOLUTION Let X denote the number of sixes in three tosses of a die. Then, X can take values 0, 1, 2, 3.

Let S_i denote the event of getting a six in i th toss, $i = 1, 2, 3$.

Then,

$$P(S_i) = \frac{1}{6} \text{ and } P(\overline{S_i}) = \frac{5}{6}; i = 1, 2, 3.$$

Now, $$P(X=0) = P(\overline{S_1} \cap \overline{S_2} \cap \overline{S_3})$$

$$\Rightarrow \quad P(X=0) = P(\overline{S_1})\, P(\overline{S_2})\, P(\overline{S_3}) \qquad [\because \overline{S_1}, \overline{S_2}, \overline{S_3} \text{ are independent events}]$$

$$\Rightarrow \quad P(X=0) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$$

$$P(X=1) = P\left((S_1 \cap \overline{S_2} \cap \overline{S_3}) \cup (\overline{S_1} \cap S_2 \cap \overline{S_3}) \cup (\overline{S_1} \cap \overline{S_2} \cap S_3)\right)$$

$$\Rightarrow \quad P(X=1) = P(S_1 \cap \overline{S_2} \cap \overline{S_3}) + P(\overline{S_1} \cap S_2 \cap \overline{S_3}) + P(\overline{S_1} \cap \overline{S_2} \cap S_3)$$

$$\Rightarrow \quad P(X=1) = P(S_1)\, P(\overline{S_2})\, P(\overline{S_3}) + P(\overline{S_1})\, P(S_2)\, P(\overline{S_3}) + P(\overline{S_1})\, P(\overline{S_2})\, P(S_3)$$

$$\Rightarrow \quad P(X=1) = \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{75}{216}$$

$$P(X=2) = P\left((S_1 \cap S_2 \cap \overline{S_3}) \cap (\overline{S_1} \cap S_2 \cap S_3) \cap (S_1 \cap \overline{S_2} \cap S_3)\right)$$

$$\Rightarrow \quad P(X=2) = P(S_1 \cap S_2 \cap \overline{S_3}) + P(\overline{S_1} \cap S_2 \cap S_3) + P(S_1 \cap \overline{S_2} \cap S_3)$$

$$\Rightarrow \quad P(X=2) = P(S_1)\, P(S_2)\, P(\overline{S_3}) + P(\overline{S_1})\, P(S_2)\, P(S_3) + P(S_1)\, P(\overline{S_2})\, P(S_3)$$

$$\Rightarrow \quad P(X=2) = \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{15}{216}$$

and, $$P(X=3) = P(S_1 \cap S_2 \cap S_3) = P(S_1)\, P(S_2)\, P(S_3) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$$

Thus, the probability distribution of X is given by

X :	0	1	2	3
$P(X)$:	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Computation of mean and variance

x_i	$P(X=x_i)=p_i$	$p_i x_i$	$p_i x_i^2$
0	$\frac{125}{216}$	0	0
1	$\frac{75}{216}$	$\frac{75}{216}$	$\frac{75}{216}$
2	$\frac{15}{216}$	$\frac{30}{216}$	$\frac{60}{216}$
3	$\frac{1}{216}$	$\frac{3}{216}$	$\frac{9}{216}$
		$\Sigma p_i x_i = \frac{108}{216} = \frac{1}{2}$	$\Sigma p_i x_i^2 = \frac{144}{216}$

We have,

$$\Sigma p_i x_i = \frac{108}{216} = \frac{1}{2} \text{ and } \Sigma p_i x_i^2 = \frac{144}{216}$$

$\therefore$ $$\text{Mean} = \overline{X} = \Sigma p_i x_i = \frac{1}{2}$$

and, $$\text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{144}{216} - \left(\frac{1}{2}\right)^2 = \frac{90}{216} = \frac{5}{12}$$

Hence, $\text{Mean} = \frac{1}{2}$ and $\text{Variance} = \frac{5}{12}$

EXAMPLE 9 *A die is tossed twice. A "success" is "getting an odd number" on a random toss. Find the variance of the number of successes.*

SOLUTION Let X denote the number of successes in two tosses of a die. Then, X can take values 0, 1, 2.

Let S_i and F_i denote the success and failure respectively in i th toss. Then,

$$P(S_i) = \text{Probability of getting an odd number in i th toss} = \frac{3}{6} = \frac{1}{2}$$

and, $P(F_i)$ = Probability of not getting an odd number in i th toss

$\Rightarrow$ $$P(F_i) = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

Now, $P(X=0)$ = Probability of getting no success in two tosses of a die

$\Rightarrow$ $P(X=0) = P(F_1 \cap F_2)$

$\Rightarrow$ $P(X=0) = P(F_1)\, P(F_2)$ [By Multiplication Theorem]

$\Rightarrow$ $P(X=0) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ $\left[\because P(F_1) = P(F_2) = \frac{1}{2}\right]$

$P(X=1)$ = Probability of getting one success in two tosses of a die

$\Rightarrow$ $P(X=1) = P\big((S_1 \cap F_2) \cup (F_1 \cap S_2)\big)$

$\Rightarrow$ $P(X=1) = P(S_1 \cap F_2) + P(F_1 \cap S_2)$

$\Rightarrow \quad P(X=1) = P(S_1)\,P(F_2) + P(F_1)\,P(S_2) = \frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2} = \frac{1}{2}.$

and, $\quad P(X=2)$ = Probability of getting two successes in two tosses of a die

$\Rightarrow \quad P(X=2) = P(S_1 \cap S_2) = P(S_1)\,P(S_2) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

Thus, the probability distribution of X i.e. the number of successes in two tosses of a die, is given by

X:	0	1	2
$P(X)$:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Computation of variance

x_i	$p_i = P(X=x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{1}{4}$	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{2}$	1
		$\Sigma p_i x_i = 1$	$\Sigma p_i x_i^2 = \frac{3}{2}$

We have, $\Sigma p_i x_i = 1$ and $\Sigma p_i x_i^2 = \frac{3}{2}$

$\therefore \quad \text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{3}{2} - 1 = \frac{1}{2}$

EXAMPLE 10 *Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the mean and standard deviation of the numbet of aces.*

SOLUTION Let A_i $(i = 1, 2)$ denote the event of getting an ace in i th draw. Since the cards are drawn with replacement, therefore,

$P(A_i)$ = Probability of getting an ace in i th draw

$\Rightarrow \quad P(A_i) = \frac{{}^4C_1}{{}^{52}C_1} = \frac{4}{52} = \frac{1}{13}$

and, $\quad P(\overline{A_i}) = 1 - P(A_i) = 1 - \frac{1}{13} = \frac{12}{13}, \; i = 1, 2.$

Let X denote the number of aces in two draws. Then, X can take values 0, 1, 2.

Now, $\quad P(X=0)$ = Probability of getting no ace in two draws

$\Rightarrow \quad P(X=0) = P(\overline{A_1} \cap \overline{A_2}) = P(\overline{A_1})\,P(\overline{A_2}) = \frac{12}{13}\times\frac{12}{13} = \frac{144}{169}$

$P(X=1)$ = Probability of getting an ace in either of the two draws

$\Rightarrow \quad P(X=1) = P\left((A_1 \cap \overline{A_2}) \cup (\overline{A_1} \cap A_2)\right)$

$\Rightarrow \quad P(X=1) = P(A_1 \cap \overline{A_2}) + P(\overline{A_1} \cap A_2)$

$\Rightarrow \quad P(X=1) = P(A_1)P(\overline{A_2}) + P(\overline{A_1})P(A_2) = \frac{1}{13}\times\frac{12}{13}+\frac{12}{13}\times\frac{1}{13} = \frac{24}{169}$

and, $\quad P(X=2)$ = Probability of getting ace in each draw

$\Rightarrow \quad P(X=2) = P(A_1 \cap A_2) = P(A_1)P(A_2) = \frac{1}{13}\times\frac{1}{13} = \frac{1}{169}$

Thus, the probability distribution of X is given by

X :	0	1	2
P(X) :	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

$\therefore \quad \Sigma p_i x_i = 0\times\frac{144}{169}+1\times\frac{24}{169}+2\times\frac{1}{169} = \frac{26}{169}$

and, $\quad \Sigma p_i x_i^2 = 0\times\frac{144}{169}+1\times\frac{24}{169}+4\times\frac{1}{169} = \frac{28}{169}$

Hence, $\quad \overline{X} = \text{Mean} = \Sigma p_i x_i = \frac{26}{169} = \frac{2}{13}$

and, $\quad \text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{28}{169} - \left(\frac{2}{13}\right)^2 = \frac{24}{169}$

$\therefore \quad \text{S.D.} = \sqrt{\text{Var}(X)} = \sqrt{\frac{24}{169}} = \frac{2\sqrt{6}}{13}$

Hence, Mean $= \frac{2}{13}$ and S.D. $= \frac{2\sqrt{6}}{13}$.

EXAMPLE 11 *Two cards are drawn successively without replacement from a well-shuffled deck of 52 cards. Compute the variance of the number of aces.* **[CBSE 2010]**

SOLUTION Let A_i denote the event of getting an ace in ith draw, where $i = 1, 2$. Further, let X denote the number of aces in two draws. Then, X can take values 0, 1, 2.

Now, $\quad P(X=0)$ = Probability of getting no ace in two successive draws

$\Rightarrow \quad P(X=0) = P(\overline{A_1} \cap \overline{A_2}) = P(\overline{A_1})\ P(\overline{A_2}/\overline{A_1}) = \frac{48}{52}\times\frac{47}{51} = \frac{564}{663}$

$P(X=1)$ = Proabability of getting an ace in one of the two draws

$\Rightarrow \quad P(X=1) = P\left((A_1 \cap \overline{A_2}) \cup (\overline{A_1} \cap A_2)\right)$

$\Rightarrow \quad P(X=1) = P(A_1 \cap \overline{A_2}) + P(\overline{A_1} \cap A_2)$

$\Rightarrow \quad P(X=1) = P(A_1)P(\overline{A_2}/A_1) + P(\overline{A_1})P(A_2/\overline{A_1})$

$\Rightarrow \quad P(X=1) = \frac{4}{52}\times\frac{48}{51}+\frac{48}{52}\times\frac{4}{51} = \frac{96}{663}$

$P(X=2)$ = Probability of getting an ace in each draw

$\Rightarrow \quad P(X=2) = P(A_1 \cap A_2) = P(A_1)P(A_2/A_1) = \frac{4}{52}\times\frac{3}{51} = \frac{3}{663}$

Thus, the probability distribution of X is given by

X :	0	1	2
P(X) :	$\frac{564}{663}$	$\frac{96}{663}$	$\frac{3}{663}$

$\therefore \quad \Sigma p_i x_i = 0 \times \frac{564}{663} + 1 \times \frac{96}{663} + 2 \times \frac{3}{663} = \frac{102}{663}$

and, $\quad \Sigma p_i x_i^2 = \frac{564}{663} \times 0 + \frac{96}{663} \times 1 + \frac{3}{663} \times 4 = \frac{108}{663}$

Hence, $\quad \text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2$

$\Rightarrow \quad \text{Var}(X) = \frac{108}{663} - \left(\frac{102}{663}\right)^2 = \frac{108 \times 663 - (102)^2}{(663)^2} = \frac{61200}{663 \times 663} = \frac{400}{2873}$

EXAMPLE 12 *From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. If the items in the sample are drawn one by one without replacement, find*

(i) The probability distribution of X.
(ii) Mean of X
(iii) Variance of X.

SOLUTION (i) Clearly, X can assume values 0, 1, 2, 3 such that

$$P(X = 0) = \frac{{}^7C_4}{{}^{10}C_4} = \frac{1}{6}, \qquad P(X = 1) = \frac{{}^3C_1 \times {}^7C_3}{{}^{10}C_4} = \frac{1}{2}$$

$$P(X = 2) = \frac{{}^3C_2 \times {}^7C_1}{{}^{10}C_4} = \frac{3}{10}, \text{ and } P(X = 3) = \frac{{}^3C_3 \times {}^7C_1}{{}^{10}C_4} = \frac{1}{30}$$

So, the probability distribution of X is as given below.

X :	0	1	2	3
P (X) :	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Computation of mean and variance

x_i	$P(X = x_i) = p_i$	$p_i x_i$	$p_i x_i^2$
0	$\frac{1}{6}$	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{6}{5}$
3	$\frac{1}{30}$	$\frac{1}{10}$	$\frac{3}{10}$
		$\Sigma p_i x_i = \frac{12}{10}$	$\Sigma p_i x_i^2 = 2$

We have, $\Sigma p_i x_i = \frac{12}{10} = \frac{6}{5}$

$\therefore \quad \overline{X} = \text{Mean} = \Sigma p_i x_i = \frac{6}{5}$

Now, $\quad \text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = 2 - \left(\frac{6}{5}\right)^2 = 2 - \frac{36}{25} = \frac{14}{25}$

Hence, $\quad \text{Mean} = \frac{6}{5}$ and $\text{Variance} = \frac{14}{25}$.

EXAMPLE 13 *A random variable X has the following probability distribution :*

x_i :	-2	-1	0	1	2	3
p_i :	0.1	k	0.2	2k	0.3	k

(i) Find the value of k.
(ii) Calculate the mean of X.
(iii) Calculate the variance of X.

SOLUTION (i) Since sum of the probabilities in a frequency distribution is always unity.

$\therefore \quad 0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$\Rightarrow \quad 0.6 + 4k = 1 \Rightarrow 4k = 0.4 \Rightarrow k = 0.1$

Calculation of mean and variance

x_i	p_i	$p_i x_i$	$p_i x_i^2$
-2	0.1	-0.2	0.4
-1	0.1	-0.1	0.1
0	0.2	0	0
1	0.2	0.2	0.2
2	0.3	0.6	1.2
3	0.1	0.3	0.9
		$\Sigma p_i x_i = 0.8$	$\Sigma p_i x_i^2 = 2.8$

We have, $\Sigma p_i x_i = 0.8$ and $\Sigma p_i x_i^2 = 2.8$

$\therefore \quad \text{Mean} = 0.8$

and, $\quad \text{Variance} = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = 2.8 - (0.8)^2 = 2.8 - 0.64 = 2.16$

EXAMPLE 14 *A coin weighted so that* $P(H) = \frac{3}{4}$ *and* $P(T) = \frac{1}{4}$ *is tossed three times. Let X be the random variable which denotes the longer string of heads which occurs. Find the probability distribution, mean and variance of X.*

SOLUTION The random variable X is defined on the sample space S given by

$$S = \{TTT, HTT, THT, TTH, THH, HTH, HHT, HHH\}$$

Note that the string of heads means the sequence of consecutive heads.

Since X denotes the longest string of heads. Therefore,

$$X(TTT) = 0, X(THT) = 1, X(HTT) = 1, X(TTH) = 1, X(HTH) = 1,$$
$$X(HHT) = 2, \ X(THH) = 2 \text{ and } X(HHH) = 3.$$

Now, $\quad P(X = 0) = P(TTT) = P(T)\,P(T)\,P(T) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$

$P(X = 1) = P(THT \cup HTT \cup TTH \cup HTH$

$\Rightarrow \quad P(X = 1) = P(THT) + P(HTT) + P(TTH) + P(HTH)$

$\Rightarrow \quad P(X = 1) = P(T)\,P(H)\,P(T) + P(H)\,P(T)\,P(T) + P(T)\,P(T)\,P(H) + P(H)\,P(T)\,P($

$\Rightarrow \quad P(X = 1) = 3\left(\frac{1}{4} \times \frac{3}{4} \times \frac{1}{4}\right) + \frac{3}{4} \times \frac{1}{4} + \frac{3}{4} = \frac{18}{64}$

$P(X = 2) = P(THH \cup HHT)$

$\Rightarrow \quad P(X = 2) = P(THH) + P(HHT)$

$\Rightarrow \quad P(X = 2) = P(T)\,P(H)\,P(H) + P(H)\,P(H)\,P(T)$

$\Rightarrow \quad P(X = 2) = 2\left(\frac{1}{4} \times \frac{3}{4} \times \frac{3}{4}\right) = \frac{18}{64}$

and, $\quad P(X = 3) = P(HHH)$

$\Rightarrow \quad P(X = 3) = P(H)\,P(H)\,P(H) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$

So, the probability distribution is as follows :

x_i :	0	1	2	3
p_i :	$\frac{1}{64}$	$\frac{18}{64}$	$\frac{18}{64}$	$\frac{27}{64}$

Calculation of mean and variance

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	$\frac{1}{64}$	0	0
1	$\frac{18}{64}$	$\frac{18}{64}$	$\frac{18}{64}$
2	$\frac{18}{64}$	$\frac{36}{64}$	$\frac{72}{64}$
3	$\frac{27}{64}$	$\frac{81}{64}$	$\frac{243}{64}$
		$\Sigma p_i x_i = \frac{135}{64}$	$\Sigma p_i x_i^2 = \frac{333}{64}$

We have, $\Sigma p_i x_i = \frac{135}{64}$ and $\Sigma p_i x_i^2 = \frac{333}{64}$

$\therefore \quad \text{Mean} = \Sigma p_i x_i = \frac{135}{64} = 2.1$

and, $\quad \text{Variance} = \Sigma p_i x_i^2 - (\text{Mean})^2 = \frac{333}{64} - (2.1)^2 = 5.2 - 4.41 = 0.79$

Hence, Mean = 21 and Variance = 0.79.

EXAMPLE 15 *A fair coin is tossed until a head or five tails occur. If X denotes the number of tosses of the coin, find the mean of X.*

SOLUTION The sample space related to the given random experiment is given by

$S = \{H, TH, TTH, TTTH, TTTTH, TTTTT\}$

Clearly, X assumes values 1, 2, 3, 4, 5 such that

$P(X = 1) = P(H) = \frac{1}{2}$

$$P(X=2) = P(TH) = P(T)\,P(H) = \frac{1}{4}$$

$$P(X=3) = P(TTH) = P(T)\,P(T)\,P(H) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$$

$$P(X=4) = P(TTTH) = P(T)\,P(T)\,P(T)\,P(H) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{16}$$

and, $P(X=5) = P(TTTTH \cup TTTTT)$

$\Rightarrow$ $P(X=5) = P(TTTTH) + P(TTTTT)$

$\Rightarrow$ $P(X=5) = P(T)\,P(T)\,P(T)\,P(T)\,P(H) + P(T)\,P(T)\,P(T)\,P(T)\,P(T)$

$\Rightarrow$ $P(X=5) = \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 = \frac{1}{16}$

So, the probability distribution of X is given by

x_i:	1	2	3	4	5
p_i:	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

Now,

$$\text{Mean} = \Sigma p_i x_i = \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + \frac{1}{16}\times 5 = \frac{31}{16} = 1.9.$$

EXAMPLE 16 *There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance.*

SOLUTION Clearly, X can take values from 3 to 9.

We have,

$P(X=3)$ = Probability of getting 3 as the sum

$\Rightarrow$ $P(X=3) = P$ {(Getting 1 in first draw and 2 in second draw) or (Geting 2 in first draw and 1 in second draw)}

$\Rightarrow$ $P(X=3) = \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} = \frac{1}{10}$

$P(X=4) = P$ {(Getting 1 in first draw and 3 in second draw) or (Getting 3 in first draw and 1 in second draw)}

$\Rightarrow$ $P(X=4) = \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} = \frac{1}{10}$

$$P(X=5) = \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} = \frac{1}{5}$$

$$P(X=6) = \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} = \frac{1}{5}$$

$$P(X=7) = \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} = \frac{1}{5}$$

$$P(X=8) = \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} = \frac{1}{10}$$

$$P(X=9) = \frac{1}{5}\times\frac{1}{4} + \frac{1}{5}\times\frac{1}{4} = \frac{1}{10}$$

Thus, the probability distribution of X is as given below:

X:	3	4	5	6	7	8	9
$P(X)$:	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

Computation of mean and variance

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
3	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{9}{10}$
4	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{16}{10}$
5	$\frac{1}{5}$	$\frac{5}{5}$	$\frac{25}{5}$
6	$\frac{1}{5}$	$\frac{6}{5}$	$\frac{36}{5}$
7	$\frac{1}{5}$	$\frac{7}{5}$	$\frac{49}{5}$
8	$\frac{1}{10}$	$\frac{8}{10}$	$\frac{64}{10}$
9	$\frac{1}{10}$	$\frac{9}{10}$	$\frac{81}{10}$
Total	$\Sigma p_i = 1$	$\Sigma p_i x_i = \frac{60}{10} = 6$	$\Sigma p_i x_i^2 = \frac{390}{10} = 39$

We have, $\Sigma p_i x_i = 6$ and $\Sigma p_i x_i^2 = 39$

$\therefore \quad \overline{X} = \text{Mean} = \Sigma p_i x_i = 6$

and, $\quad \text{Var}(X) = \Sigma p_i x_i^2 - \left(\Sigma p_i x_i\right)^2 = 39 - 6^2 = 3$

EXAMPLE 17 *In a game, a person is paid Rs. 5 if he gets all heads or all tails when three coins are tossed, and he will pay Rs. 3 if either one or two heads show. What can he expect to win on the average per game?*

SOLUTION Let X be the amount received by the person. Then, X can take values 5 and -3 such that

$P(X = 5) =$ Probability of getting all heads or all tails when three coins are tossed

$\Rightarrow \quad P(X = 5) = \frac{2}{8} = \frac{1}{4}$

$P(X = -3) =$ Probability of getting one or two heads

$\Rightarrow \quad P(X = -3) = \frac{6}{8} = \frac{3}{4}$

$\therefore$ Expected amount to win, on the average, per game

$= \overline{X} = \Sigma p_i x_i = 5 \times \frac{1}{4} + -3 \times \frac{3}{4} = -1$

Thus, the person will, on the average, lose Re 1 per toss of the coins.

EXAMPLE 18 *Let X denote the number of vowels in word selected at random from this sentence. Find the expected value and standard deviation of the random variable X. (Consider X as a word with one letter).*

SOLUTION There are 12 words in the following sentence.

"Find the expected value and standard deviation of the random variable X".

Clearly, X can take values 0, 1, 2, 3, 4, 5.

We have,

$P(X=0) = P$ (Selecting a word containing no vowel)

$\Rightarrow$ $P(X=0) = P$ (Selecting X) $= \frac{1}{12}$

$P(X=1) = P$ (Selecting a word containing one vowel)

$\Rightarrow$ $P(X=1) = P$ (Selecting a word from the words 'The', 'Find', 'and', of 'The')

$\Rightarrow$ $P(X=1) = \frac{5}{12}$

$P(X=2) = P$ (Selecting a word containing two vowels)

$\Rightarrow$ $P(X=2) = P$ (Selecting a word from the words 'standard', 'random')

$\Rightarrow$ $P(X=2) = \frac{2}{12}$

$P(X=3) = P$ (Selecting a word containing three vowels)

$\Rightarrow$ $P(X=3) = P$ (Selecting a word from the word 'expected', 'value')

$\Rightarrow$ $P(X=3) = \frac{2}{12}$

$P(X=4) = P$ (Selecting a word containing four vowels)

$\Rightarrow$ $P(X=4) = P$ (Selecting the word 'variable')

$\Rightarrow$ $P(X=4) = \frac{1}{12}$

$P(X=5) = P$ (Selecting a word containing five vowels)

$\Rightarrow$ $P(X=5) = P$ (Selecting the word 'deviation')

$\Rightarrow$ $P(X=5) = \frac{1}{12}$

So, the probability distribution of X is as given below:

$X:$	0	1	2	3	4	5
$P(X):$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

$\therefore$ $E(X) = 0 \times \frac{1}{12} + 1 \times \frac{5}{12} + 2 \times \frac{2}{12} + 3 \times \frac{2}{12} + 4 \times \frac{1}{12} + 5 \times \frac{1}{12}$

$\Rightarrow$ $E(X) = \frac{0+5+4+6+4+5}{12} = 2.$

$E(X^2) = 0^2 \times \frac{1}{12} + 1^2 \times \frac{5}{12} + 2^2 \times \frac{2}{12} + 3^2 \times \frac{2}{12} + 4^2 \times \frac{1}{12} + 5^2 \times \frac{1}{12}$

$\Rightarrow$ $E(X^2) = \frac{0+5+8+18+16+25}{12} = \frac{72}{12} = 6$

$\therefore$ $\text{Var}(X) = E(X^2) - [E(X)]^2 = 6 - 2^2 = 2.$

$\Rightarrow$ Standard deviation of $X = \sqrt{\text{Var }(X)} = \sqrt{2}$

Hence, $E(X) = 2$ and standard deviation $= \sqrt{2}$.

EXAMPLE 19 *In a game a man wins a rupee for a six and looses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/looses.* **[NCERT]**

SOLUTION The man may get six in the first throw and then he quits the game. He may get a number other than six in the first throw and in the second throw he may get six and quits the game. In the first two throws he gets a number other than six and in third throw he may get a six. He may not get six in any one of three throws.

Let X be the amount he wins/looses. Then, X can take values $1, 0, -1, -3$ such that

$$P(X=1) = P\text{ (Getting six in first throw)} = \frac{1}{6}$$

$$P(X=0) = P\text{ (Getting an other number in first throw and six in second throw)}$$

$$\Rightarrow \quad P(X=0) = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$$

$$P(X=-1) = P\text{ (Getting numbers other than 6 in first two throws and a six in third throw)}$$

$$\Rightarrow \quad P(X=-1) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}$$

$$P(X=-3) = P\text{ (Getting a number other than six in first three throws)}$$

$$\Rightarrow \quad P(X=-3) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$$

Thus, the probability distribution of X is as given below:

X :	1	0	-1	-3
$P(X)$:	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{25}{216}$	$\frac{125}{216}$

$$\therefore \quad E(X) = 1 \times \frac{1}{6} + 0 \times \frac{5}{36} + (-1) \times \frac{25}{216} + (-3) \times \frac{125}{216}$$

$$\Rightarrow \quad E(X) = \frac{36 + 0 - 25 - 375}{216} = -\frac{364}{216} = -\frac{91}{54}$$

EXERCISE 31.2

1 Find the mean and standard deviation of each of the following probability distributions :

(i)

x_i:	3	11
p_i:	1/2	1/2

(ii)

x_i :	1	3	4	5
p_i :	0.4	0.1	0.2	0.3

(iii)

x_i :	-5	-4	1	2
p_i :	1/4	1/8	1/2	1/8

(iv)

x_i :	-1	0	1	2	3
p_i :	0.3	0.1	0.1	0.3	0.2

(v)

x_i :	1	2	3	4
p_i :	0.4	0.3	0.2	0.1

(vi)

x_i :	0	1	3	5
p_i :	0.2	0.5	0.2	0.1

(vii)

x_i :	-2	-1	0	1	2
p_i :	0.1	0.2	0.4	0.2	0.1

(viii)

x_i :	-3	-1	0	1	3
p_i :	0.05	0.45	0.20	0.25	0.05

2. Find the mean and variance of the number of heads in three tosses of a coin.
3. Two cards are drawn simultaneously from a pack of 52 cards. Compute the mean and standard deviation of the number of kings. [CBSE 2008]
4. Find the mean, variance and standard deviation of the number of tails in three tosses of a coin.
5. Compute the man and variance of the probability distribution of the number of doublets in four throws of a pair of dice.
6. A pair of fair dice is thrown. Let X be the random variable which denotes the minimum of the two numbers which appear. Find the probability distribution, mean and variance of X.
7. A fair coin is tossed four times. Let X denote the number of heads occurring. Find the probability distribution, mean and variance of X.
8. A fair die is tossed. Let X denote twice the number appearing. Find the probability distribution, mean and variance of X.
9. A fair die is tossed. Let X denote 1 or 3 according as an odd or an even number appears. Find the probability distribution, mean and variance of X.
10. A fair coin is tossed four times. Let X denote the longest string of heads occurring. Find the probability distribution, mean and variance of X.
11. Find the mean variance and standard deviation of the following probability distribution

x_i:	a	b
p_i:	p	q

where $p + q = 1$.
12. Two cards are selected at random from a box which contains five cards numbered 1, 1, 2, 2, and 3. Let X denote the sum and Y the maximum of the two numbers drawn. Find the probability distribution, mean and variance of X and Y.
13. A die is tossed twice. A 'success' is getting an odd number on a toss. Find the variance of the number of successes.
14. Find the mean of the random variable X which denotes the number of doublets in four throws of a pair of dice.
15. Two eggs are accidently mixed up with ten good ones. Three eggs are drawn at random with replacement from this lot. Compute the mean for the number of bad eggs drawn.
16. A box contains 13 bulbs, out of which 5 are defective. 3 bulbs are randomly drawn, one by one without replacement, from the box. Find the probability distribution of the number of defective bulbs. [CBSE 2005]
17. In roulette, Fig. 31.2, the wheel has 13 numbers 0, 1, 2, ..., 12 marked on equally spaced slots. A player sets Rs 10 on a given number. He receives Rs 100 from the organiser of the game if the ball comes to rest in this slot; otherwise he gets nothing. If X denotes the player's net gain/loss, find E (X).

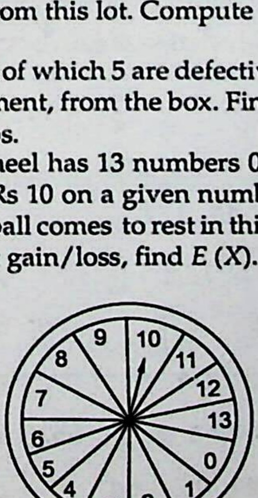

Fig. 31.2

18. Find the probability distribution of the number of doublets in three throws of a pair of dice and hence find its mean. [CBSE 2010]

ANSWERS

1. (i) 4, 3.2 (ii) – 1, 2.9 (iii) 3, 1.7 (iv) 1, 1.5
   (v) Mean = 2, S.D = 1 (vi) Mean = 1.6, S.D = 1.497
   (vii) Mean = 0, S.D = 1.095 (viii) Mean = – 0.2, S.D = 1.249

2. Mean = 1.5, Var = 3/4 3. Mean = $\frac{34}{221}$, Var = $\frac{400}{2873}$

4. Mean = 1.5, Var = $\frac{3}{4}$, S.D. = 0.87 5. Mean = $\frac{2}{3}$, Var = $\frac{5}{9}$

6.

x_i:	1	2	3	4	5	6	Mean = 2.5
p_i:	11/36	9/36	7/36	5/36	3/36	1/36	Var = 2.1

7.

x_i:	0	1	2	3	4	Mean = 2
p_i:	1/16	4/16	6/16	4/16	1/16	Var = 1

8.

x_i:	2	4	6	8	10	12	Mean = 7
p_i:	1/6	1/6	1/6	1/6	1/6	1/6	Var = 11.7

9.

x_i:	1	3	Mean = 2
p_i:	1/2	1/2	Var = 1

10.

x_i:	0	1	2	3	4	Mean = 1.7
p_i:	1/16	7/16	5/16	2/16	1/16	Var = 0.9

11. Mean = $ap + bq$ Var = $pq\,(a-b)^2$ $\sigma = |a-b|\sqrt{pq}$

12.

x_i:	2	3	4	5	Mean = 3.6
p_i:	0.1	0.4	0.3	0.2	Var = 0.84
y_i:	1	2	3		Mean = 2.3
p_i:	0.1	0.5	0.4		Var = 0.41

13. 1/2 14. 2/3 15. 1/2

16.

X:	0	1	2	3
P (X):	$\frac{28}{143}$	$\frac{70}{143}$	$\frac{40}{143}$	$\frac{5}{143}$

17. $E(X) = -\frac{30}{13}$

18.

X:	0	1	2	3	
P (X):	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$,	Mean = 1/2

HINTS TO SELECTED PROBLEM

17. The probability distribution of X is

X:	-10	90
$P(X)$:	$\frac{12}{13}$	$\frac{1}{13}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the values of 'a' for which the following distribution of probabilities becomes a probability distribution:

$X = x_i$:	-2	-1	0	1
$P(X = x_i)$:	$\frac{1-a}{4}$	$\frac{1+2a}{4}$	$\frac{1-2a}{4}$	$\frac{1+a}{4}$

2. For what value of k the following distribution is a probability distribution?

$X = x_i$:	0	1	2	3
$P(X = x_i)$:	$2k^4$	$3k^2 - 5k^3$	$2k - 3k^2$	$3k - 1$

3. If X denotes the number on the upper face of a cubical die when it is thrown, find the mean of X.
4. If the probability distribution of a random variable X is given by

$X = x_i$:	1	2	3	4
$P(X = x_i)$:	$2k$	$4k$	$3k$	k

Write the value of k.

5. Find the mean of the following probability distribution:

X:	1	2	3
$P(X = x_i)$:	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{5}{8}$

6. If the probability distribution of a random variable X is as given below:

$X = x_i$:	1	2	3	4
$P(X = x_i)$:	c	$2c$	$4c$	$4c$

Write the value of $P(X \leq 2)$.

7. A random variable has the following probability distribution:

$X = x_i$:	1	2	3	4
$P(X = x_i)$:	k	$2k$	$3k$	$4k$

Write the value of $P(X \geq 3)$.

ANSWERS

1. $-\frac{1}{2} \leq a \leq \frac{1}{2}$ 2. $k = \frac{1}{2}$ 3. 3.5 4. $k = 0.1$ 5. $\frac{19}{8}$

6. 0.3 7. $\frac{7}{10}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If a random variable X has the following probability distribution:

X :	0	1	2	3	4	5	6	7	8
P(X) :	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

, then the value of a is

(a) $\frac{7}{81}$ (b) $\frac{5}{81}$ (c) $\frac{2}{81}$ (d) $\frac{1}{81}$

2. A random variable X has the following probability distribution :

X :	1	2	3	4	5	6	7	8
P(X) :	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events $E = \{X \text{ is a prime number}\}$, $F = \{X < 4\}$, the probability $P(E \cup F)$ is

(a) 0.50 (b) 0.77 (c) 0.35 (d) 0.87

3. A random variable X takes the values $0, 1, 2, 3$ and its mean is 1.3. If $P(X=3) = 2P(X=1)$ and $P(X=2) = 0.3$, then $P(X=0)$ is

(a) 0.1 (b) 0.2 (c) 0.3 (d) 0.4

4. A random variable has the following probability distribution.

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	$2p$	$2p$	$3p$	p^2	$2p^2$	$7p^2$	$2p$

The value of p is

(a) 1/10 (b) −1 (c) −1/10 (d) none of these.

5. If X is a random-variable with probability distribution as given below :

X :	0	1	2	3
$P(X=x)$:	k	3 k	3 k	k

The value of k and its variance are

(a) 1/8, 22/27 (b) 1/8, 23/27

(c) 1/8, 24/27 (d) 1/8, 3/4

ANSWERS

1. (d) 2. (b) 3. (d) 4. (a) 5. (d)

SUMMARY

1. Let S be the sample space associated with a given random experiment. Then, a real valued function X which assigns to each event $w \in S$ to a unique real number $X(w)$ is called a random variable.

   In other words, a random variable is a real valued function having domain as the sample space associated with a random experiment.

2. If a random variable X takes values $x_1, x_2, ..., x_n$ with respective probabilities $p_1, p_2, ..., p_n$, then

X:	x_1	x_2	x_3	......	x_n
P(X):	p_1	p_2	p_3	......	p_n

is known as the probability distribution of X.

3. The probability distribution of a random variable X is defined only when we have the various values of the random variable e.g. $x_1, x_2, \ldots, x_n$ together with respective probabilities $p_1, p_2, \ldots, p_n$ satisfying $\sum_{i=1}^{n} p_i = 1$.

4. If X is a random variable with the probability distribution

X :	x_1	x_2	......	x_n
$P(X)$:	p_1	p_2	......	p_n

then,

$$P(X \le x_i) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_i)$$
$$= p_1 + p_2 + \ldots + p_i$$
$$P(X < x_i) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_{i-1})$$
$$= p_1 + p_2 + \ldots p_{i-1}$$
$$P(X \ge x_i) = P(X = x_i) + P(X = x_{i+1}) + \ldots + P(X = x_n)$$
$$= p_i + p_{i+1} + \ldots + p_n$$
$$P(X > x_i) = P(X = x_{i+1}) + P(X = x_{i+2}) + \ldots + P(X = x_n)$$
$$= p_{i+1} + p_{i+2} + \ldots + p_n$$

Also,

$$P(X \ge x_i) = 1 - P(X < x_i),\ P(X > x_i) = 1 - P(X \le x_i),$$
$$P(X \le x_i) = 1 - P(X > x_i) \text{ and } P(X < x_i) = 1 - P(X \ge x_i)$$
$$P(x_i \le X \le x_j) = P(X = x_i) + P(X = x_{i+1}) + \ldots + P(X = x_j)$$
$$P(x_i < X < x_j) = P(X = x_{i+1} + P(X = x_{i+2}) + \ldots + P(X = x_{j-1})$$

5. If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with respective probabilities $p_1, p_2, p_3, \ldots, p_n$, then the mean $\overline{X}$ of X is defined as

$$\overline{X} = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n \quad \text{or,} \quad \overline{X} = \sum_{i=1}^{n} p_i x_i$$

The mean of a random variable X is also known as its mathematical expectation or expected value and is denoted by $E(X)$.

6. If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with the respective probabilities $p_1, p_2, \ldots, p_n$, then variance of X is defined as

$$\text{Var}(X) = p_1 (x_1 - \overline{X})^2 + p_2 (x_2 - \overline{X})^2 + \ldots + p_n (x_n - \overline{X})^2$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i (x_i - \overline{X})^2, \text{ where } \overline{X} = \sum_{i=1}^{n} p_i x_i \text{ is the mean of X.}$$

Also, $\text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i \right)^2$

BINOMIAL DISTRIBUTION

32.1 INTRODUCTION

In the previous chapter, we have studied about discrete random variable and its probability distribution. In this chapter we shall study a particular type of probability distribution, known as binomial distribution. It was discovered by James Bernouli in the year 1700 and was first published posthumously in 1713. This distribution has been used to describe a wide variety of processes in business and social sciences as well as other areas. In binomial distribution a random experiment is performed repeatedly under identical conditions and the distribution determines the probability of occurrence of one set of dichotomous alternatives i.e. success or failure.

32.2 BERNOULLI TRIALS

In our day to day life we come across many experiments which are dichotomous in nature i.e. they produce one of the two possible outcomes in a trial. For example, a tossed coin shows a 'head' or 'tail', a manufactured item can be 'defective' or 'non-defective' etc. In such type of experiments, it is customary to call one of the outcomes a 'success' and the other 'not success', or 'failure'. If this type of experiments are repeated under identical conditions, then the outcome of any trial is independent of the outcome of any other trial. In each of such trials, the probability of success or failure remains constant. Such independent trials which have only two outcomes usually referred as 'success' or 'failure' are called Bernoulli trials as defined below.

BERNOULLI TRIALS *Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions:*

(i) *They are finite in number.*

(ii) *They are independent of each other.*

(iii) *Each trial has exactly two outcomes: success or failure.*

(iv) *The probability of success or failure remains same in each trial.*

For example, if four balls are drawn successively with replacement from a bag containing 7 red and 6 black balls, then the probability of getting a black ball in each trial remains same equal to $\frac{6}{13}$. So, trials are Bernoulli trials. If the balls are drawn successively without replacement, then the probability of getting a black ball in first trial is $\frac{6}{13}$ and in second trial it is $\frac{5}{12}$, if the first ball drawn is black or $\frac{6}{12}$, if the first ball drawn is red and so on. Clearly, the probability of getting a black ball is not same for all trials. So, the trials are not Bernoulli trials.

32.3 BINOMIAL DISTRIBUTION

Consider a random experiment and an event A associated with it. If the experiment results in the event A, let us call it a "success", denoted by S. If on the other hand, the event A does not occur let us say that the experiment has resulted in a "failure", denoted by F. Let $P(S) = p$ and $P(F) = q$. Then, $p + q = 1$. Suppose the experiment is performed 4 times under identical condition and we want to find the probability of 3 successes. Three successes in 4 trials can occur in ${}^4C_3$ mutually exclusive ways as given below

$$SSS\,F,\ S\,FSS,\ F\,SSS,\ SS\,FS$$

By addition theorem of probability, we have

Probability of 3 successes in 4 trials $= P\,(SSS\,F) + P\,(S\,FSS) + P\,(F\,SSS) + P\,(SS\,FS)$

Since 4 trials are independent. Therefore, by multiplication theorem for independent events, we have

$$P\,(SSS\,F) = P\,(S)\,P\,(S)\,P\,(S)\,P\,(F) = p^3 q = p^3 q^{4-3}$$

Similarly, we have

$$P\,(S\,FSS) = p^3 q^{4-3},\ P\,(F\,SSS) = p^3 q^{4-3} \text{ and } P\,(SS\,FS) = p^3 q^{4-3}$$

Thus,

$$\text{Probability of 3 successes in 4 trials} = p^3 q^{4-3} + p^3 q^{4-3} + p^3 q^{4-3} + p^3 q^{4-3}$$
$$= {}^4C_3\, p^3 q^{4-3}$$

Proceeding on the same lines we can easily show that the probabilities of 0, 1, 2 and 4 successes are given by

$${}^4C_0\, p^0 q^{4-0},\ {}^4C_1\, p^1 q^{4-1},\ {}^4C_2\, p^2 q^{4-2} \text{ and } {}^4C_4\, p^4 q^{4-4} \text{ respectively.}$$

Now, if X denotes the number of successes in 4 trials, then X can take values 0, 1, 2, 3 and 4 such that

$$P\,(X=0) = {}^4C_0\, p^0 q^{4-0},\ P\,(X=1) = {}^4C_1\, p^1 q^{4-1},\ P\,(X=2) = {}^4C_2\, p^2 q^{4-2}$$
$$P\,(X=3) = {}^4C_3\, p^3 q^{4-3} \text{ and } P\,(X=4) = {}^4C_4\, p^4 q^{4-4}$$

In other words, we have

$$P\,(X=r) = {}^4C_r\, p^r q^{4-r};\ r = 0, 1, 2, 3, 4.$$

This result can be generalised to the case where the experiment is repeated n times under identical conditions. The probability of r successes in n independent trials in a specific order, say $\underbrace{\underline{SSSFSFFSS\ldots SF}}_{r-\text{successes},\ (n-r)\text{ failures}}$ is given by

$$P\,\underbrace{\underline{(SSSFSFFSS\ldots SF)}}_{r\text{ successes},\ n-r\text{ failures}}$$

$$= P(S)\,P(S)\,P(S)\,P(F)\ \ldots\ P(S)\,P(F) \qquad \text{[By multiplication theorem]}$$

$$= pppq \ldots p\ q = p^r q^{n-r}$$

But, r successes in n trials can occur in ${}^nC_r$ mutually exclusive ways and the probability of each such way is $p^r q^{n-r}$. So by addition theorem of probability, the probability of r successes in n trials in any order is given by

$${}^nC_r\, p^r q^{n-r}$$

Let X denote the random variable which associates every outcome to the number of successes in it. Then, X assumes values 0, 1, 2, ..., n such that

$$P\,(X=r) = {}^nC_r\, p^r q^{n-r},\ r = 0, 1, 2, \ldots, n.$$

The probability distribution of the random variable X is therefore given by

X :	0	1	2	...	r	...	n
$P(X)$:	${}^nC_0\, p^0\, q^{n-0}$	${}^nC_1\, p^1\, q^{n-1}$	${}^nC_2\, p^2\, q^{n-2}$	...	${}^nC_r\, p^r\, q^{n-r}$	...	${}^nC_n\, p^n\, q^{n-n}$

From the probability distribution of the random variable X, we observe that the probabilities of the random variable taking values 0, 1, 2, ..., n are given by the terms in the binomial expansion of $(q+p)^n$. That is why we say that the probability distribution of the random variable X is the binomial distribution or, that X is a Binomial random variable.

DEFINITION (BINOMIAL DISTRIBUTION) *A random variable X which takes values 0, 1, 2, ..., n is said to follow binomial distribution if its probability distribution function is given by*

$$P(X=r) = {}^nC_r\, p^r\, q^{n-r},\ r = 0, 1, 2, \ldots, n,$$

where $p, q > 0$ such that $p+q=1$.

The two constants n and p in the distribution are known as the parameters of the distribution.

The notation $X \sim B(n, p)$ is generally used to denote that the random variable X follows binomial distribution with parameters n and p.

We have,

$$P(X=0) + P(X=1) + \ldots + P(X=n)$$

$$= {}^nC_0\, p^0\, q^{n-0} + {}^nC_1\, p^1\, q^{n-1} + \ldots + {}^nC_n\, p^n\, q^{n-n}$$

$$= (q+p)^n = 1^n = 1$$

Thus, the assignment of probabilities to the random variable X is permissible.

NOTE *If n trials constitute an experiment and the experiment is repeated N times, then the frequencies of 0, 1, 2, ..., n successes are given by*

$$N \cdot P(X=0),\ N \cdot P(X=1),\ N \cdot P(X=2),\ \ldots,\ N \cdot P(X=n).$$

ILLUSTRATIVE EXAMPLES

Type I ON FINDING THE PROBABILITY OF THE OCCURRENCE OF AN EVENT GIVEN NUMBER OF TIMES IN A SERIES OF INDEPENDENT TRIALS

EXAMPLE 1 *A die is thrown 6 times. If "getting an odd number" is a "success", what is the probability of*
(i) 5 successes (ii) at least 5 successes (iii) at most 5 successes (iv) at least one success (v) no success.

SOLUTION Let p denote the probability of getting an odd number in a single throw of the die. Then,

$$p = \frac{3}{6} = \frac{1}{2} \text{ and } q = 1-p = 1-\frac{1}{2} = \frac{1}{2}$$

Let X denote the number of successes in 6 trials. Then, X is a binomial variate with parameter $n=6, p=1/2$.

The probability of r successes is given by

$$P(X=r) = {}^6C_r\,(1/2)^{6-r}\,(1/2)^r, \text{ where } r = 0, 1, 2, \ldots, 6$$

or, $$P(X=r) = {}^6C_r\,(1/2)^6, \quad \text{where } r = 0, 1, 2, \ldots 6 \qquad \ldots\text{(i)}$$

(i) Probability of 5 successes $= P(X=5) = {}^6C_5\left(\frac{1}{2}\right)^6 = \frac{3}{32}$ [Using (i)]

(ii) Probability of at least 5 successes $= P(X \geq 5)$

$$= P(X=5) + P(X=6)$$

$$= {}^6C_5\left(\frac{1}{2}\right)^6 + {}^6C_6\left(\frac{1}{2}\right)^6 \quad \text{[Using (i)]}$$

$$= (6+1)\frac{1}{64} = \frac{7}{64}$$

(iii) Probability of at most 5 successes $= P(X \leq 5) = 1 - P(X > 5)$

$$= 1 - P(X=6)$$

$$= 1 - {}^6C_6\left(\frac{1}{2}\right)^6 \quad \text{[Using (i)]}$$

$$= 1 - \frac{1}{64} = \frac{63}{64}$$

(iv) Probability of at least one success $= P(X \geq 1) = 1 - P(X=0)$

$$= 1 - {}^6C_0\left(\frac{1}{2}\right)^6 \quad \text{[Using (i)]}$$

$$= 1 - \frac{1}{64} = \frac{63}{64}$$

(v) Probability of no success $= P(X=0) = {}^6C_0\left(\frac{1}{2}\right)^6$ [Using (i)]

$$= \frac{1}{64}$$

EXAMPLE 2 *A coin is tossed 5 times. What is the probability of getting at least 3 heads?*

SOLUTION Let p denote the probability of getting head in a single toss of a coin. Then,

$$p = \frac{1}{2} \text{ and so, } q = \frac{1}{2}$$

Let X denote the number of heads in 5 tosses of a coin. Then, X is a binomial variate with parameters $n = 5, p = 1/2$ such that

$$P(X=r) = {}^5C_r\,(1/2)^{5-r}\,(1/2)^r = {}^5C_r\,(1/2)^5, \text{ where } r = 0, 1, 2, ..., 5 \quad ...(i)$$

Now,

Probability of at least 3 heads

$$= P(X \geq 3)$$

$$= P(X=3) + P(X=4) + P(X=5)$$

$$= {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_4\left(\frac{1}{2}\right)^5 + {}^5C_5\left(\frac{1}{2}\right)^5$$

$$= \left\{{}^5C_3 + {}^5C_4 + {}^5C_5\right\}\left(\frac{1}{2}\right)^5 \quad \text{[Using (i)]}$$

$$= (10+5+1) \times \frac{1}{32} = \frac{1}{2}$$

EXAMPLE 3 *A pair of dice is thrown 7 times. If getting a total of 7 is considered a success, what is the probability of*
(i) no success? (ii) 6 successes? (iii) at least 6 successes? (iv) at most 6 successes?

[CBSE 2004]

SOLUTION Let p denote the probability of getting a total of 7 in a single throw of a pair of dice. Then,

$$p = \frac{6}{36} = \frac{1}{6} \qquad \left[\because \text{ The sum can be 7 in any one of the ways : } (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\right]$$

$$\therefore \qquad q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Let X denote the number of successes in 7 throws of a pair of dice. The X is a binomial variate with parameters $n = 7$ and $p = 1/6$ such that

Now,

$$P(X = r) = {}^7C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{7-r}, \quad r = 0, 1, 2, \ldots, 7 \qquad \ldots\text{(i)}$$

(i) Probability of no success $= P(X = 0) = {}^7C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{7-0} = \left(\frac{5}{6}\right)^7$ [Using (i)]

(ii) Probability of 6 successes $= P(X = 6) = {}^7C_6 \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{7-6}$ [Using (i)]

$$= 35\,(1/6)^7$$

(iii) Probability of at least 6 successes $= P(X \geq 6)$

$$= P(X = 6) + P(X = 7)$$

$$= {}^7C_6 \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{7-6} + {}^7C_7 \left(\frac{1}{6}\right)^7 \left(\frac{5}{6}\right)^0 \qquad \text{[Using (i)]}$$

$$= 7 \cdot \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^7 = \left(\frac{1}{6}\right)^6 \left(\frac{35}{6} + \frac{1}{6}\right) = \left(\frac{1}{6}\right)^5$$

(iv) Probability of at most 6 successes $= P(X \leq 6)$

$$= 1 - P(X > 6)$$

$$= 1 - P(X = 7)$$

$$= 1 - {}^7C_7 \left(\frac{1}{6}\right)^7 \left(\frac{5}{6}\right)^0 \qquad \text{[Using (i)]}$$

$$= 1 - \left(\frac{1}{6}\right)^7$$

EXAMPLE 4 *An urn contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one with replacement, what is the probability that*
(i) all are white? *(ii) only 3 are white?*
(iii) none is white? *(iv) at least three are white?*

SOLUTION Let p denote the probability of drawing a white ball from an urn containing 5 white, 7 red and 8 black balls. Then,

$$p = \frac{^5C_1}{^{20}C_1} = \frac{5}{20} = \frac{1}{4}$$

So, $$q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$$

Let X denote the number of white balls in 4 draws with replacement. Then, X is a binomial variate with parameter $n = 4$ and $p = \frac{1}{4}$ such that

$$P(X = r) = \text{Probability that } r \text{ balls are white } = {}^4C_r \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{4-r} \ ; \ r = 0, 1, 2, 3, 4 \qquad \text{...(i)}$$

Now,

(i) Probability that all are white $= P(X = 4)$

$$= {}^4C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{4-4} = \left(\frac{1}{4}\right)^4 \qquad \text{[Using (i)]}$$

(ii) Probability that only 3 are white $= P(X = 3) = {}^4C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^{4-3}$ [Using (i)]

$$= 3\left(\frac{1}{4}\right)^3$$

(iii) Probability that none is white $= P(X = 0) = {}^4C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^4$ [Using (i)]

$$= \left(\frac{3}{4}\right)^4$$

(iv) Probability that at least three are white

$= P(X \geq 3)$

$= P(X = 3) + P(X = 4)$

$= {}^4C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^{4-3} + {}^4C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^0$ [Using (i)]

$= 13\left(\frac{1}{4}\right)^4$

EXAMPLE 5 *A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps he is just one step away from the starting point.*

SOLUTION Let p denote the probability that the man takes a step forward. Then, $p = 0.4$

$\therefore$ $$q = 1 - p = 1 - 0.4 = 0.6$$

Let X denote the number of steps taken in the forward direction. Since the steps are independent of each other, therefore X is a binomial variate with parameters $n = 11$ and $p = 0.4$ such that

$$P(X = r) = {}^{11}C_r (0.4)^r (0.6)^{11-r} \ ; \ r = 0, 1, 2, \ldots, 11 \qquad \text{...(i)}$$

Since the man is one step away from the initial point, he is either one step forward or one step backward from the initial point at the end of eleven steps. If he is one step forward, then he must have taken six steps forward and five steps backward and if he is one step backward, then he must have taken five steps forward and six steps backward. Thus, either $X = 6$ or $X = 5$.

$\therefore$ Required probability $= P[(X=5) \text{ or } (X=6)]$

$\Rightarrow$ Required probability $= P(X=5) + P(X=6)$

$\Rightarrow$ Required probability $= {}^{11}C_5 (0.4)^5 (0.6)^{11-5} + {}^{11}C_6 (0.4)^6 (0.6)^{11-6}$ [Using (i)]

$\Rightarrow$ Required probability $= {}^{11}C_5 (0.4)^5 (0.6)^5 [0.6 + 0.4]$ $[\because {}^{11}C_5 = {}^{11}C_6]$

$\Rightarrow$ Required probability $= 462 (0.4)^5 (0.6)^5 = 462 (0.24)^5$

EXAMPLE 6 *An urn contains 25 balls of which 10 balls bear a mark 'A' and the remaining 15 balls bear a mark 'B'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that*

(i) all will bear 'A' mark

(ii) not more that 2 will bear 'B' mark

(iii) the number of balls with 'A' mark and 'B' mark will be equal

(iv) at least one ball will bear 'B' mark

SOLUTION Let p denote the probability of drawing a ball which bears mark 'A'. Then

$$p = \frac{10}{25} = \frac{2}{5}$$

Let X denote the number of balls which bear mark 'A' in 6 draws. Then, X is a binomial variate with parameters $n = 6$ and $p = \frac{2}{5}$.

Also $q = 1 - p = 1 - \frac{2}{5} = \frac{3}{5}$

Now, $P(X=r)$ = Probability of getting r balls bearing mark 'A'

$$= {}^6C_r \left(\frac{2}{5}\right)^r \left(\frac{3}{5}\right)^{6-r}, \quad r = 0, 1, 2, \dots, 6 \quad \text{...(i)}$$

(i) Probability that all balls bear 'A' mark $= P(X=6)$

$$= {}^6C_6 \left(\frac{2}{5}\right)^6 \left(\frac{3}{5}\right)^{6-6} \quad \text{[Using (i)]}$$

$$= (2/5)^6$$

(ii) Not more than 2 balls will bear 'B' mark means that there can be either no ball or one ball or two balls of 'B' mark. This implies that there can be either 6 or 5 or 4 balls of 'A' mark.

$\therefore$ Required probability

$= P(X \geq 4)$

$= P(X=4) + P(X=5) + P(X=6)$

$$= {}^6C_4 \left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^2 + {}^6C_5 \left(\frac{2}{5}\right)^5 \left(\frac{3}{5}\right) + {}^6C_6 \left(\frac{2}{5}\right)^6 \left(\frac{3}{5}\right)^0$$

$$= 7\left(\frac{2}{5}\right)^4$$

ALITER Let p denote the probability that a ball drawn bears mark 'B'. Then,

$$p = \frac{15}{25} = \frac{3}{5}$$

Let Y denote the number of balls which bear mark 'B' in 6 draws. Then, Y is a binomial variate with parameters $n = 6$ and $p = 3/5$ such that

$$P(X = r) = {}^6C_r \left(\frac{3}{5}\right)^r \left(\frac{2}{5}\right)^{6-r}, \ r = 0, 1, 2, \ldots, 6 \qquad \ldots\text{(ii)}$$

Required probability

$= P(Y \leq 2)$

$= P(Y = 0) + P(Y = 1) + P(Y = 2)$

$= {}^6C_0 \left(\frac{3}{5}\right)^0 \left(\frac{2}{5}\right)^6 + {}^6C_1 \left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^5 + {}^6C_2 \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^4$ [Using (ii)]

$= 7(2/5)^4.$

(iii) Required probability

$= P(X = 3) = {}^6C_3 \left(\frac{2}{5}\right)^3 \left(\frac{3}{5}\right)^{6-3}$ [Using (i)]

$= 20 \left(\frac{2}{5}\right)^3 \left(\frac{3}{5}\right)^3$

(iv) Probability that at least one ball will bear 'B' mark

$= P(Y \geq 1)$

$= 1 - P(Y = 0) = 1 - {}^6C_0 \left(\frac{3}{5}\right)^0 \left(\frac{2}{5}\right)^6$ [Using (ii)]

$= 1 - \left(\frac{2}{5}\right)^6$

EXAMPLE 7 *In a hurdles race, a player has to cross 10 hurdles. The probability that he will clear each hurdle. is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?*

[NCERT]

SOLUTION Let X denote the number of hurdles knocked down by the player. Then, X follows binomial distribution with $n = 10$, $p = 1 - \frac{5}{6} = \frac{1}{6}$ and $q = \frac{5}{6}$.

$$\therefore \quad P(X = r) = {}^{10}C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{10-r}; r = 0, 1, 2, \ldots, 10$$

Required probability $= P(X < 2)$

$= P(X = 0) + P(X = 1)$

$= \left(\frac{5}{6}\right)^{10} + {}^{10}C_1 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^9$

$= \left(\frac{5}{6}\right)^9 \left\{\frac{5}{6} + \frac{10}{6}\right\} = \frac{5^{10}}{2 \times 6^9}$

EXAMPLE 8 *Five dice are thrown simultaneously. If the occurrence of an even number in a single dice is considered a success, find the probability of at most 3 successes.*

SOLUTION Let X denote the number of successes in 5 throws of a die and let p be the probability of getting an even number in a single throw of a die. Then,

$$p = \frac{3}{6} = \frac{1}{2} \text{ and } q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

The probability of r successes in five throws of die is given by

$$P(X = r) = {}^5C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r} = {}^5C_r \left(\frac{1}{2}\right)^5, \text{ where } r = 0, 1, 2, \ldots, 5$$

$\therefore$ Required probability $= P(X \le 3) = 1 - P(X > 3)$

$\Rightarrow$ Required probability $= 1 - \{P(X = 4) + P(X = 5)\}$

$\Rightarrow$ Required probability $= 1 - \left\{ {}^5C_4 \left(\frac{1}{2}\right)^5 + {}^5C_5 \left(\frac{1}{2}\right)^5 \right\}$

$\Rightarrow$ Required probability $= 1 - \left(\frac{5}{32} + \frac{1}{32}\right) = \frac{26}{32} = \frac{13}{16}$

EXAMPLE 9 *An unbiased die is thrown again and again until three sixes are obtained. Find the probability of obtaining 3rd six in the sixth throw of the die.* **[CBSE 2009]**

SOLUTION Let p be the probability of obtaining a 'six' in a single throw of the die. Then,

$$p = \frac{1}{6} \text{ and } q = 1 - \frac{1}{6} = \frac{5}{6}$$

Obtaining third six in the sixth throw of the die means that in first five throws there are 2 sixes and the third six is obtained in sixth throw. Therefore,

Required probability

$= P$ (Getting 2 sixes in first 5 throws) P (Getting 'six' in sixth throw)

$$= ({}^5C_2\, p^2 q^{5-2})(p) = {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 \times \frac{1}{6} = 10 \times \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^3 = \frac{625}{23328}$$

Type II PROBLEMS ON FINDING THE PROBABILITY DISTRIBUTION

EXAMPLE 10 *Find the probability distribution of the number of heads when three coins are tossed.*

SOLUTION Let X denote the number of heads obtained in three tosses of a coin, then X can take values 0, 1, 2 and 3.

Let p denote the probability of obtaining a head in a single toss of a coin. Then,

$$p = \frac{1}{2} \text{ and } q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

Since three trials are independent, therefore X is a binomial variate with parameters $n = 3$ and $p = 1/2$ such that

$$P(X = r) = {}^3C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{3-r} = {}^3C_r \left(\frac{1}{2}\right)^3, \; r = 0, 1, 2, 3.$$

Thus, the probability distribution of X is given by

X :	0	1	2	3
$P(X)$:	${}^3C_0\left(\frac{1}{2}\right)^3$	${}^3C_1\left(\frac{1}{2}\right)^3$	${}^3C_2\left(\frac{1}{2}\right)^3$	${}^3C_3\left(\frac{1}{2}\right)^3$

or,

X :	0	1	2	3
$P(X)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

EXAMPLE 11 *A bag contains 3 red and 4 black balls. One ball is drawn and then replaced in the bag and the process is repeated. Every time the ball drawn is red we say that the draw has resulted in success. Let X be the number of successes in 3 draws. Assuming that at each draw each ball is equally likely to be selected, find the probability distribution of X.*

SOLUTION Let p denote the probability of success in a draw. Then,

$$p = \text{Probability of getting a red ball in a draw} = \frac{3}{7}$$

$$\therefore \quad q = 1 - p \Rightarrow q = 1 - \frac{3}{7} = \frac{4}{7}$$

Since the ball drawn in each draw is replaced in the bag, therefore three trials are independent. Consequently X, the number of successes, can take values 0, 1, 2 and 3 and is a binomial variate with parameters $n = 3$, $p = 3/7$ such that

$$P(X = r) = \text{Probability of } r \text{ successes} = {}^3C_r\left(\frac{3}{7}\right)^r\left(\frac{4}{7}\right)^{3-r}, \ r = 0, 1, 2, 3.$$

Thus, the probability distribution of X is given by

X :	0	1	2	3
$P(X)$:	${}^3C_0\left(\frac{3}{7}\right)^0\left(\frac{4}{7}\right)^3$	${}^3C_1\left(\frac{3}{7}\right)^1\left(\frac{4}{7}\right)^2$	${}^3C_2\left(\frac{3}{7}\right)^2\left(\frac{4}{7}\right)^1$	${}^3C_3\left(\frac{3}{7}\right)^3$

or,

X :	0	1	2	3
$P(X)$:	$\left(\frac{4}{7}\right)^3$	$\frac{9}{7}\left(\frac{4}{7}\right)^2$	$\frac{12}{7}\left(\frac{3}{7}\right)^2$	$\left(\frac{3}{7}\right)^3$

EXAMPLE 12 *Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of aces.*

SOLUTION Let p denote the probability of getting an ace in a draw. Then,

$$p = \frac{4}{52} = \frac{1}{13}. \text{ So, } q = 1 - p \Rightarrow q = 1 - \frac{1}{13} = \frac{12}{13}$$

Let X denote the number of aces in 2 draws. Then, X can take values 0, 1 and 2. Since two cards are drawn successively with replacement, therefore X is a binomial variate with parametres $n = 2$ and $p = 1/13$ such that

$$P(X = r) = {}^2C_r\left(\frac{1}{13}\right)^r\left(\frac{12}{13}\right)^{2-r}, r = 0, 1, 2$$

$\therefore \quad P(X=0) = {}^2C_0\left(\frac{1}{13}\right)^0\left(\frac{12}{13}\right)^2 = \frac{144}{169}, P(X=1) = {}^2C_1\left(\frac{1}{13}\right)\left(\frac{12}{13}\right) = \frac{24}{169}$

and, $\quad P(X=2) = {}^2C_2\left(\frac{1}{13}\right)^2\left(\frac{12}{13}\right)^{2-2} = \frac{1}{169}$

Thus, the probability distribution of X is given by

X:	0	1	2
$P(X)$:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

EXAMPLE 13 *Find the probability distribution of the number of doublets in 4 throws of a pair of dice.* **[CBSE 2008]**

SOLUTION Let p denote the probability of getting a doublet in a single throw of a pair of dice. Then,

$$p = \frac{6}{36} = \frac{1}{6} \Rightarrow q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Let X denote the number of doublets in 4 throws of a pair of dice. Then, X is a binomial variate with parameters $n = 4$ and $p = 1/6$ such that

$$P(X=r) = \text{Probability of getting } r \text{ doublets}$$

$$\Rightarrow \quad P(X=r) = {}^4C_r\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{4-r}, r = 0, 1, 2, 3, 4$$

$$\therefore \quad P(X=0) = {}^4C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4 = \left(\frac{5}{6}\right)^4, P(X=1) = {}^4C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 = \frac{2}{3}\left(\frac{5}{6}\right)^3,$$

$$P(X=2) = {}^4C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = \frac{1}{6}\left(\frac{5}{6}\right)^2, P(X=3) = {}^4C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1 = \frac{10}{3}\left(\frac{1}{6}\right)^3$$

and, $\quad P(X=4) = {}^4C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^0 = \left(\frac{1}{6}\right)^4$

Thus, the probability distribution of X is given by

X:	0	1	2	3	4
$P(X)$:	$(5/6)^4$	$(2/3)(5/6)^3$	$(1/6)(5/6)^2$	$(10/3)(1/6)^3$	$(1/6)^4$

Type III ON FINDING THE NUMBER OF TRIALS WHEN PROBABILITY OF OCCURRENCE OF CERTAIN EVENT IS GIVEN

EXAMPLE 14 *The probability of a man hitting a target is $1/4$. How many times must he fire so that the probability of his hitting the target at least once is greater than $2/3$?*

SOLUTION Suppose the man fires n times and let X denote the number of times he hits the target. Then,

$$P(X=r) = {}^nC_r\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{n-r}, r = 0, 1, 2, ..., n$$

Now, $\quad P(X \geq 1) > \frac{2}{3}$

$\Rightarrow \quad 1 - P(X=0) > \frac{2}{3}$

$$\Rightarrow \quad 1 - {}^nC_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^n > \frac{2}{3}$$

$$\Rightarrow \quad 1 - \left(\frac{3}{4}\right)^n > \frac{2}{3} \Rightarrow \left(\frac{3}{4}\right)^n < \frac{1}{3}$$

Clearly, $\frac{3}{4} \not< \frac{1}{3}, \left(\frac{3}{4}\right)^2 \not< \frac{1}{3}, \left(\frac{3}{4}\right)^3 \not< \frac{1}{3}$, but $\left(\frac{3}{4}\right)^4 = \frac{81}{256} < \frac{1}{3}$

$$\therefore \quad \left(\frac{3}{4}\right)^n < \frac{1}{3} \Rightarrow n = 4, 5, 6, \ldots$$

Hence, the man must fire at least 4 times.

EXAMPLE 15 *How many dice must be thrown so that there is a better than even chance of obtaining a six?*

SOLUTION Let n dice be thrown, and let X denote the number of sixes. Then,

$$P(X = r) = {}^nC_r\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^r, \; r = 0, 1, 2, \ldots, n.$$

We have to find the smallest value of n for which $P(X = 0)$ is less than $\frac{1}{2}$.

Now, $\quad P(X = 0) < \frac{1}{2} \Rightarrow \left(\frac{5}{6}\right)^n < \frac{1}{2}$

Clearly, $\frac{5}{6} \not< \frac{1}{2}, \left(\frac{5}{6}\right)^2 \not< \frac{1}{2}, \left(\frac{5}{6}\right)^3 \not< \frac{1}{2}$, but $\left(\frac{5}{6}\right)^4 = \frac{625}{1296} < \frac{1}{2}$

$$\therefore \quad P(X = 0) < \frac{1}{2} \Rightarrow \left(\frac{5}{6}\right)^n < \frac{1}{2} \Rightarrow n = 4, 5, \ldots.$$

Thus, at least 4 dice must be thrown.

EXAMPLE 16 *The probability of a man hitting a target is 1/2. How many times must he fire so that the probability of hitting the target at least once is more than 90%.*

SOLUTION Suppose he fires n times. Let X denote the number of times he hits the target in n trials. Then,

$$P(X = r) = {}^nC_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{n-r} = {}^nC_r\left(\frac{1}{2}\right)^n, \; r = 0, 1, 2, \ldots, n$$

Now, $\quad P(X \geq 1) > \frac{90}{100}$

$$\Rightarrow \quad 1 - P(X = 0) > \frac{90}{100}$$

$$\Rightarrow \quad P(X = 0) < 1 - \frac{90}{100}$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^n < \frac{1}{10} \Rightarrow {}^nC_0\left(\frac{1}{2}\right)^n < \frac{1}{10}$$

Clearly, $\frac{1}{2} \not< \frac{1}{10}, \left(\frac{1}{2}\right)^2 \not< \frac{1}{10}, \left(\frac{1}{2}\right)^3 \not< \frac{1}{10}$, but $\left(\frac{1}{2}\right)^4 < \frac{1}{10}$

$$\therefore \quad \left(\frac{1}{2}\right)^n < \frac{1}{10} \Rightarrow n = 4, 5, 6, \ldots$$

Thus, he must fire at least 4 times.

Type IV ON DETERMINING THE FREQUENCY OF NUMBER OF SUCCESSES

EXAMPLE 17 *Assuming that half the population are consumers of chocolate, so that the chance of an individual being a consumer is 1/2, and assuming that 100 investigators each take ten individuals to see whether they are consumers, how many investigators would you expect to report that 3 people or less were consumers?*

SOLUTION Let X denote the number of consumers in a group of ten individuals. Then, X follows binomial distribution with $n = 10, p = 1/2$ and $q = 1/2$.

$$\therefore \quad P(X = r) = {}^{10}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{10-r} = {}^{10}C_r \left(\frac{1}{2}\right)^{10}, \; r = 0, 1, 2, \ldots, 10.$$

$$\Rightarrow \quad P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$\Rightarrow \quad P(X \leq 3) = {}^{10}C_0 \left(\frac{1}{2}\right)^{10} + {}^{10}C_1 \left(\frac{1}{2}\right)^{10} + {}^{10}C_2 \left(\frac{1}{2}\right)^{10} + {}^{10}C_3 \left(\frac{1}{2}\right)^{10}$$

$$\Rightarrow \quad P(X \leq 3) = \left(\frac{1}{2}\right)^{10} \left\{ {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 \right\}$$

$$\Rightarrow \quad P(X \leq 3) = \frac{1}{2^{10}}(1 + 10 + 45 + 120) = \frac{176}{2^{10}}$$

$\therefore$ Number of investigators reporting that there are 3 persons or less are consumers.

$$= 100\, P(X \leq 3) = 100 \times \frac{126}{2^{10}} = \frac{17600}{2^{10}} = 17 \text{ approximately.}$$

EXAMPLE 18 *Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six.*

SOLUTION Let X denote the number of dice showing five or six in a set of six dice. Then, X follows binomial distribution with $n = 6$, p = probability of getting 5 or 6 in a single throw of a die $= \frac{2}{6} = \frac{1}{3}$ and $q = \frac{2}{3}$.

$$\therefore \quad P(X = r) = {}^6C_r (1/3)^r (2/3)^{6-r}, \; r = 0, 1, 2, 3, \ldots, 6.$$

$$\Rightarrow \quad P(X \geq 3) = 1 - P(X < 3)$$

$$\Rightarrow \quad P(X \geq 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$\Rightarrow \quad P(X \geq 3) = 1 - [\,{}^6C_0 (2/3)^6 + {}^6C_1 (1/3)(2/3)^5 + {}^6C_2 (1/3)^2 (2/3)^4]$$

$$\Rightarrow \quad P(X \geq 3) = 1 - \left(\frac{2}{3}\right)^6 \left[1 + 3 + \frac{15}{4}\right] = 1 - \left(\frac{2}{3}\right)^6 \left(\frac{31}{4}\right)$$

$$\Rightarrow \quad P(X \geq 3) = 1 - \frac{64 \times 31}{729 \times 4} = 1 - \frac{496}{729} = \frac{233}{729}$$

Thus, the frequency that at least three dice show five or six when six dice are thrown 729 times

$$= 729 \, . \, P(X \geq 3) = 729 \times \frac{233}{729} = 233$$

EXERCISE 32.1

1. There are 6% defective items in a large bulk of items. Find the probability that a sample of 8 items will include not more than one defective items.

2. A coin is tossed 5 times. What is the probability of getting at least 3 heads .
3. A coin is tossed 5 times. What is the probability that tail appears an odd number of times .
4. A pair of dice is thrown 6 times. If getting a total of 9 is considered a success, what is the probability of at least 5 successes .
5. A fair coin is tossed 6 times. What is the probability of getting at least 3 heads.
6. Find the probability of 4 turning up at least once in two tosses of a fair die.
7. A coin is tossed 5 times. What is the probability that head appears an even number of times.
8. The probability of a man hitting of target is 1/4. If he fires 7 times, what is the probability of his hitting the target at least twice .
9. Assume that on an average one telephone number out of 15 called between 2 P.M. and 3 P.M. on week days is busy. What is the probability that if six randomly selected telephone numbers are called, at least 3 of them will be busy.
10. If getting 5 or 6 in a throw of an unbiased die is a success and the random variable 'X' denotes the number of successes in six throws of the die, find $P(X \geq 4)$.
11. Eight coins are thrown simultaneously. Find the chance of obtaining at least six heads.
12. Five cards are drawn successively with replacement from a well shuffled pack of 52 cards. What is the probability that
    (i) all the five cards are spades? (ii) only 3 cards are spades?
    (iii) none is spade?
13. A bag contains 7 red, 5 white and 8 black balls. If four balls are drawn one by one with replacement, what is the probability that
    (i) none is white? (ii) all are white? (iii) any two are white?
14. A box contains 100 tickets each bearing one of the numbers from 1 to 100. If 5 tickets are drawn successively with replacement from the box, find the probability that all the tickets bear numbers divisible by 10.
15. A bag contains 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0 ?
16. There are 5 percent defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item.
17. The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs
    (i) none (ii) not more than one (iii) more than one
    (iv) at least one will fuse after 150 days of use.
18. In a hurdle race, a player has to cross 10 hurdels. The probability that he will cross each hurdle is 5/6. What is the probability that he will knock down fewer than 2 hurdels.
19. A bag contains 7 green, 4 white and 5 red balls. If four balls are drawn one by one with replacement, what is the probability that one is red?
20. A bag contains 2 white, 3 red and 4 blue balls. Two balls are drawn at random from the bag. If X denotes the number of white balls among the two balls drawn, describe the probability distribution of X.
21. An urn contains 4 white and three red balls. Find the probability distribution of the number of red balls in three draws, with replacement from the urn.
22. Find the probability distribution of the number of doublets in 4 throws of a pair of dice.
23. Find the probability distribution of the number of sixes in three tosses of a die.
    [CBSE 2007 C]

24. A coin is tossed 5 times. If X is the number of heads observed, find the probability distribution of X.
25. An unbiased die is thrown twice. A success is getting a number greater than 4. Find the probability distribution of the number of successes.
26. A man wins a rupee for head and loses a rupee for tail when coin is tossed. Suppose that he tosses once and quits if he wins but tries once more if he loses on the first toss. Find the probability distribution of the number of rupees the man wins.
27. Five dice are thrown simultaneously. If the occurrence of 3, 4 or 5 in a single die is considered a success, find the probability of at least 3 successes.
28. The items produced by a company contain 10% defective items. Show that the probability of getting 2 defective items in a sample of 8 items is $\frac{28 \times 9^6}{10^8}$.
29. A card is drawn and replaced in an ordinary pack of 52 cards. How many times must a card be drawn so that (i) there is at least an even chance of drawing a heart, (ii) the probability of drawing a heart is greater than 3/4 ?
30. The mathematics department has 8 graduate assistants who are assigned to the same office. Each assistant is just likely to study at home as in the office. How many desks must there be in the office so that each assistant has a desk at least 90% of the time?
31. An unbiased coin is tossed 5 times. Find, by using binomial distribution, the probability of getting at least 6 heads. **[CBSE 2000]**
32. Six coins are tossed simultaneously. Find the probability of getting
    (i) 3 heads (ii) no heads (ii) at least one head **[CBSE 2003]**
33. Suppose that a radio tube inserted into a certain type of set has probability 0.2 of functioning more than 500 hours. If we test 4 tubes at random what is the probability that exactly three of these tubes function for more than 500 hours?
34. The probability that a certain kind of component will survive a given shock test is $\frac{3}{4}$. Find the probability that among 5 components tested
    (i) exactly 2 will survive (ii) at most 3 will survive
35. Assume that the probability that a bomb dropped from an aeroplane will strike a certain target is 0.2. If 6 bombs are dropped find the probability that
    (i) exactly 2 will strike the target. (ii) at least 2 will strike the target.
36. It is known that 60% of mice inoculated with a serum are protected from a certain disease. If 5 mice are inoculated, find the probability that
    (i) none contract the disease (ii) more than 3 contract the disease.
37. An experiment succeeds twice as often as it fails. Find the probability that in the next 6 trials there will be at least 4 successes.
38. In a hospital, there are 20 kidney dialysis machines and that the chance of any one of them to be out of service during a day is 0.02. Determine the probability that exactly 3 machines will be out of service on the same day.
39. The probability that a student entering a university will graduate is 0.4. Find the probability that out of 3 students of the university:
    (i) none will graduate,
    (ii) only one will graduate,
    (iii) all will graduate. **[CBSE 2005]**
40. Ten eggs are drawn successively, with replacement, from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg. **[NCERT]**
41. In a 20-question true-false examination suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer 'true'; if it falls tails, he answers 'false'. Find the probability that he answers at least 12 questions correctly. **[NCERT]**

42. Suppose X has a binomial distribution with $n = 6$ and $p = \frac{1}{2}$. Show that $X = 3$ is the most likely outcome. [NCERT]

43. In a multiple choice examination with three possible answers for each of the five questions out of which only one is correct, what is the probability that a candidate would get four or more correct answers just by guessing? [NCERT, CBSE 2010]

44. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize
(i) at least once (ii) exactly once (iii) at least twice? [NCERT]

45. The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 0.99? [NCERT]

46. How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%. [NCERT]

47. Suppose that 90% of people are right-handed. What is the probability that at most 6 of a random smaple of 10 people are right handed? [NCERT]

ANSWERS

1. $1.42 \times (0.94)^7$ 2. $\frac{1}{2}$ 3. $\frac{1}{2}$ 4. $\frac{49}{9^6}$ 5. $\frac{21}{32}$ 6. $\frac{11}{36}$ 7. $\frac{1}{2}$

8. $\frac{4547}{8192}$ 9. $1 - \left[\left(\frac{14}{15}\right)^6 + \frac{6}{15}\left(\frac{14}{15}\right)^5 + \frac{1}{15}\left(\frac{14}{15}\right)^4\right]$ 10. $\frac{73}{729}$ 11. $\frac{37}{256}$

12. (i) $\frac{1}{1024}$ (ii) $\frac{45}{512}$ (iii) $\frac{243}{1024}$ 13. (i) $\frac{81}{256}$ (ii) $\frac{1}{256}$ (iii) $\frac{27}{128}$

14. $\left(\frac{1}{10}\right)^5$ 15. $\left(\frac{9}{10}\right)^4$ 16. $\left(\frac{19}{20}\right)^9\left(\frac{29}{20}\right)$ 17. (i) $\left(\frac{19}{20}\right)^5$ (ii) $\frac{6}{5}\left(\frac{19}{20}\right)^4$

(iii) $1 - \frac{6}{5}\left(\frac{19}{20}\right)^4$ (iv) $1 - \left(\frac{19}{20}\right)^5$ 18. $\frac{5}{2}\left(\frac{5}{6}\right)^9$ 19. $\frac{5}{4}\left(\frac{11}{16}\right)^3$

20.

X :	0	1	2
P(X) :	$\frac{21}{36}$	$\frac{14}{36}$	$\frac{1}{36}$

21.

X :	0	1	2	3
P(X) :	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$

22.

X :	0	1	2	3	4
P(X) :	$\frac{625}{1296}$	$\frac{500}{1296}$	$\frac{150}{1296}$	$\frac{20}{1296}$	$\frac{1}{1296}$

23.

X :	0	1	2	3
$P(X)$:	$\frac{125}{216}$	$\frac{25}{72}$	$\frac{5}{72}$	$\frac{1}{216}$

24.

X :	0	1	2	3	4	5
$P(X)$:	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

25.

X :	0	1	2
$P(X)$:	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

26.

X :	1	0	-2
$P(X)$:	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

27. $\frac{1}{2}$ 29. 3, 5 30. 6 31. $\frac{219}{256}$

32. (i) $\frac{5}{6}$ (ii) $\frac{1}{64}$ (iii) $\frac{63}{64}$ 33 0.0256

34. (i) 0.0879 (ii) 0.3672 35. (i) 0.2167 (ii) 0.3378

36. (i) 0.0778 (ii) 0.087 37. $\frac{496}{729}$ 38. 0.0071

39. (i) 0.216 (ii) 0.432 (iii) 0.064

40. $1-\frac{9^{10}}{10^{10}}$ 41. $\frac{{}^{20}C_{12}+{}^{20}C_{13}+\ldots+{}^{20}C_{20}}{2^{20}}$

43. $\frac{11}{243}$ 44. (i) $1-\left(\frac{99}{100}\right)^{50}$ (ii) $\frac{1}{2}\left(\frac{99}{100}\right)^{49}$ (iii) $1-\frac{(99)^{49}\times 149}{100^{50}}$

45. 4 46. 4 47. $1-\sum_{r=7}^{10} {}^{10}C_r\ (0.9)^r\ (0.1)^{10-r}$

HINTS TO SELECTED PROBLEMS

1. We have $p=6/100, q=94/100, n=8$.

   Required probability $= P(X \le 1) = P(X=0)+P(X=1)$

2. We have, $p=1/2, q=1/2, n=5$.
   Req. prob. $= P(X \ge 3) = P(X=3)+P(X=4)+P(X=5)$
3. We have, $p=q=1/2, n=5$. Req. prob. $= P(X=1)+P(X=3)+P(X=5)$
4. We have, $p=4/36, q=8/9, n=6$. Req. prob. $= P(X \ge 5)$
5. We have, $n=6, p=1/2=q$. Required probability $= P(X \ge 3)$
6. We have, $p=1/6, q=5/6, n=2$. Required probability $= P(X \ge 1) = 1-P(X=0)$

$$= P(X = 4) + P(X = 5) + P(X = 6)$$

$$= {}^6C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 + {}^6C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right) + {}^6C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^0 = \left(\frac{2}{3}\right)^4 \times \frac{31}{9}$$

41. Here, $n = 20,\ p = \frac{1}{2},\ q = \frac{1}{2}$

$$\therefore \quad P(X = r) = {}^{20}C_r \left(\frac{1}{2}\right)^{20}$$

$$\text{Required probability} = P(X \geq 12) = \sum_{r=12}^{20} {}^{20}C_r \left(\frac{1}{2}\right)^{20}$$

42. Show that $P(X = 3)$ is the maximum among $P(X = 0), P(X = 1), \ldots P(X = 6)$.

43. Here, $n = 5, p = \frac{1}{3}, q = \frac{2}{3}$. $P(X = r) = {}^5C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{5-r}$; $r = 0, 1, 2, \ldots, 5$.

Required probability $= P(X \geq 4) = P(X = 4) + P(X = 5)$

44. Here, $p = \frac{1}{100}, n = 50, q = \frac{99}{100}$

$$\therefore \quad P(X = r) = {}^{50}C_r \left(\frac{1}{100}\right)^r \left(\frac{99}{100}\right)^{50-r}$$

(i) Required probability $= P(X \geq 1) = 1 - P(X = 0) = 1 - \left(\frac{99}{100}\right)^{50}$

(ii) Required probability $= P(X = 1) = {}^{50}C_1 \times \frac{1}{100} \times \left(\frac{99}{100}\right)^{50}$

(iii) Required probability $= P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$

$$= 1 - \left(\frac{99}{100}\right)^{50} - {}^{50}C_1 \times \frac{1}{100} \times \left(\frac{99}{100}\right)^{49}$$

45. Let the shooter fire n times and let X denote the number of times the shooter hits the target. Then, X follows binomial distribution with $p = \frac{3}{4}$ and $q = \frac{1}{4}$ such that

$$P(X = r) = {}^nC_r \left(\frac{3}{4}\right)^r \left(\frac{1}{4}\right)^{n-r} = {}^nC_r \frac{3^r}{4^n}$$

It is given that

$$P(X \geq 1) > 0.99$$

$$\Rightarrow \quad 1 - P(X = 0) > 0.99$$

$$\Rightarrow \quad 1 - \frac{1}{4^n} > 0.99 \Rightarrow \frac{1}{4^n} < 0.01 \Rightarrow 4^n > \frac{1}{0.01} \Rightarrow 4^n > 100$$

The least value of n satisfying this inequality is 4. Hnece, the shooter must fire at least 4 times.

46. Suppose the man tosses a fair coin n times and X denote the number of heads in n

tosses. Then,

$$P(X-r) = {}^nC_r \left(\frac{1}{2}\right)^n \qquad \left[\because p = q = \frac{1}{2}\right]$$

It is given that

$$P(X \geq 1) > 0.9$$

$\Rightarrow \quad 1 - P(X = 0) > 0.9$

$\Rightarrow \quad 1 - {}^nC_0 \left(\frac{1}{2}\right)^n > 0.9 \Rightarrow \left(\frac{1}{2}\right)^n < \frac{1}{10} \Rightarrow 2^n > 10 \Rightarrow n = 4, 5, 6, \ldots$

Hence, the man must toss the coin at least 4 times.

32.3 MEAN AND VARIANCE OF BINOMIAL DISTRIBUTION

Let X be a binomial variate with parameters n and p.

$$P(X=r) = {}^nC_r\, p^r q^{n-r}, \; r = 0, 1, 2, \ldots, n.$$

$\therefore \quad \text{Mean} = \sum_{r=0}^{n} r \,.\, P(X=r) \qquad [\because \text{Mean} = \Sigma\, p_i x_i]$

$\Rightarrow \quad \text{Mean} = \sum_{r=0}^{n} r \,.\, {}^nC_r\, p^r q^{n-r}$

$\Rightarrow \quad \text{Mean} = \sum_{r=0}^{n} r \,.\, \frac{n}{r}\, {}^{n-1}C_{r-1}\, p \,.\, p^{r-1} q^{n-r} \qquad \left[\because {}^nC_r = \frac{n}{r}\, {}^{n-1}C_{r-1}\right]$

$\Rightarrow \quad \text{Mean} = np \sum_{r=0}^{n} {}^{n-1}C_{r-1}\, p^{r-1} q^{(n-1)-(r-1)}$

$\Rightarrow \quad \text{Mean} = np\,(q+p)^{n-1}$

$\Rightarrow \quad \text{Mean} = np \qquad [\because p + q = 1]$

and, $\quad \text{Variance} = \left(\sum_{r=0}^{n} r^2 P(X=r)\right) - (\text{Mean})^2 \qquad [\because \text{Var.} = (\Sigma\, p_i x_i^2) - (\text{Mean})^2]$

$\Rightarrow \quad \text{Variance} = \sum_{r=0}^{n} [r(r-1)+r]\, {}^nC_r\, p^r q^{n-r} - (np)^2 \qquad \left[\begin{array}{l}\because r^2 = r(r-1) + r \\ \text{and Mean} = np\end{array}\right]$

$\Rightarrow \quad \text{Variance} = \sum_{r=0}^{n} r(r-1)\, {}^nC_r\, p^r q^{n-r} + \sum_{r=0}^{n} r \,.\, {}^nC_r\, p^r q^{n-r} - (np)^2$

$\Rightarrow \quad \text{Variance} = \sum_{r=0}^{n} r(r-1) \frac{n}{r} \,.\, \frac{n-1}{r-1}\, {}^{n-2}C_{r-2}\, p^2 \,.\, p^{r-2} q^{n-r} + np - (np)^2$

$\Rightarrow \quad \text{Variance} = n(n-1)\, p^2 \left(\sum_{r=0}^{n} {}^{n-2}C_{r-2}\, p^{r-2} q^{n-r}\right) + np - n^2p^2$

$\Rightarrow \quad \text{Variance} = n(n-1)\, p^2 (q+p)^{n-2} + np - n^2p^2$

$\Rightarrow \quad \text{Variance} = n^2p^2 - np^2 + np - n^2p^2 \qquad [\because q + p = 1]$

$\Rightarrow \quad \text{Variance} = np - np^2 = np\,(1-p) = npq.$

Hence, mean and variance of a binomial variate with parameters n and p are np and npq respectively.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Prove that the mean of a binomial distribution is always greater than the variance.*

SOLUTION Let X be a binomial variate with parameters n and p. Then,

$$\text{Mean} = np, \text{ and Variance} = npq$$

$$\therefore \quad \text{Mean} - \text{Variance} = np - npq = np(1-q) = np^2$$

$$\Rightarrow \quad \text{Mean} - \text{Variance} > 0 \qquad [\because n \in N \text{ and } p > 0 \therefore np^2 > 0]$$

$$\Rightarrow \quad \text{Mean} > \text{Variance}.$$

EXAMPLE 2 *A die is thrown 20 times. Getting a number greater than 4 is considered a success. Find the mean and variance of the number of successes.*

SOLUTION Clearly, the distribution of the 'number of successes' is a binomial distribution with $n = 20$ and,

$$p = \text{Probability of getting a number greater than } 4 = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \quad q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

Now, $\quad \text{Mean} = np$ and Variance $= npq$

$$\Rightarrow \quad \text{Mean} = 20 \times \frac{1}{3} = 6.66 \text{ and Variance} = 20 \times \frac{1}{3} \times \frac{2}{3} = 4.44$$

Hence, $\quad$ Mean $= 6.66$ and Variance $= 4.44$.

EXAMPLE 3 *The mean and variance of a binomial distribution are 4 and 4/3 respectively, find $P(X \geq 1)$.* **[CBSE 2004, 2005]**

SOLUTION Let X be a binomial variate with parameters n and p. Then,

$$\text{Mean} = np \text{ and Variance} = npq$$

$$\Rightarrow \quad np = 4 \text{ and } npq = \frac{4}{3} \qquad \left[\because \text{Mean} = 4, \text{Var}(X) = \frac{4}{3} \text{ (Given)}\right]$$

$$\Rightarrow \quad \frac{npq}{np} = \frac{\frac{4}{3}}{4} \Rightarrow q = \frac{1}{3} \Rightarrow p = 1 - \frac{1}{3} = \frac{2}{3} \qquad [\because p = 1 - q]$$

Putting $p = 2/3$ in $np = 4$, we get

$$n \times \frac{2}{3} = 4 \Rightarrow n = 6.$$

Thus, we have

$$n = 6, p = \frac{2}{3} \text{ and } q = \frac{1}{3}$$

$$\therefore \quad P(X = r) = {}^nC_r p^r q^{n-r} \Rightarrow P(X = r) = {}^6C_r \left(\frac{2}{3}\right)^r \left(\frac{1}{3}\right)^{6-r}, r = 0, 1, 2, \ldots, 6.$$

Now, $\quad P(X \geq 1) = 1 - P(X < 1)$

$$\Rightarrow \quad P(X \geq 1) = 1 - P(X = 0)$$

$$\Rightarrow \quad P(X \geq 1) = 1 - {}^6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^6 = 1 - \left(\frac{1}{3}\right)^6 = 1 - \frac{1}{729} = \frac{728}{729}$$

EXAMPLE 4 *A die is tossed thrice. Getting an even number is considered as success. What is the variance of the binomial distribution?*

SOLUTION Here, $n = 3$

p = Probability of getting an even number in a single throw of a die

$\Rightarrow \quad p = \frac{3}{6} = \frac{1}{2}$

and, $\quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore \quad \text{Variance} = npq = 3 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$

EXAMPLE 5 *Find the binomial distribution for which the mean is 4 and variance 3.*

SOLUTION Let X be a binomial variate with parameters n and p. Then,

Mean = 4, Variance = 3

$\Rightarrow \quad np = 4 \text{ and } npq = 3 \Rightarrow \frac{npq}{np} = \frac{3}{4} \Rightarrow q = \frac{3}{4}$

$\therefore \quad p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$

Putting $p = \frac{1}{4}$ in $np = 4$, we get : $n = 16$

Then, we have

$n = 16, p = \frac{1}{4}$ and $q = \frac{3}{4}$

Hence, the binomial distribution is given by

$$P(X = r) = {}^{16}C_r \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{16-r}, r = 0, 1, 2, ..., 16.$$

EXAMPLE 6 *If the sum of the mean and variance of a binomial distribution for 5 trials is 1.8, find the distribution.* **[CBSE 2004]**

SOLUTION Let n and p be the parameters of the distribution. Then,

Mean = np and Variance = npq

It is given that

$n = 5$ and, Mean + Variance = 1.8

$\Rightarrow \quad np + npq = 1.8$

$\Rightarrow \quad 5p + 5pq = 1.8$

$\Rightarrow \quad 5p + 5p(1 - p) = 1.8$

$\Rightarrow \quad 5p^2 - 10p + 1.8 = 0$

$\Rightarrow \quad p^2 - 2p + 0.36 = 0$

$\Rightarrow \quad (p - 0.2)(p - 1.8) = 0 \Rightarrow p = 0.2 \qquad [\because p \ngtr 1]$

Thus, we have

$n = 5, p = 0.2$ and $q = 0.8$.

Therefore, if X denotes the binomial variate, then

$$P(X = r) = {}^{5}C_r (0.2)^r (0.8)^{5-r}, r = 0, 1, 2, 3, 4, 5.$$

This is the required binomial distribution.

EXAMPLE 7 *The sum and the product of the mean and variance of a binomial distribution are 24 and 128 respectively. Find the distribution.*

SOLUTION Let n and p be the parameters of the given binomial distribution. Then,

Mean $= np$ and Varaince $= npq$, where $q = 1 - p$

Now, Mean + Variance = 24 and, Mean × Variance = 128

$$\Rightarrow \quad np + npq = 24 \text{ and } np \times npq = 128$$

$$\Rightarrow \quad np(1+q) = 24 \text{ and } n^2p^2 \times q = 128$$

$$\Rightarrow \quad np = \frac{24}{1+q} \text{ and } n^2p^2 = \frac{128}{q}$$

$$\Rightarrow \quad \left(\frac{24}{1+q}\right)^2 = \frac{128}{q}$$

$$\Rightarrow \quad 576q = 128(1+q)^2$$

$$\Rightarrow \quad 9q = 2(1+2q+q^2)$$

$$\Rightarrow \quad 2q^2 - 5q + 2 = 0$$

$$\Rightarrow \quad (2q-1)(q-2) = 0$$

$$\Rightarrow \quad q = \frac{1}{2} \qquad [\because q \neq 2]$$

$$\therefore \quad p = 1 - q = \frac{1}{2}$$

Putting $p = \frac{1}{2}, q = \frac{1}{2}$ in $np + npq = 24$, we get

$$\frac{n}{2} + \frac{n}{4} = 24 \Rightarrow n = 32$$

Let X be the binomial variate. Then, the probability distribution of X is given by

$$P(X = r) = {}^{32}C_r \left(\frac{1}{2}\right)^{32-r} \left(\frac{1}{2}\right)^r; r = 0, 1, 2, ..., 32.$$

EXAMPLE 8 *The sum of mean and variance of a binomial distribution is 15 and the sum of their squares is 117. Determine the distribution.*

SOLUTION Let n and p be the parameters of the distribution. Then,

Mean $= np$ and Variance $= npq$.

$$\text{Now,} \quad np + npq = 15 \qquad \text{[Given]}$$

$$\text{and,} \quad n^2p^2 + n^2p^2q^2 = 117 \qquad \text{[Given]}$$

$$\Rightarrow \quad np(1+q) = 15 \text{ and } n^2p^2(1+q^2) = 117$$

$$\Rightarrow \quad n^2p^2(1+q)^2 = 225 \text{ and } n^2p^2(1+q^2) = 117$$

$$\Rightarrow \quad \frac{n^2p^2(1+q)^2}{n^2p^2(1+q^2)} = \frac{225}{117} \Rightarrow \frac{(1+q)^2}{(1+q^2)} = \frac{225}{117}$$

$$\Rightarrow \quad \frac{1+q^2+2q}{1+q^2} = \frac{225}{117} \Rightarrow 1 + \frac{2q}{1+q^2} = \frac{225}{117}$$

$$\Rightarrow \quad \frac{2q}{1+q^2} = \frac{12}{13} \Rightarrow \frac{2q}{1+q^2} \frac{108}{117}$$

$$\Rightarrow \quad \frac{1+q^2}{2q} = \frac{13}{12}$$

$$\Rightarrow \quad \frac{1+q^2+2q}{1+q^2-2q} = \frac{13+12}{13-12} \qquad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \quad \left(\frac{1+q}{1-q}\right)^2 = 25 \Rightarrow \frac{1+q}{1-q} = 5$$

$$\Rightarrow \quad 6q = 4 \Rightarrow q = \frac{2}{3} \Rightarrow p = 1-q = 1-\frac{2}{3} = \frac{1}{3}$$

Putting $p = \frac{1}{3}, q = \frac{2}{3}$ in $np + npq = 15$, we get

$$\frac{n}{3} + \frac{2n}{9} = 15 \Rightarrow \frac{5n}{9} = 15 \Rightarrow n = 27$$

Thus, $n = 27,\ p = \frac{1}{3}$ and $q = \frac{2}{3}$

Hence, the distribution is given by

$$P(X=r) = {}^{27}C_r\left(\frac{1}{3}\right)^r\left(\frac{2}{3}\right)^{27-r} \quad \ldots \quad 27.$$

EXAMPLE 9 *If the probability of* … *mean and standard deviation for the distribution of defective bolts i* …

SOLUTION We have, $n = 500$ …

$\therefore$ Mean $= np = 50$ …

And, $S.D. = \sqrt{\text{Variance}} =$ … $= 6.71$

EXAMPLE 10 *If X follows binomial distribution with mean 4 and variance 2, find* $P(|X-4| \leq 2)$.

SOLUTION Let n and p be the parameters of the binomial distribution. Then,

Mean = 4 and, Variance = 2

$$\Rightarrow \quad np = 4 \text{ and } npq = 2$$

$$\Rightarrow \quad \frac{npq}{np} = \frac{2}{4} \Rightarrow q = \frac{1}{2} \Rightarrow p = \frac{1}{2}$$

Putting $p = 1/2$ in $np = 4$, we get : $n = 8$.

Thus, X is a binomial variate with parameters $n = 8$ and $p = \frac{1}{2}$

$$\therefore \quad P(X=r) = {}^8C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{8-r} = {}^8C_r\left(\frac{1}{2}\right)^8, r = 0, 1, 2, \ldots, 8.$$

Now,

$$P(|X-4| \leq 2) = P(-2 \leq X-4 \leq 2)$$

$$\Rightarrow \quad P(|X-4| \leq 2) = P(2 \leq X \leq 6)$$

$$\Rightarrow \quad P(|X-4| \leq 2) = P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$$

$$\Rightarrow \quad P(|X-4| \leq 2) = {}^8C_2\left(\frac{1}{2}\right)^8 + {}^8C_3\left(\frac{1}{2}\right)^8 + {}^8C_4\left(\frac{1}{2}\right)^8 + {}^8C_5\left(\frac{1}{2}\right)^8 + {}^8C_6\left(\frac{1}{2}\right)^8$$

$$\Rightarrow \quad P(|X-4| \leq 2) = \left({}^8C_2 + {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6\right)\left(\frac{1}{2}\right)^8 = \frac{119}{128}$$

EXAMPLE 11 *A perfect cubic die is thrown a large number of times in sets of 8. The occurrence of 5 or 6 is called a success. In what proportion of the sets would you expect 3 successes.*

SOLUTION Let there be n sets of 8 dice.

We have, $p =$ Probability of getting 5 or a 6 with six faced die $= \frac{2}{6} = \frac{1}{3}$

$$\therefore \quad q = 1 - \frac{1}{3} = \frac{2}{3}$$

Let X denote the number of successes in one set of 8 dice. Then,

$$P(X = r) = {}^8C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{8-r}, r = 0, 1, 2, ..., 8.$$

$$\therefore \quad P(X = 3) = {}^8C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^5 = \frac{1792}{6561}$$

Total number of sets in which we get 3 successes $= N \,.\, P(X = 3) = \frac{1792}{6561} N$

So, percentage of 3 successes in 100 sets $= \frac{1792}{6561} N \times \frac{1}{N} \times 100 = 27.31\%$.

EXAMPLE 12 *Find the expectation of the number of heads in 15 tosses of a coin.*

SOLUTION Let p be the probability of getting a head in a single toss. Then, $p = \frac{1}{2}$.

Clearly, the distribution of the number of heads is a binomial distribution with $n = 15, p = \frac{1}{2}$.

$$\therefore \quad \text{Expectation} = E(X) = np = 15 \times \frac{1}{2} = 7.5$$

EXAMPLE 13 *In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/loses.*

SOLUTION Let n denote the number of throws required to get a six and X denote the amount won/lost.

The man may get a six in the very first throw of the die or in 2nd throw or in the third row (as he has decided to throw a die at most thrice).

Thus, we have the following probability distribution for X.

Number of throws (n) :	1	2	3
Amount won/lost (X):	1	0	-1
Probability ($P(X)$) :	$\frac{1}{6}$	$\frac{5}{6} \times \frac{1}{6}$	$\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

$$\therefore \quad E(X) = 1 \times \frac{1}{6} + 0 \times \frac{5}{36} + (-1) \times \frac{25}{216} = \frac{11}{216}$$

EXAMPLE 14 *If two dice are rolled 12 times, obtain the mean and the variance of the distribution of successes, if getting a total greater than 4 is considered a success.* **[CBSE 2002C]**

SOLUTION Let X denote the number of successes in 12 trials. Then, X follows biomial distribution with parameters $n = 12$ and p.

We have,

$p =$ Probability of getting a total greater than 4 in a single thrawof a pair of dice.

$\Rightarrow \quad p = 1 - \frac{6}{36} = \frac{5}{6}$

$\Rightarrow \quad q = 1 - p = 1 - \frac{5}{6} = \frac{1}{6}$

$\therefore$ Mean $= np = \frac{5}{6} \times 12 = 10$ and, Variance $= npq = 12 \times \frac{5}{6} \times \frac{1}{6} = \frac{5}{3}$.

EXERCISE 32.2

1. Can the mean of a binomial distribution be less than its variance ?
2. Determine the binomial distribution whose mean is 9 and variance 9/4.
3. If the mean and variance of a binomial distribution are respectively 9 and 6, find the distribution.
4. Find the binomial distribution when the sum of its mean and variance for 5 trials is 4.8.
5. Determine the binomial distribution whose mean is 20 and variance 16.
6. In a binomial distribution the sum and product of the mean and the variance are $\frac{25}{3}$ and $\frac{50}{3}$ respectively. Find the distribution.
7. The mean of a binomial distribution is 20, and the standard deviation 4. Calculate parameters of the binomial distribution.
8. If the probability of a defective bolt is 0.1, find the (i) mean, (ii) standard deviation, for the distribution of bolts in a total of 400 bolts.
9. Find the binomial distribution whose mean is 5 and variance $\frac{10}{3}$.
10. If on an average 9 ships out of 10 arrive safely to ports, find the mean and S.D. of ships returning safely out of a total of 500 ships.
11. The mean and variance of a binomial variate with parameters n and p are 16 and 8 respectively. Find $P(X = 0)$, $P(X = 1)$ and $P(X \geq 2)$.
12. In eight throws of a die 5 or 6 is considered a success, find the mean number of successes and the standard deviation.
13. Find the expected number of boys in a family with 8 children, assuming the sex distribution to be equally probable.
14. The probability is 0.02 that an item produced by a factory is defective. A shipment of 10,000 items is sent to its warehouse. Find the expected number of defective items and the standard deviation.
15. A dice is thrown thrice. A success is 1 or 6 in a throw. Find the mean and variance of the number of successes.
16. If a random variable X follows binomial distribution with mean 3 and variance $\frac{3}{2}$, find $P(X \leq 5)$. [CBSE 2002]
17. If X follows binomial distribution with mean 4 and variance 2 find $P(X \geq 5)$. [CBSE 2001C]
18. The mean and variance of a binomial distribution are $\frac{4}{3}$ and $\frac{8}{9}$ respectively. Find $P(X \geq 1)$. [CBSE 2004]
19. If the sum of the mean and variance of a binomial distribution for 6 trials is $\frac{10}{3}$, find the distribution. [CBSE 2004]

ANSWERS

2. $P(X = r) = {}^{12}C_r \left(\frac{3}{4}\right)^r \left(\frac{1}{4}\right)^{12-r}, r = 0, 1, 2, \ldots, 12$

3. $P(X=r) = {}^{27}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{27-r}, r = 0, 1, 2, ..., 27$

4. $P(X=r) = {}^{5}C_r \left(\frac{4}{5}\right)^r \left(\frac{1}{5}\right)^{5-r}, r = 0, 1, 2, ..., 5$

5. $P(X=r) = {}^{100}C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{100-r}; r = 0, 1, 2, \ldots, 100$

6. $P(X=r) = {}^{15}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{15-r}, r = 0, 1, 2, ..., 15$

7. $n = 100, p = \frac{1}{5}$ 8. (i) 40 (ii) 6 9. $P(X = r) = {}^{15}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{15-r}$

10. Mean = 450, S.D. = 6.71 11. $\left(\frac{1}{2}\right)^{32}, \left(\frac{1}{2}\right)^{27}, 1 - \frac{33}{2^{32}}$ 12. 2.66, 1.33

13. 4 14. 200, 14 15. Mean = 1, Variance = 2/3 16. $\frac{63}{64}$

17. $\frac{93}{256}$ 18. $\frac{65}{81}$ 19. $P(X=r) = {}^{6}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{6-r}$ $r = 0, 1, 2, ..., 6$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. In a binomial distribution, if $n = 20$, $q = 0.75$, then write its mean.
2. If in a binomial distribution mean is 5 and variance is 4, write the number of trials.
3. In a group of 200 items, if the probability of getting a defective item is 0.2, write the mean of the distribution.
4. If the mean of a binomial distribution is 20 and its standard deviation is 4, find p.
5. The mean of a binomial distribution is 10 and its standard deviation is 2, write the value of q.
6. If the mean and variance of a random variable X having a binomial distribution are 4 and 2 respectively, find $P(X = 1)$.
7. If the mean and variance of a binomial variate X are 2 and 1 respectively, find $P(X > 1)$.
8. If in a binomial distribution $n = 4$ and $P(X = 0) = \frac{16}{81}$, find q.
9. If the mean and variance of a binomial distribution are 4 and 3 respectively, find the probability of no-success.
10. If for a binomial distribution $P(X = 1) = P(X = 2) = \alpha$, write $P(X = 4)$ in terms of α.

ANSWERS

1. 5 2. 25 3. 40 4. $\frac{1}{5}$ 5. 0.4 6. $\frac{1}{32}$

7. $\frac{15}{16}$ 9. $\left(\frac{3}{4}\right)^{16}$ 10. $\frac{\alpha}{3}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. In a box containing 100 bulbs, 10 are defective. What is the probability that out of a sample of 5 bulbs, none is defective,
(a) $\left(\frac{9}{10}\right)^5$ (b) $\frac{9}{10}$ (c) 10^{-5} (d) $\left(\frac{1}{2}\right)^2$

2. If in a binomial distribution $n = 4$, $P(X=0) = \frac{16}{81}$, then $P(X=4)$ equals
(a) $\frac{1}{16}$ (b) $\frac{1}{81}$ (c) $\frac{1}{27}$ (d) $\frac{1}{8}$

3. A rifleman is firing at a distant target and has only 10% chance of hitting it. The least number of rounds, he must fire in order to have more than 50% chance of hitting it at least once is
(a) 11 (b) 9 (c) 7 (d) 5

4. A fair coin is tossed a fixed number of times. If the probability of getting seven heads is equal to that of getting nine heads, the probability of getting two heads is
(a) $15/2^8$ (b) $2/15$ (c) $15/2^{13}$ (d) none of these

5. A fair coin is tossed 100 times. The probability of getting tails an odd number of times is
(a) 1/2 (b) 1/8 (c) 3/8 (d) none of these

6. A fair die is thrown twenty times. The probability that on the tenth throw the fourth six appears is
(a) $\frac{{}^{20}C_{10} \times 5^6}{6^{20}}$ (b) $\frac{120 \times 5^7}{6^{10}}$
(c) $\frac{84 \times 5^6}{6^{10}}$ (d) none of these

7. If X is a binomial variate with parameters n *and* p, where $0 < p < 1$ such that $\frac{P(X=r)}{P(X=n-r)}$ is independent of n *and* r, then p equals
(a) 1/2 (b) 1/3 (c) 1/4 (d) none of these

8. Let X denote the number of times heads occur in n tosses of a fair coin. If $P(X=4)$, $P(X=5)$ *and* $P(X=6)$ are in AP; the value of n is
(a) 7, 14 (b) 10, 14 (c) 12, 7 (d) 14, 12

9. One hundred identical coins, each with probability p of showing heads are tossed once. If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of heads showing on 51 coins, the value of p is
(a) 1/2 (b) 51/101 (c) 49/101 (d) none of these

10. A fair coin is tossed 99 times. If X is the number of times heads occur, then $P(X=r)$ is maximum when r is
(a) 49, 50 (b) 50, 51 (c) 51, 52 (d) none of these

11. The least number of times a fair coin must be tossed so that the probability of getting at least one head is at least 0.8, is
(a) 7 (b) 6 (c) 5 (d) 3

12. If the mean and variance of a binomial variate X are 2 and 1 respectively, then the probability that X takes a value greater than 1 is
(a) 2/3 (b) 4/5 (c) 7/8 (d) 15/16

13. A biased coin with probability $p, 0 < p < 1$, of heads is tossed until a head appears for the first time. If the probability that the number of tosses required is even is $2/5$, then p equals
(a) $1/3$ (b) $2/3$ (c) $2/5$ (d) $3/5$

14. If X follows a binomial distribution with parameters $n = 8$ *and* $p = 1/2$, then $P(|X-4| \leq 2)$ equals
(a) $\frac{118}{128}$ (b) $\frac{119}{128}$ (c) $\frac{117}{128}$ (d) none of these

15. If X follows a binomial distribution with parameters $n = 100$ *and* $p = 1/3$, then $P(X = r)$ is maximum when $r =$
(a) 32 (b) 34 (c) 33 (d) 31

16. A fair die is tossed eight times. The probability that a third six is observed in the eight throw is
(a) $\frac{{}^7C_2 \times 5^5}{6^7}$ (b) $\frac{{}^7C_2 \times 5^5}{6^8}$ (c) $\frac{{}^7C_2 \times 5^5}{6^6}$ (d) none of these

17. Fifteen coupons are numbered 1 to 15. Seven coupons are selected at random, one at a time with replacement. The probability that the largest number appearing on a selected coupon is 9, is
(a) $\left(\frac{3}{5}\right)^7$ (b) $\left(\frac{1}{15}\right)^7$ (c) $\left(\frac{8}{15}\right)^7$ (d) none of these

18. A five-digit number is written down at random. The probability that the number is divisible by 5 and no two consecutive digits are identical, is
(a) $\frac{1}{5}$ (b) $\frac{1}{5}\left(\frac{9}{10}\right)^3$ (c) $\left(\frac{3}{5}\right)^4$ (d) none of these

19. A coin is tossed 10 times. The probability of getting exactly six heads is
(a) $\frac{512}{513}$ (b) $\frac{105}{512}$ (c) $\frac{100}{153}$ (d) ${}^{10}C_6$ (e) ${}^{10}C_6 \times 6!$

20. The mean and variance of a binomial distribution are 4 and 3 respectively, then the probability of getting exactly six successes in this distribution, is
(a) ${}^{16}C_6\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^6$ (b) ${}^{16}C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^{10}$
(c) ${}^{12}C_6\left(\frac{1}{20}\right)\left(\frac{3}{4}\right)^6$ (d) ${}^{12}C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^6$

21. In a binomial distribution, the probability of getting success is $1/4$ and standard deviation is 3. Then, its mean is
(a) 6 (b) 8 (c) 12 (d) 10

22. A coin is tossed 4 times. The probability that at least one head turns up, is
(a) $\frac{1}{16}$ (b) $\frac{2}{16}$ (c) $\frac{14}{16}$ (d) $\frac{15}{16}$

23. For a binomial variate X, if $n = 3$ and $P(X = 1) = 8\,P(X = 3)$, then $p =$
(a) $4/5$ (b) $1/5$ (c) $1/3$ (d) $2/3$

24. A coin is tosed n times. The probability of getting at least once is greater than 0.8. Then, the least value of n, is
(a) 2 (b) 3 (c) 4 (d) 5

25. The probability of selecting a male or a female is same. If the probability that in an office of n persons $(n-1)$ males being selected is $\frac{3}{2^{10}}$, the value of n is

(a) 5 (b) 3 (c) 10 (d) 12

ANSWERS

1. (a)	2. (b)	3. (c)	4. (c)	5. (a)	6. (c)	7. (a)	8. (a)
9. (b)	10. (a)	11. (d)	12. (d)	13. (a)	14. (b)	15. (c)	16. (b)
17. (d)	18. (c)	19. (b)	20. (b)	21. (c)	22. (d)	23. (d)	24. (b)
25. (d)							

SUMMARY

1. A random variable X which takes values 0, 1, 2, ..., n is said to follow binomial distribution if its probability distribution function is given by

$$P(X=r) = {}^nC_r\, p^r\, q^{n-r},\ r = 0, 1, 2, \ldots, n,$$

where $p, q > 0$ such that $p + q = 1$.

The two constants n and p in the distribution are known as the parameters of the distribution.

The notation $X \sim B(n, p)$ is generally used to denote that the random variable X follows binomial distribution with parameters n and p.

We have,

$$P(X=0) + P(X=1) + \ldots + P(X=n)$$

$$= {}^nC_0\, p^0\, q^{n-0} + {}^nC_1\, p^1\, q^{n-1} + \ldots + {}^nC_n\, p^n\, q^{n-n}$$

$$= (q+p)^n = 1^n = 1$$

Thus, the assignment of probabilities to the random variable X is permissible.

2. If n trials constitute an experiment and the experiment is repeated N times, then the frequencies of 0, 1, 2, ..., n successes are given by

$$N \cdot P(X=0),\ N \cdot P(X=1),\ N \cdot P(X=2),\ \ldots,\ N \cdot P(X=n).$$

3. The mean and variance of a binomial variate with parameters n and p are np and npq respectively.

4. If $(n+1)p$ is not an integer:

$$P(X = 0),\ P(X = 1),\ \ldots\ldots,\ P(X = n).$$

(i) Then, $P(X = r)$ is maximum when $r = m = [(n+1)p]$.

(ii) If $(n+1)p$ is an integer, then
$P(X = r)$ is maximum when $r = m - 1$ or $r = m$, where $m = (n+1)p$ is an integer.

APPENDIX

CURVE SKETCHING

A.1 INTRODUCTION

In this chapter, we shall use the results of differential calculus to find an approximate shape of the curves from their equations without plotting a large number of points. For the said task we shall make use of various concepts of differential calculus such as continuity, monotonicity, maxima and minima, points of inflexion etc.

A.2 CURVE SKETCHING

Following points are very helpful to draw a rough sketch of a curve.

I SYMMETRY

(i) *Symmetry about x-axis* : If all powers of y in the equation of the given curve are even, then it is symmetric about x-axis i.e., the shape of the curve above x-axis is exactly identical to its shape below x-axis.

For example, $y^2 = 4\,ax$ is symmetric about x-axis.

(ii) *Symmetry about y-axis* : If all powers of x in the equation of the given curve are even, then it is symmetric about y-axis.

For example, $x^2 = 4ay$ is symmetric about y-axis.

(iii) *Symmetry in opposite quadrants* : If by putting $-x$ for x and $-y$ for y, the equation of a curve remains same, then it is symmetric in opposite quadrants.

For example, $x^2 + y^2 = a^2$ and $xy = a^2$ are symmetric in opposite quadrants.

(iv) *Symmetry about the line* $y = x$: If the equation of a given curve remains unaltered by interchanging x and y, then it is symmetric about the line $y = x$ which passes through the origin and makes an angle of 45° with the positive direction of x-axis.

II ORIGIN AND TANGENTS AT THE ORIGIN

See whether the curve passes through the origin or not. If the point (0, 0) satisfies the equation of the curve, then it passes through the origin and in such a case to find the equation(s) of the tangent(s) at the origin, equate the lowest degree term to zero.

For example, $y^2 = 4ax$ passes through the origin. The lowest degree term in this equation is $4ax$. Equating $4ax$ to zero, we get $x = 0$. So, $x = 0$ i.e. y-axis is tangent at the origin to $y^2 = 4ax$.

III POINTS OF INTERSECTION OF THE CURVE WITH THE COORDINATE AXES

By putting $y = 0$ in the equation of the given curve, find points where the curve crosses the x-axis. Similarly, by putting $x = 0$ in the equation of the given curve we can find points where the curve crosses the y-axis.

For example, to find points where the curve $xy^2 = 4a^2(2a - x)$ meets x-axis, we put $y = 0$ in the equation which gives $4a^2(2a - x) = 0$ or, $x = 2a$. So, the curve

$xy^2 = 4a^2(2a - x)$, meets x-axis at $(2a, 0)$. This curve does not intersect y-axis, because by putting $x = 0$ in the equation of the given curve get an absurd result.

IV REGIONS WHERE THE CURVE DOES NOT EXIST

Determine the regions in which the curve does not exist. For this, find the value of y in terms of x from the equation of the curve and find the values of x for which y is imaginary. Similarly, find the value of x in terms of y and determine the values of y for which x is imaginary. The curve does not exist for these values of x and y.

For example, the values of y obtained from $y^2 = 4ax$ are imaginary for negative values of x. So, the curve does not exist on the left side of y-axis. Similarly, the curve $a^2 y^2 = x^2 (a - x)$ does not exist for $x > a$ as the values of y are imaginary for $x > a$.

V SPECIAL POINTS

Find the points at which $\frac{dy}{dx} = 0$. At these points the tangent to the curve is parallel to x-axis.

Find the points at which $\frac{dx}{dy} = 0$. At these points the tangent to the curve is parallel to y-axis.

VI SIGN OF $\frac{dy}{dx}$, AND POINTS OF MAXIMA AND MINIMA

Find the interval in which $\frac{dy}{dx} > 0$. In this interval, the function is monotonically increasing.

Find the interval in which $\frac{dy}{dx} < 0$. In this interval, the function is monotonically decreasing.

Put $\frac{dy}{dx} = 0$ and check the sign of $\frac{d^2 y}{dx^2}$ at the points so obtained to find the points of maxima or minima.

Keeping the above facts in mind and plotting some points on the curve one can easily have a rough sketch of the curve.
Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Sketch the curve* $y = x^3$

SOLUTION We observe the following points about the given curve.

(i) The equation of the curve remains unchanged if x is replaced by $-x$ and y by $-y$, so it is symmetric in opposite quadrants. Conse-quently the shape of the curve is similar in the first and the third quadrants.

(ii) The curve passes through the origin. Equating the lowest degree term y to zero, we get $y = 0$ i.e. x-axis is the tangent at the origin.

(iii) Putting $y = 0$ in the equation of the curve, we get $x = 0$. Similarly, when $x = 0$, we get $y = 0$. So, the curve meets the coordinate axes at $(0, 0)$ only.

(iv) $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$, $\frac{d^2y}{dx^2} = 6x$ and $\frac{d^3 y}{dx^3} = 6$.

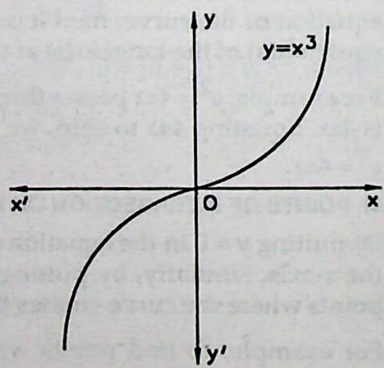

Fig. A.1

Clearly, $\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$ at the origin but $\frac{d^3y}{dx^3} = 6 \neq 0$. So, origin is a point of inflexion.

(v) As x increases from 0 to ∞, y also increases from 0 to ∞. Keeping all the above points in mind, we obtain a sketch of the curve as shown in Fig. A.1.

EXAMPLE 2 *Sketch the curve $y = x^3 - 4x$.*

SOLUTION We make the following observations about the given curve.

(i) The equation of the curve remains same if x is replaced by $-x$ and y by $-y$, so it is symmetric in opposite quadrants. Consequently, the curve in the first quadrant is identical to the curve in third quadrant and the curve in second quadrant is similar to the curve in fourth quadrant.

(ii) The curve passes through the origin. Equating the lowest degree term $y + 4x$ to zero, we get $y + 4x = 0$ or $y = -4x$. So, $y = -4x$ is tangent to the curve at the origin.

(iii) Putting $y = 0$ in the equation of the curve, we obtain $x^3 - 4x = 0 \Rightarrow x = 0, = \pm 2$. So, the curve meets x-axis at $(0, 0)$ $(2, 0)$ and $(-2, 0)$.
Putting $x = 0$ in the equation of the curve, we get $y = 0$. So, the curve meets y-axis at $(0, 0)$ only.

(iv) $y = x^3 - 4x \Rightarrow \frac{dy}{dx} = 3x^2 - 4$

Now, $\frac{dy}{dx} > 0$

$\Rightarrow$ $3x^2 - 4 > 0 \Rightarrow x^2 - \frac{4}{3} > 0$

$\Rightarrow$ $\left(x - \frac{2}{\sqrt{3}}\right)\left(x + \frac{2}{\sqrt{3}}\right) > 0$

$\Rightarrow$ $x < -\frac{2}{\sqrt{3}}$ or $x > \frac{2}{\sqrt{3}}$

and, $\frac{dy}{dx} < 0 \Rightarrow -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$

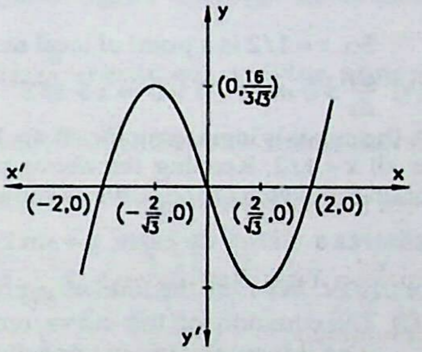

Fig. A.2

So, the curve is decreasing in the interval $(-2/\sqrt{3}, 2/\sqrt{3})$ and increasing for $x > 2/\sqrt{3}$ or $x < -2/\sqrt{3}$. $x = -2/\sqrt{3}$ is a point of local maximum and $x = 2/\sqrt{3}$, is a point of local minimum. When $x = 2/\sqrt{3}$, $y = 8/3\sqrt{3} - 8/\sqrt{3} = -16/3\sqrt{3}$. When $x = -2/\sqrt{3}$, $y = 16/3\sqrt{3}$. Keeping the above points in mind, we sketch the curve as shown in Fig. A.2.

EXAMPLE 3 *Sketch the curve $y = (x-1)(x-2)(x-3)$.*

SOLUTION We note the following points about the given curve.

(i) The curve does not have any type of symmetry about the coordinate axes and also in the opposite quadrants.

(ii) The curve does not pass through the origin.

(iii) Putting $y = 0$ in the equation of the curve, we get $(x-1)(x-2)(x-3) = 0 \Rightarrow x = 1, 2, 3$. So, the curve meets x-axis at $(1, 0)$, $(2, 0)$ and $(3, 0)$.

Putting $x = 0$ in the equation of the curve, we get $y = -6$. So, the curve crosses y-axis at $(0, -6)$.

We observe that $x < 1 \Rightarrow y < 0$

$1 < x < 2 \Rightarrow y > 0$,

$2 < x < 3 \Rightarrow y < 0$ and $x > 3 \Rightarrow y > 0$.

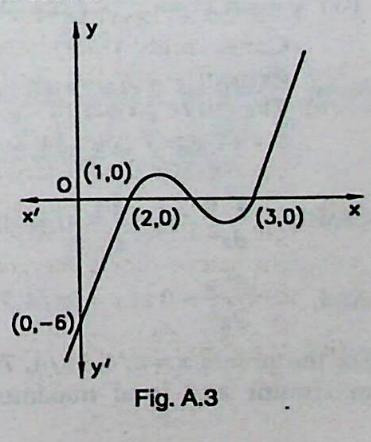

Fig. A.3

Clearly, y decreases as x decreases for all $x < 1$ and y increases as x increases for $x > 3$. Keeping all the above points in mind, we sketch the curve as shown in Fig. A.3.

EXAMPLE 4 *Sketch the curve* $y = x^2 - x$.

SOLUTION We note the following points about the curve.

(i) The curve does not have any kind of symmetry.

(ii) The curve passes through the origin and the tangent at the origin is obtained by equating the lowest degree term to zero. The lowest degree term is $x + y$. Equating it to zero, we get $x + y = 0$ as the equation of the tangent at the origin.

(iii) Putting $y = 0$ in the equation of the curve, we get $x^2 - x = 0 \Rightarrow x(x-1) = 0$ $\Rightarrow x = 0, 1$. So, the curve crosses x-axis at (0, 0) and (1, 0).
Putting $x = 0$, in the equation of the curve, we obtain $y = 0$. So, the curve meets y-axis at (0, 0) only.

(iv) $y = x^2 - x \Rightarrow \frac{dy}{dx} = 2x - 1$ and $\frac{d^2y}{dx^2} = 2$.

Now, $\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{2}$. At $x = \frac{1}{2}, \frac{d^2y}{dx^2} = 2 > 0$.

So, $x = 1/2$ is a point of local minima.

(v) $\frac{dy}{dx} > 0 \Rightarrow 2x - 1 > 0 \Rightarrow x > 1/2$

So, the curve is increasing for all $x > 1/2$ and decreasing for all $x < 1/2$. Keeping the above points in mind, we obtain the sketch of the curve as shown in Fig. A.4.

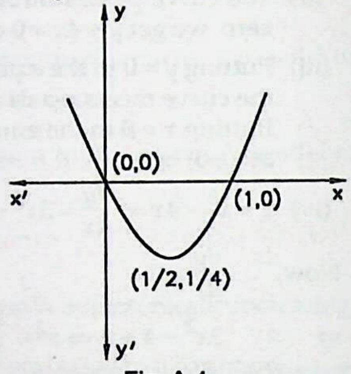

Fig. A.4

EXAMPLE 5 *Sketch the curve* $y = \sin 2x$.

SOLUTION We note the following points about the curve.

(i) The equation of the curve remains unchanged if x is replaced by $-x$ and by $-y$, so it is symmetric in opposite quadrants. Consequently, the shape of the curve is similar in opposite quadrants.

(ii) The curve passes through the origin.

(iii) Putting $x = 0$ in the equation of the curve, we get $y = 0$. So, the curve crosses the y-axis at (0, 0) only.
Putting $y = 0$ in the equation of the curve, we get $\sin 2x = 0 \Rightarrow 2x = n\pi, n \in Z$ $\Rightarrow x = n\pi/2, n \in Z$. So, the curve cuts x-axis at the points ..., $(-\pi, 0)$ $(-\pi/2, 0)$ $(0, 0)$, $(\pi/2, 0)$, $(\pi, 0)$, ...

(iv) $y = \sin 2x \Rightarrow \frac{dy}{dx} = 2\cos 2x$ and $\frac{d^2y}{dx^2} = -4\sin 2x$

Now, $\frac{dy}{dx} = 0 \Rightarrow 2\cos 2x = 0 \Rightarrow \cos 2x = 0 \Rightarrow 2x = \pm\pi/2, \pm 3\pi/2, \ldots$

$$\Rightarrow x = \pm\pi/4, \pm 3\pi/4, \ldots$$

Clearly, $\frac{d^2y}{dx^2} < 0$ at $x = \pi/4, 5\pi/4 \ldots$ and at $x = -3\pi/4, -7\pi/4, \ldots$

And, $\frac{d^2y}{dx^2} > 0$ at $x = 3\pi/4, 7\pi/4, \ldots$ and at $x = -\pi/4, -5\pi/4 \ldots$

So, the points $x = \pi/4, 5\pi/4, 7\pi/4$, and $x = -3\pi/4, -7\pi/4, \ldots$ are the points of local maximum and local maximum value at these points is 1. Similarly the points

$x = -\pi/4, -5\pi/4 \ldots$ and $x = 3\pi/4, 7\pi/4, \ldots$ are points of local minimum and local minimum value at these points is -1.

(v) $\sin 2(x+\pi) = \sin 2x$ for all x. So, the periodicity of the function is π. This means that the pattern of the curve repeats at intervals of length π.

Thus, keeping the above ideas in mind, we sketch the curve $y = \sin 2x$ as shown in Fig. A.5.

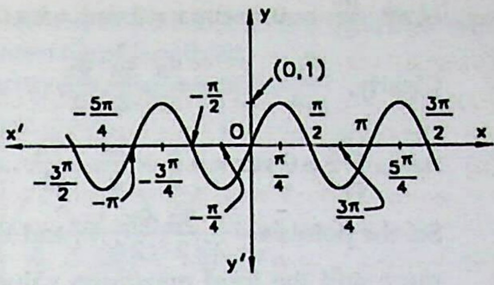

Fig. A.5

EXAMPLE 6 *Sketch the curve* $y = 2\sin 2x$.

SOLUTION We note the following points about the curve.

(i) The equation of the curve remains same if x is replaced by $-x$ and y by $-y$, so it is symmetric in opposite quadrants.

(ii) The curve passes through the origin.

(iii) The curve meets the coordinate axes at the same points where $y = \sin 2x$ meets them.

(iv) $\frac{dy}{dx} = 0$ gives the same points as given by $\frac{d}{dx}(\sin 2x) = 0$ and the signs of $\frac{d^2}{dx^2}(2\sin 2x)$ and $\frac{d^2}{dx^2}(\sin 2x)$ are same for any value of x. This means that the points of local maxima or minima of $y = 2\sin 2x$ are the same as the points of local maxima or minima of $y = \sin 2x$, but the maximum value will be 2 and minimum value will be -2.

Keeping the above ideas in mind, we obtain the rough sketch of the curve as shown in Fig. A.6.

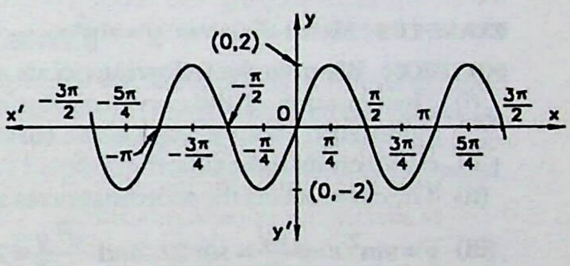

Fig. A.6

EXAMPLE 7 *Sketch the curve* $y = \sin x$.

SOLUTION We note the following points about the curve.

(i) The equation of the curve does not alter if x is replaced by $-x$ and y by $-y$. So, the curve is symmetric in opposite quadrants.

(ii) Point $(0, 0)$ satisfies the equation $y = \sin x$. So, the curve passes through the origin.

(iii) Putting $y = 0$ in the equation of the curve, we obtain $\sin x = 0 \Rightarrow x = n\pi, n \in Z$. So, the curve meets the x-axis at ... $(3\pi, 0), (-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0), \ldots$ Putting $x = 0$ in the equation of the curve, we obtain $y = \sin 0 = 0$. So, the curve meets y-axis at $(0, 0)$ only.

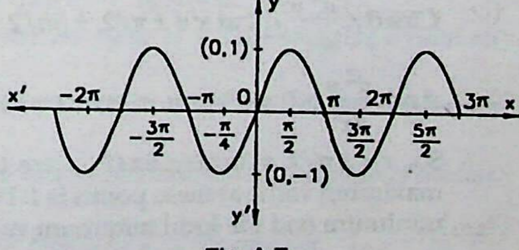

Fig. A.7

(iv) $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$ and $\frac{d^2y}{dx^2} = -\sin x$.

Now, $\frac{dy}{dx} = 0 \Rightarrow \cos x = 0 \Rightarrow x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$

Clearly, $\frac{d^2y}{dx^2} < 0$ at $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$ and at $x = -\frac{3\pi}{2}, -\frac{7\pi}{2}, \ldots$

And, $\frac{d^2y}{dx^2} > 0$ at $x = -\frac{\pi}{2}, -\frac{5\pi}{2}, -\frac{9\pi}{2}, \ldots$ and $x = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

So, the points $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$, and $x = -\frac{3\pi}{2}, -\frac{7\pi}{2}, \ldots$ are the points of local maximum and the local maximum value at these points is 1. Similarly, the points $x = -\frac{\pi}{2}, -\frac{5\pi}{2}, \ldots$, and $x = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$ are the points of local minimum and the local minimum value at these points is -1.

(v) $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$.

Clearly, $\frac{dy}{dx} > 0$ when $-\pi/2 < x < \pi/2$ and $\frac{dy}{dx} < 0$ when $\pi/2 < x < 3\pi/2$. So, the given curve is increasing for $-\pi/2 < x < \pi/2$ and decreasing for $\pi/2 < x < 3\pi/2$.

(vi) Sin $(2\pi + x) = \sin x$ for all x. So, the periodicity of the function is 2π. This means that the pattern of the curve repeats at the intervals of length 2π.

Keeping the above facts in mind, we may sketch the curve $y = \sin x$ as shown in Fig. A.7.

EXAMPLE 8 *Sketch the curve* $y = \sin^2 x$.

SOLUTION We note the following points about the curve.

(i) The equation of the curve remains same if x is replaced by $-x$. So, the curve is symmetric about y-axis i.e. the curve on the left side of y-axis is identical to the curve on its right side.

(ii) The curve meets the coordinate axes at the same points where $y = \sin x$ meets them.

(iii) $y = \sin^2 x \Rightarrow \frac{dy}{dx} = \sin 2x$ and $\frac{d^2y}{dx^2} = 2\cos 2x$

Now, $\frac{dy}{dx} = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = n\pi,\ n \in Z \Rightarrow x = n\pi/2, n \in Z$

$$\Rightarrow x = \pm\pi/2, \pm\pi, \pm 3\pi/2, \pm 2\pi, \ldots$$

Clearly, $\frac{d^2y}{dx^2} < 0$ at $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots$

And, $\frac{d^2y}{dx^2} > 0$ at $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

So, $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots$ are the points of local maximum and the local maximum value at these points is 1. Points $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ are points of local minimum and the local minimum value at these points is zero.

(iv) $y = \sin^2 x \Rightarrow \frac{dy}{dx} = \sin 2x$

Clearly, $\frac{dy}{dx} > 0$ when $0 < x < \pi/2$ and $\frac{dy}{dx} < 0$ when $\pi/2 < x < \pi$.

So, the given curve is increasing in the interval $[0, \pi/2]$ and decreasing in the interval $[\pi/2, \pi]$.

(v) $\sin^2(\pi + x) = \sin^2 x$ for all x. So, the periodicity of the function is π. This means that the shape of the curve repeats at the intervals of length 2π.
With these ideals, we may sketch the curve as shown in Fig. A.8.

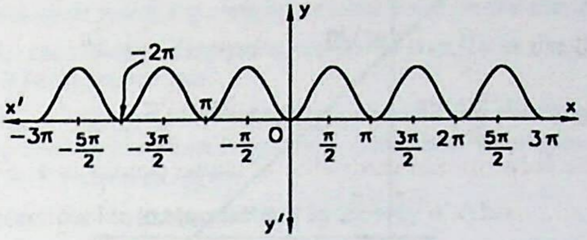

Fig. A.8

EXERCISE A.1

Sketch the following curves :

1. $y = x^3 + 1$
2. $y = x^2 - 1$
3. $y = x^4 - 1$
4. $y = \sqrt{9 - x^2}$
5. $y = 2 \cos x$
6. $y = \cos x$
7. $y = -\sin 2x$
8. $y = \sin^3 x$
9. $y = \cos^2 x$
10. $y = \sin^2 2x$
11. $y = 2 \cos 2x$

A.3 SKETCHING OF SOME STANDARD CURVES

In section A.1, we have learnt about sketching of curves using calculus. In this section, we shall discuss sketching of some standard curves viz. straight line, circle, parabola, ellipse and hyperbola, without taking the help of various points learnt in the previous section. Sketching of these curves will be very helpful in finding the areas of bounded regions.

A.3.1 STRAIGHT LINE

As we know that every first degree equation in x, y represents a straight line. The general equation of a straight line is $ax + by + c = 0$. If $c = 0$, then the line passes through the origin. In order to draw the graph of the line represented by $ax + by + c = 0$, $c \neq 0$, we may follow the following algorithm.

ALGORITHM

STEP I *Obtain the first degree equation, say,* $ax + by + c = 0, \quad c \neq 0$.

STEP II *Put* $y = 0$ *in the given equation and obtain the value of* x. *This will give a point on* x*-axis. Mark this point as A.*

STEP III *Put* $x = 0$ *in the given equation and obtain the value of* y. *This will give a point on* y*-axis. Mark this point as B.*

STEP IV *Draw line passing through A* and *B. The line so obtained will be the graph of the line represented by the given equation.*

ILLUSTRATION 1 *Draw a rough sketch of the line represented by the equation*
$3x + 2y - 6 = 0$.

SOLUTION We have,

$$3x + 2y - 6 = 0 \qquad \text{...(i)}$$

Putting $y = 0$ in (i) we get $x = 2$. So, the line (i) meets x-axis at A (2, 0).

Now, putting $x = 0$ in (i), we get $y = 3$. So, the line (i) cuts y-axis at B (0, 3).

Marking these points on the coordinate axes and drawing a line passing through them, we obtain the graph of the line represented by equation (i) as shown in Fig. A.9.

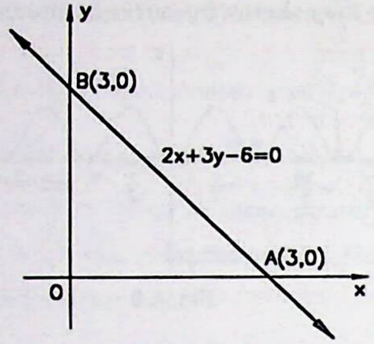

Fig. A.9

A first degree equation of the form $ax + by = 0$ or $y = mx$ always represents a straight line passing through the origin. In order to draw a rough sketch of a line passing through the origin, we may follow the following algorithm:

ALGORITHM

STEP I *Obtain the first degree equation. Let the equation be* $ax + by = 0$.

STEP II *Find the slope m of the line represented by the given equation.*
If $m > 0$, *then the line makes an acute angle with x-axis. If* $m < 0$, *then the line makes an obtuse angle with x-axis.*

STEP III *Draw a line passing through the origin and making an angle* $\tan^{-1}(m)$ *with the positive direction of x-axis.*
The line so obtained represents the graph of the line represented by the given equation.

ILLUSTRATION 2 *Draw the rough sketches of the following lines:*
(i) $y = x$ (ii) $y = -x$ (iii) $y - 2x = 0$

SOLUTION The equations $y = x$, $y = -x$ and $y - 2x = 0$ represent lines passing through the origin and making angles $\tan^{-1}(1) = 45°$, $\tan^{-1}(-1) = 135°$ and $\tan^{-1}(2)$ respectively with the positive directions of x-axis. So, their graphs are as shown in Fig. A.10.

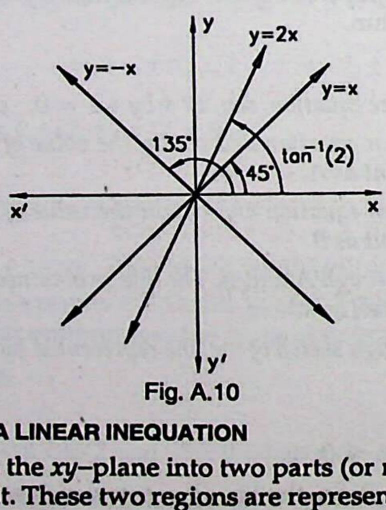

Fig. A.10

REGION REPRESENTED BY A LINEAR INEQUATION

Every straight line divides the xy–plane into two parts (or regions), one lying below it and the other lying above it. These two regions are represented by the two inequations obtained from the equation of the given line. In order to determine the region represented by a given linear inequation, we follow the following algorithm.

ALGORITHM

STEP I *Obtain the inequation and convert the inequation into an equation by replacing the inequality sign by equality sign.*

STEP II *Draw the straight line represented by the linear equation obtained in step I.*

STEP III *Choose a convenient point, e.g., origin, or some point on the coordinate axes.*

STEP IV *Substitute the coordinates of the point, chosen in step III, in the given inequation and see whether it holds true or not.*

STEP V *If the inequation holds good, then the region containing the chosen point will be the region represented by the given inequation. Otherwise, the region on the other side of the line will be the required region.*

ILLUSTRATION 3 *Mark the region represented by* $3x + 4y \leq 12$.

SOLUTION Converting the given inequation into equation, we obtain

$$3x + 4y = 12.$$

The line represented by this equation meets the coordinate axes at A (4, 0) and B (0, 3) as shown in Fig. A.11. Clearly, it divides the plane into two parts. One part containing the origin and other part on the other side of the line. We observe that O (0, 0) satisfies the inequation $3x + 4y \leq 12$. So, the region represented by the given inequation is the region containing the origin as shown in Fig. A.11.

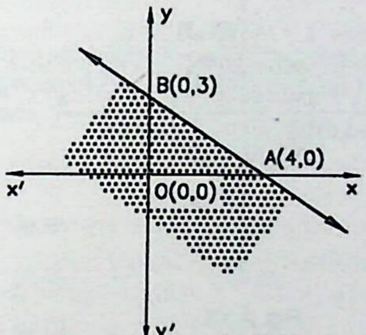

Fig. A.11

ILLUSTRATION 4 *Mark the region represented by* $y \leq x$.

SOLUTION We have,

$$y \leq x.$$

Converting the inequation into equation, we get

$$y = x.$$

Clearly, it represents a straight line passing through the origin and making 45° with x-axis. We observe that the points (2, 0), (3, 1), (0, − 5), (0, − 2) etc. satisfy the inequation $y \leq x$. So, the region containing these points is the region represented by the inequation $y \leq x$ as shown in Fig. A.12.

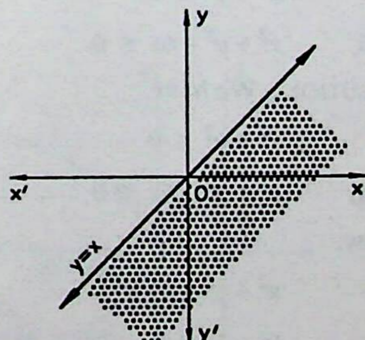

Fig. A.12

A.3.2 CIRCLE

An equation of the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

always represents a circle with centre at $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

Also, note that an equation of the form

$$(x-a)^2+(y-b)^2 = r^2$$

represents a circle with centre at (a, b) and radius r.

The inequation $(x-a)^2+(y-b)^2 \leq r^2$

represents the interior of the circle $(x-a)^2+(y-b)^2 = r^2$ and its exterior is represented by the inequation

$$(x-a)^2+(y-b)^2 \geq r^2$$

ILLUSTRATION 1 *Mark the region represented by* $x^2+y^2 \leq 9$.

SOLUTION We have,

$$x^2+y^2 \leq 9$$

Clearly, it represents the interior of the region lying inside the circle $x^2+y^2=9$ as shown in Fig. A.13.

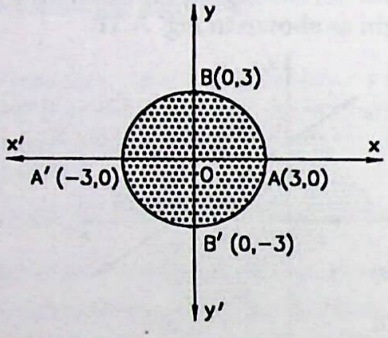

Fig. A.13

ILLUSTRATION 2 *Mark the region represented by the inequations*

$$x^2+y^2 \leq 9$$

and, $\quad x^2+y^2-6x \leq 0.$

SOLUTION We have,

$$x^2+y^2 \leq 9$$

and, $\quad x^2+y^2-6x \leq 0$

Now,

$$x^2+y^2-6x \leq 0$$

$\Rightarrow \quad (x-3)^2+(y-0)^2 \leq 3^2.$

Thus, we have

$$x^2+y^2 \leq 9 \qquad \text{...(i)}$$

and, $\quad (x-3)^2+(y-0)^2 \leq 3^2 \qquad$ ...(ii)

Clearly, inequation (i) represents the interior of the circle $x^2 + y^2 = 9$ and inequation (ii) represents the interior of the circle $(x-3)^2 + (y-0)^2 = 3^2$. The common region is shaded in Fig. A.14. It is the region Represented by both the inequations.

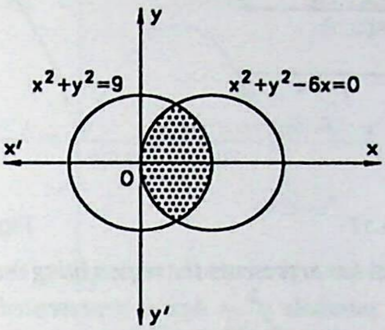

Fig. A.14

A.3.3 PARABOLA

We have learnt about four standard forms of parabola in earlier classes. Rough sketches of these parabolas are given in Fig. A.18 and various terms associated to them are given below for ready reference.

Equation	*Vertex*	*Focus*	*Latustrectum*	*Directrix*
$y^2 = 4ax$	(0, 0)	$(a, 0)$	$4a$	$x = -a$
$y^2 = -4ax$	(0, 0)	$(-a, 0)$	$4a$	$x = a$
$x^2 = 4ay$	(0, 0)	$(0, a)$	$4a$	$y = -a$
$x^2 = -4ay$	(0, 0)	$(0, -a)$	$4a$	$y = a$

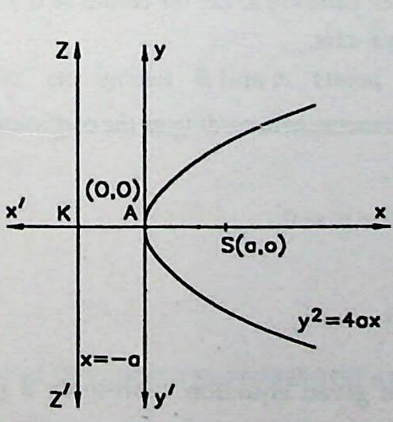

Fig. A.15

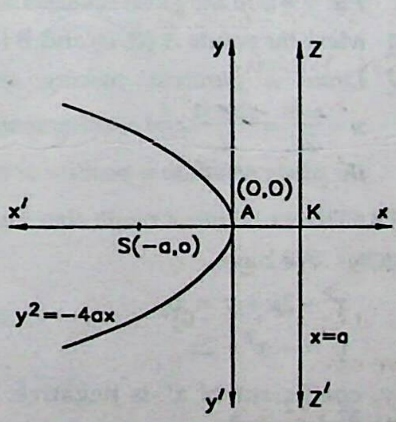

Fig. A.16

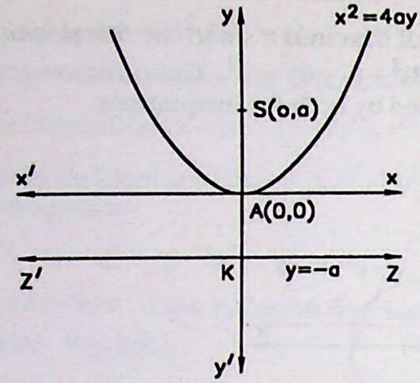

Fig. A.17

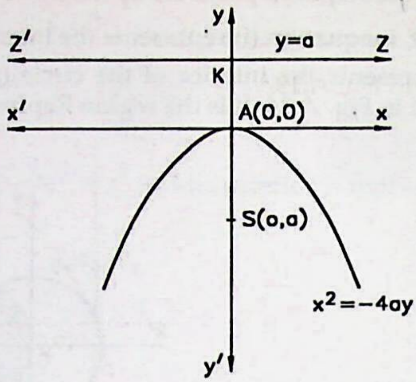

Fig. A.18

<u>REMARK</u> *The inequation $y^2 \leq 4ax$ represents the region lying inside the parabola $y^2 = 4ax$ and the region lying outside the parabola $y^2 = 4ax$ is represented by the inequation $y^2 \geq 4ax$. Similarly, the inequations $y^2 \leq -4ax$, $x^2 \leq 4ay$ and $x^2 \leq -4ay$ represent the regions lying inside the parabolas $y^2 = -4ax$, $x^2 = 4ay$ and $x^2 = -4ay$ respectively.*

SKETCHING OF CURVES REPRESENTED BY $y = ax^2 + bx + c$.

The equation $y = ax^2 + bx + c$ always represents a parabola having vertex at

$$\left(\frac{-b}{2a}, \frac{-D}{4a}\right) \text{ and axis } x = -\frac{b}{2a}.$$

The parabola opens upward or downward according as $a > 0$ or < 0. It meets x-axis at $(\alpha, 0)$ and $(\beta, 0)$, where α and β are the roots of the equation $ax^2 + bx + c = 0$. If the roots of this equation are not real, then the parabola does not cross x-axis. In order to draw rough sketch of the parabolas given by the equations of the form $y = ax^2 + bx + c$, we may follow the following algorithm.

ALGORITHM

<u>STEP I</u> *Obtain the equation and observe the sign of the coefficient of x^2 in it.*

<u>STEP II</u> *Put $y = 0$ in the given equation and get the values of x. Let the values be α and β.*

<u>STEP III</u> *Mark the points $A(\alpha, 0)$ and $B(\beta, 0)$ on x-axis.*

<u>STEP IV</u> *Draw a parabola passing through points A and B having its vertex on $x = \frac{-b}{2a} = \frac{\alpha + \beta}{2}$ and opening upward or downward according as the coefficient of x^2 in the given equation is positive or negative.*

ILLUSTRATION 1 *Draw a rough sketch of $x^2 - 2x + y = 0$.*

SOLUTION We have,

$$x^2 - 2x + y = 0$$

$$\Rightarrow \quad y = -x^2 + 2x \qquad \text{...(i)}$$

Clearly, coefficient of x^2 is negative. So, the given equation represents a parabola opening downward.

Putting $y = 0$ in (i), we get

$$-x^2 + 2x = 0 \Rightarrow x = 0, 2.$$

Therefore, the parabola cuts x-axis at $(0, 0)$ and $(2, 0)$.

Thus, the required parabola opens downward, crosses x-axis at (0, 0) and (2, 0) and its axis of symmetry is the line $x = \frac{0+2}{2}$ i.e. $x = 1$. The rough sketch of this parabola is as given in Fig. A.19.

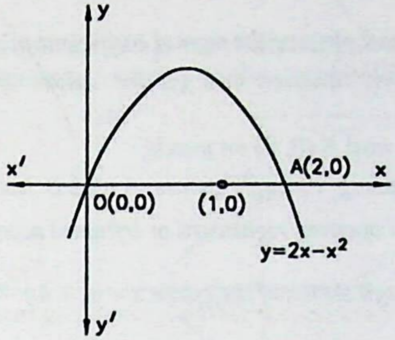

Fig. A.19

REMARK *In the above algorithm, if the values of* α and β *are imaginary, then the equation* $y = ax^2 + bx + c$ *represents a parabola having vertex at* $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ *and opens upward or downward according as* $a > 0$ or $a < 0$.

ILLUSTRATION 2 *Draw a rough sketch of the curve* $y = x^2 + 2$.

SOLUTION We have,

$$y = x^2 + 2.$$

Since coefficient of x^2 is positive. Therefore, $y = x^2 + 2$ represents a parabola opening upward. We observe that $x^2 + 2 = 0$ gives imaginary values of x. So, the parabola does not cross x-axis. The coordinates of the vertex are (0, 2). The rough sketch of the curve is as shown in Fig. A.20.

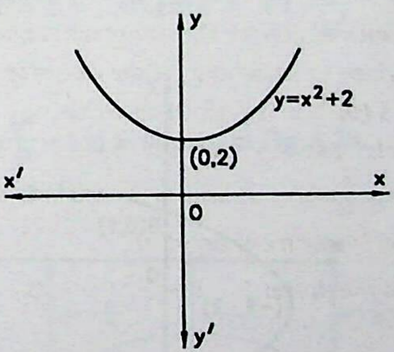

Fig. A.20

SKETCHING OF CURVES REPRESENTED BY $x = ay^2 + by + c$.

The equation $x = ay^2 + by + c$ also represents a parabola having vertex at $\left(\frac{-D}{4a}, \frac{-b}{2a}\right)$ axis $y = \frac{-b}{2a}$ and it opens leftward or rightward according as $a < 0$ or $a > 0$. It crosses y-axis at

$(0, \alpha)$ and $(0, \beta)$, where α and β are the roots of the equation $ay^2 + by + c = 0$. If α, β are not real, then the parabola does not cross y–axis and it opens rightward if $a > 0$ and leftward if $a < 0$. In order to draw a rough sketch of the parabolas given by the equations of the form $x = ay^2 + by + c$, we may follow the following algorithm.

ALGORITHM

STEP I *Obtain the equation and observe the sign of coefficient of y^2 i.e. of a.*

STEP II *Put $x = 0$ in the given equation and get the values of y. Let the values of y be α and β.*

STEP III *Mark points A $(0, \alpha)$ and B $(0, \beta)$ on y-axis.*

STEP IV *Draw a parabola passing through points A and B having its vertex on the line $y = \frac{-b}{2a} = \left(\frac{\alpha + \beta}{2}\right)$ and opening rightward or leftward according as $a > 0$ or, $a < 0$.*

ILLUSTRATION 1 *Draw a rough sketch of the curve $x = y^2 + 4y - 5$*

SOLUTION We have,

$$x = y^2 + 4y - 5 \qquad \text{...(i)}$$

Clearly, coefficient of $y^2 > 0$. So, the given equation represents a parabola opening rightward.

Putting $x = 0$ in (i), we get

$$y^2 + 4y - 5 = 0$$

$$\Rightarrow \quad (y + 5)(y - 1) = 0$$

$$\Rightarrow \quad y = -5, 1.$$

So, the parabola cuts y-axis at A $(0, -5)$ and B $(0, 1)$. The vertex of the parabola is on the line $y = \frac{-5+1}{2}$ i.e. $y = -2$. The coordinates of the vertex are $(-9, -2)$.

Thus, the given equation represents a parabola opening rightward having its vertex at $(-9, -2)$ and cuts y-axis at $(0, -5)$ and $(0, 1)$. The rough sketch is as given in Fig. A.21.

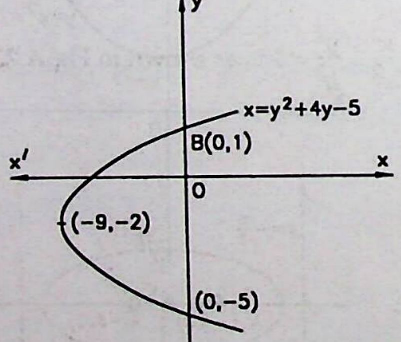

Fig. A.21

ILLUSTRATION 2 *Draw a rough sketch of the curve $x = 2y^2 + 5$*

SOLUTION We have,

$$x = 2y^2 + 5$$

Clearly, it represents a parabola opening rightward. Since $2y^2 + 5 = 0$ gives imaginary values of y. So, the curve does not cut y-axis. The coordinates of the vertex are (5, 0) and axis is x-axis.

Thus, the given equation represents a parabola having its axis as x-axis, vertex at (5, 0) and it opens rightward. The rough sketch of the parabola is as shown in Fig. A.22.

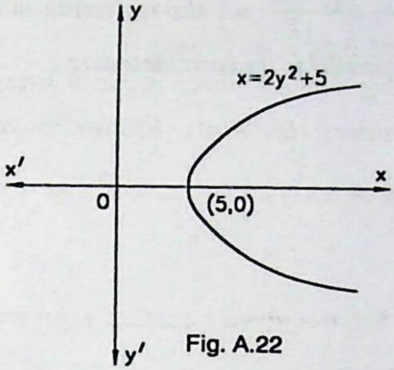

Fig. A.22

A.3.4 ELLIPSE

The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents an ellipse having the following properties:

Centre	(0, 0)
Vertices	$(a, 0), (-a, 0)$
Major axis	$2a$
Minor axis	$2b$
Directrices	$x = \pm\frac{a}{e}$, where $e = \sqrt{1 - \frac{b^2}{a^2}}$

Foci $(\pm ae, 0)$

The rough sketch of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is as shown in Fig.A.23.

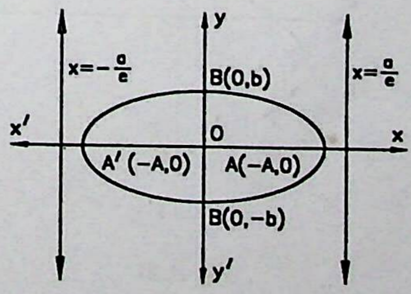

Fig. A.23

REMARK 1 *The inequation* $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$ *represents the region enclosed by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *whereas the inequation* $\frac{x^2}{a^2}+\frac{y^2}{b^2}\ge 1$ *represents the region lying outside the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

REMARK 2 *The equation* $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ *also represents an ellipse having its centre at* (h, k) *and major and minor axes parallel to the coordinate axes.*

22·57
22·109

$$I_C = \beta I_B + (\beta+1) I_{CBO}$$

$$S = \frac{dI_C}{dI_{CO}}$$

$$= \frac{d}{dI_{CO}}\left(\beta I_B + (\beta+1) I_{CBO}\right)$$

$$= (\beta+1).$$

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This text book is based on the the latest syllabus prescribed by the CBSE. The whole syllabus has been divided into two volumes. Volume - I consists of Chapter 1-19 and Volume - II consists of Chapter 20-32. In this revised edition new chapters on Relations, Functions, Binary Operations and Inverse Trigonometric Functions have been included. All other chapters have been thoroughly revised and up-dated. In each chapter all concepts and definitions have been discussed in detail with suitable **illustrative examples.** At the end of each chapters an exercise consisting of **multiple choice questions (MCQs)** have been given. For the sake of quick revision of concepts and formulae, a brief **summary** has been given at the end of each chapter.

Some new and unique features of this book

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He has undergone rigorous training from IIT, Kharagpur in computer-oriented mathematical methods and has a long experience of teaching post graduate, graduate and engineering students.

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